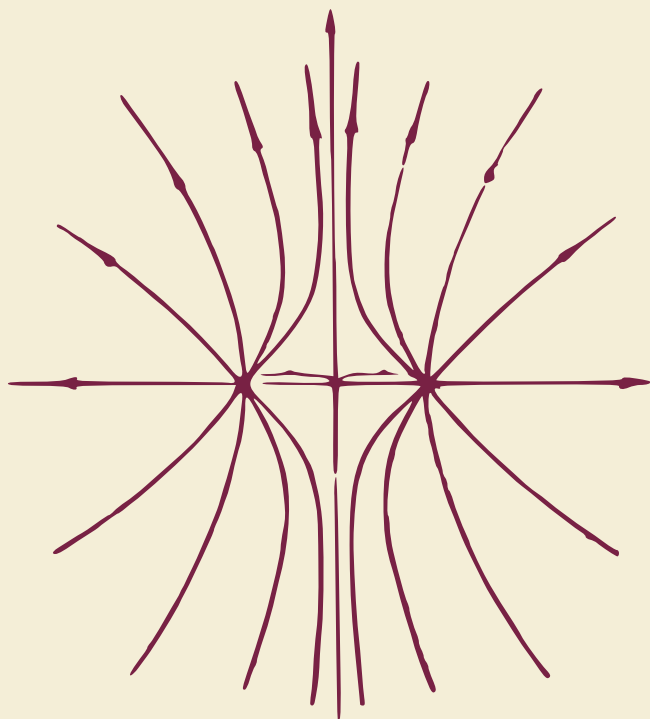
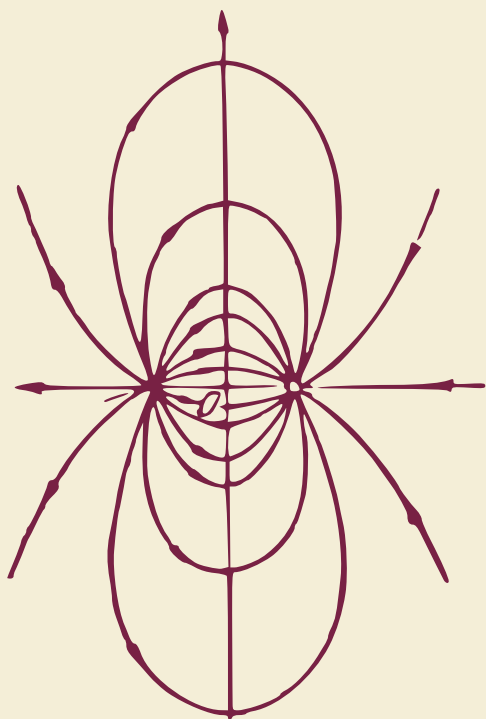


V.V. Batygin, I.N. Toptygin

PROBLEMS *in* ELECTRO- DYNAMICS



PROBLEMS in ELECTRODYNAMICS

V. V. Batygin and I. N. Toptygin

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in
ELECTRODYNAMICS

Translated by
S. Chomet

Edited by
P. J. Dean

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PREFACE

This book contains about 750 problems on classical electrodynamics and its more important applications, including over 150 problems on the special theory of relativity, and about 70 problems on vector and tensor analysis.

In addition to problems illustrating fundamental concepts and laws of electrodynamics, which can be solved by purely mathematical methods, the collection includes a large number of more complicated problems (these are indicated by asterisks). Some of the solutions involve a considerable amount of effort, while others are purely theoretical in nature and follow from a lecture course on electrodynamics (propagation of waves in anisotropic and gyrotropic media, motion of charged particles in the electromagnetic field, representation of the electromagnetic field by a set of oscillators, and so on). Finally, there are problems which are concerned with topics which are not well covered by existing texts, for example, interaction of charged particles with matter (Chapter XIII), applications of conservation laws to the analysis of collision processes and particle disintegration (Chapter XI), ferromagnetic resonance (Chapter VI), and so on.

The second part of the book gives answers and solutions to a large number of these problems.

Each section is prefaced by a short theoretical introduction in which the necessary formulae are given. These short introductions do not pretend to be complete; more extensive treatments will be found in the books listed in the bibliography.

The mathematical appendices review the basic properties of the δ -function and the cylindrical and spherical functions, which are necessary for the solution of the problems.

The present collection is based on lectures given in the Departments of Electronics and Physics and Mechanics of the Leningrad Polytechnical Institute. A large number of the problems were set to third and fourth year students.

In compiling this collection we have made frequent use of the well-known texts of L. D. Landau and E. M. Lifshits, I. E. Tamm, Ya. I. Frenkel', M. Abraham and R. Becker, W. R. Smythe, J. A. Stratton, and others. A large number of monographs, review papers, and original papers were also consulted.

V. Batygin and I. Toptygin

Problems

Chapter 1

VECTOR AND TENSOR CALCULUS

1. Vector and tensor algebra. Transformations of vectors and tensors

A vector in three-dimensional space is defined as a set of three quantities which transform in accordance with the rule

$$A'_i = \sum_{k=1}^3 \alpha_{ik} A_k \quad (\text{I.1})$$

when the system of coordinates is rotated. Here A_k are the components of the vector along the axes of the original system of coordinates, A'_i are the components along the axes of the rotated system, and α_{ik} are the transformation coefficients which are equal to the cosines of the angles between the k -th axis of the original system and the i -th axis of the rotated system.

We shall use the following summation rule: the summation sign will be omitted and the summation process will be indicated by a repeated subscript. In accordance with this convention Eqn. (I.1) may be re-written in the form

$$A'_i = \alpha_{ik} A_k.$$

A tensor of rank 2 in three-dimensional space is defined as the nine-component quantity T_{ik} ($i, k = 1, 2, 3$) which transforms in accordance with the rule

$$T'_{ik} = \alpha_{il} \alpha_{km} T_{lm} \quad (\text{I.2})$$

where the summation is to be taken over l and m . Similarly, a tensor of rank s in three-dimensional space is defined by the following transformation rule:

$$T'_{i_1 i_2 \dots i_s} = \alpha_{i_1 l_1} \alpha_{i_2 l_2} \dots \alpha_{i_s l_s} T_{l_1 l_2 \dots l_s}. \quad (\text{I.3})$$

In this expression the quantities T have s indices each.

Quantities which transform as vectors when the coordinate system is rotated may behave in two distinct ways when the system of coordinates is inverted, *i.e.* it is subjected to the transformation $x' = -x$, $y' = -y$, $z' = -z$. Vectors whose components change sign on inversion of the coordinate system are known as polar vectors, or simply vectors. Vectors whose components do not change sign on inversion of the coordinate system are called pseudovectors or axial vectors. We shall not distinguish between covariant and contravariant components of vectors and tensors because this distinction is unimportant for the problems considered in this book. The vector product of two polar vectors is an example of an axial vector. Similarly, a tensor of rank s is referred to simply as a tensor if its components transform on inversion as the products of s coordinates, *i.e.* when the result of the transformation is to multiply them by $(-1)^s$. When the result of the inversion is to multiply the components by $(-1)^{s+1}$ then the tensor is referred to as a pseudotensor.

The quantity

$$\hat{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (\text{I.4})$$

is called the transformation matrix. The determinant whose elements are equal to the elements of a given matrix is called the determinant of that matrix. Thus, the determinant of the matrix given by Eqn. (I.4) is

$$|\hat{\alpha}| = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}. \quad (\text{I.5})$$

The sum $\hat{\alpha} + \hat{\beta}$ of two matrices is defined as the matrix $\hat{\gamma}$ whose elements are equal to the sums of the corresponding elements in the component matrices. Thus

$$\gamma_{ik} = \alpha_{ik} + \beta_{ik}. \quad (\text{I.6})$$

The product $\hat{\alpha}\hat{\beta}$ of two matrices is defined as the matrix $\hat{\gamma}$ whose elements are obtained by multiplying the components α_{il} and β_{lk} in accordance with the rule

$$\gamma_{ik} = \alpha_{il}\beta_{lk} \quad (\text{I.7})$$

where the summation is to be carried out over l . The matrix $\hat{\gamma}$

represents the transformation which results after the successive application of the transformations $\hat{\beta}$ and $\hat{\alpha}$.

The unit matrix is defined by

$$\hat{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{I.8})$$

and describes the transformation yielding $A'_i = A_i$. The elements of the unit matrix may be represented by the symbol δ_{ik} which is such that

$$\delta_{ik} = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases} \quad (\text{I.9})$$

A matrix of the form

$$\hat{\alpha} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \quad (\text{I.10})$$

is called a diagonal matrix. A matrix whose elements satisfy the condition

$$\alpha_{ik}\alpha_{il} = \delta_{kl} \quad (\text{I.11})$$

is called an orthogonal matrix.

The reciprocal (or inverse) matrix $\hat{\alpha}^{-1}$ is defined by

$$\hat{\alpha}\hat{\alpha}^{-1} = \hat{\alpha}^{-1}\hat{\alpha} = \hat{1}, \quad (\text{I.12})$$

and describes the reciprocal transformation, *i.e.* if $A'_i = \alpha_{ik}A_k$ then $A_k = \alpha_{ki}^{-1}A'_i$. Finally, the transposed matrix $\tilde{\alpha}$, is obtained from $\hat{\alpha}$ by interchanging the columns and the rows, so that

$$\tilde{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}, \quad \tilde{\alpha}_{ik} = \alpha_{ki}. \quad (\text{I.13})$$

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1. Find the cosine of the angle θ between two directions $\mathbf{n}$ and $\mathbf{n}'$ which are defined in a spherical system of coordinates by the angles α , β and α' , β' respectively.

2. Prove the identities

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C});$$

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D})] \mathbf{C} - [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})] \mathbf{D} = \\ &= [\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})] \mathbf{B} - [\mathbf{B} \cdot (\mathbf{C} \times \mathbf{D})] \mathbf{A}. \end{aligned}$$

3. A set of three quantities a_i ($i = 1, 2, 3$) is given in all cartesian systems of coordinates and it is known that $a_i b_i$ is invariant with respect to rotations and reflections. Show that if b_i is a vector (pseudovector) then a_i is also a vector (pseudovector).

4. Show that if $a_i = T_{ik} b_k$ in every system of coordinates, where T_{ik} is a tensor of rank 2 while b_k is a vector, then a_i is also a vector.

5. Show that $\frac{\partial a_i}{\partial x_k}$ is a tensor of rank 2.

6. Show that if T_{ik} is a tensor of rank 2 and P_{ik} is a pseudotensor of rank 2, then $T_{ik} P_{ik}$ is a pseudoscalar.

7. Show that the symmetry of a tensor is a property which is invariant with respect to rotations, *i.e.* a tensor which is symmetric (skew-symmetric) in a given coordinate system remains symmetric (skew-symmetric) in all other systems which are rotated with respect to the original system.

8. Show that if the tensor S_{ik} is symmetric while the tensor A_{ik} is skew-symmetric, then $A_{ik} S_{ik} = 0$.

9. Show that the sum of the diagonal components of a tensor of rank 2 is an invariant quantity.

10*. In some cases it is convenient to use the cyclic components defined by $a_{\pm 1} = \mp(a_x \pm ia_y)/\sqrt{2}$, $a_0 = a_z$ instead of the cartesian components a_x, a_y, a_z of a vector. Express the scalar and vector products of two vectors in terms of their cyclic components. Find also the cyclic components of the position vector in terms of the spherical Legendre functions†.

11*. Find the components of the tensor ε_{ik}^{-1} which is reciprocal to ε_{ik} . Consider in particular the case where ε_{ik} is a symmetric tensor defined along the principal axes.

12. Suppose that in all coordinate systems the components of the vector $\mathbf{a}$ are linear functions of the components of another vector $\mathbf{b}$ so that $a_i = \varepsilon_{ik} b_k$. Show that the quantity ε_{ik} is a

† Spherical functions are defined in Appendix II.

tensor of rank 2. (More precisely, ϵ_{ik} is a tensor if $\mathbf{a}$ and $\mathbf{b}$ are both polar vectors or pseudovectors, while ϵ_{ik} is a pseudotensor if one of the vectors is polar and the other is axial.)

13. Show that the set of quantities $A_{ikl}B_{ik}$, where A_{ikl} is a tensor of rank 3 and B_{ik} is a tensor of rank 2, is a vector.

14. Find the transformation law for the set of volume integrals $T_{ik} = \int x_i x_k dV$, which describes space rotations and reflections (x_i and x_k are the cartesian coordinates).

15. Set up the transformation matrices for the basic unit vectors which represent the transformation from cartesian coordinates to spherical coordinates and *vice versa*, and also the transformation from cartesian coordinates to cylindrical coordinates and *vice versa*.

16. Write down the transformation matrices for the components of a vector which describe the reflection of the three coordinate axes and the rotation of the cartesian system of coordinates about the z -axis through an angle α .

17. Find the transformation matrix for the components of a vector, which describes the rotation of the coordinate axes defined by the Euler angles $\alpha_1, \theta, \alpha_2$ (Fig. 1), by multiplying together the matrices corresponding to rotations about the z -axis through angle α_1 , about the line ON through an angle θ , and about the z' -axis through an angle α_2 .

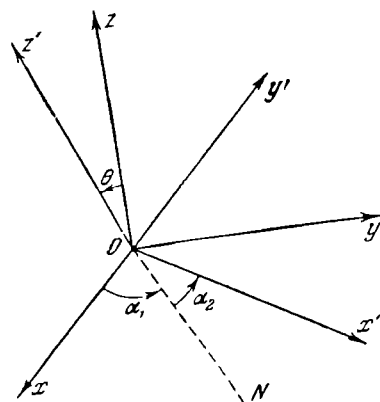


Fig. 1.

18. Find the matrix $\hat{D}(\alpha_1, \theta, \alpha_2)$ for the transformation of the cyclic coordinates of a vector (cf. Problem 10) when the coordinate system is rotated through the Euler angles α_1, θ , and α_2 (Fig. 1).

19*. Show that the matrix representing an infinitesimal rotation of a system of coordinates may be written in the form $\hat{\alpha} = \hat{1} + \hat{\epsilon}$ where $\hat{\epsilon}$ is a skew-symmetric matrix ($\epsilon_{ik} = -\epsilon_{ki}$). Elucidate the geometrical meaning of ϵ_{ik} .

20. Show that if $\hat{\alpha}$ is an orthogonal transformation matrix then the corresponding transposed matrix represents the reciprocal transformation.

21. Show that the matrix representing the reflection or rotation of the three basic unit vectors of a coordinate system

is identical to the matrix describing the transformation of the components of a vector.

22*. Show that when an even number of coordinate axes are reflected or rotated, the transformation determinant is equal to +1, while the corresponding result for an odd number of coordinate axes is -1.

23. Show that if the components of two vectors in a given system of coordinates are respectively proportional to each other, then they are also proportional in any other system of coordinates (such vectors are called parallel).

24*. The set of quantities e_{ikl} is given in all cartesian systems of coordinates and has the following property: transposition of any two subscripts of e_{ikl} gives rise to a change of sign, and $e_{123} = 1$. Show that e_{ikl} is a pseudotensor of rank 3.

25. Show that the components of a skew-symmetric tensor of rank 2 transform on rotation as the components of a vector.

26. Write down expressions for the components of the vector product of two vectors and for the curl of a vector in terms of the tensor e_{ikl} . Determine how these quantities transform on rotation and reflection.

27. Prove the following equations

$$e_{ikl}e_{lmn} = \delta_{lm}\delta_{kn} - \delta_{ln}\delta_{km}; \quad e_{ikl}e_{klm} = 2\delta_{lm}.$$

28. Re-write the following expressions in an invariant vector form:

$$e_{int}e_{irs}e_{lmp}e_{stp}a_n a_r b_m c_t; \quad e_{int}e_{krs}e_{lmp}e_{stp}a_r a'_n b'_k b'_i c'_t c'_m.$$

29. Show that $T_{ik}a_i b_k - T_{ik}a_k b_i = 2\omega \cdot (\mathbf{a} \times \mathbf{b})$ where T_{ik} is an arbitrary tensor of rank 2, $\mathbf{a}$ and $\mathbf{b}$ are vectors, and ω is a vector equivalent to the skew-symmetric part of T_{ik} .

30. Express the product $[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})][\mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}')] in the form of a sum of terms containing only scalar products of vectors.$

Hint: Use the theorem on the multiplication of determinants, or the pseudotensor e_{ikl} of rank 3 (cf. Problem 24).

31*. Show that: (1) the only vector whose components are identical in all systems of coordinates is the zero vector, (2) any tensor of rank 2 whose components are identical in all systems of coordinates is proportional to δ_{ik} , (3) e_{ikl} is a tensor of rank 3, and (4) $(\delta_{ik}\delta_{lm} + \delta_{im}\delta_{kl} + \delta_{il}\delta_{km})$ is a tensor of rank 4.

32*. Suppose that $\mathbf{n}$ is a unit vector which is such that it is equally likely to lie along any direction in space. Find the

average values of its components and of the products $\overline{n_i}$, $\overline{n_i n_k}$, $\overline{n_i n_k n_l}$, $\overline{n_i n_k n_l n_m}$ by using their transformation properties rather than by the direct evaluation of the corresponding integrals.

33. Find the averages over all directions of the following expressions: $\overline{(\mathbf{a} \cdot \mathbf{n})^2}$, $\overline{(\mathbf{a} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n})}$, $\overline{(\mathbf{a} \cdot \mathbf{n})\mathbf{n}}$, $\overline{(\mathbf{a} \times \mathbf{n})^2}$, $\overline{(\mathbf{a} \times \mathbf{n}) \cdot (\mathbf{b} \times \mathbf{n})}$, $\overline{(\mathbf{a} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n})(\mathbf{c} \cdot \mathbf{n})(\mathbf{d} \cdot \mathbf{n})}$ where $\mathbf{n}$ is a unit vector which is such that it is equally likely to lie along any direction and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are constant vectors.

Hint: Use the results of the preceding problem.

34. Obtain all the possible independent invariant forms involving the polar vectors $\mathbf{n}$, $\mathbf{n}'$, and the pseudovector $\mathbf{l}$.

35. Find the independent pseudoscalars which can be constructed from (1) two polar vectors $\mathbf{n}$, $\mathbf{n}'$ and one pseudovector $\mathbf{l}$, and (2) three polar vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$.

2. Vector analysis

In an arbitrary orthogonal system of coordinates q_1 , q_2 , q_3 the square of an element of length is given by

$$dl^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2, \quad (\text{I.14})$$

and an element of volume is given by

$$dV = h_1 h_2 h_3 dq_1 dq_2 dq_3, \quad (\text{I.15})$$

where

$$h_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2} \quad (\text{I.16})$$

are functions of the coordinates. The various differential operations may then be written down as follows:

$$\begin{aligned} (\text{grad } \varphi)_i &= \frac{1}{h_i} \frac{\partial \varphi}{\partial q_i}; \\ \text{div } \mathbf{A} &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_1 h_3 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right]; \\ \text{curl } \mathbf{A} &= \begin{vmatrix} \frac{\mathbf{e}_1}{h_2 h_3} & \frac{\mathbf{e}_2}{h_1 h_3} & \frac{\mathbf{e}_3}{h_1 h_2} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}; \\ \Delta \varphi &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \varphi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \varphi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \varphi}{\partial q_3} \right) \right]. \end{aligned} \quad (\text{I.17})$$

In the formula for curl $\mathbf{A}$ the differential operators $\partial/\partial q_i$ operate on the elements of the last row of the determinant.

In spherical polar coordinates we have

$$\left. \begin{aligned} x &= r \sin \vartheta \cos \alpha, \quad y = r \sin \vartheta \sin \alpha, \quad z = r \cos \vartheta; \\ h_r &= 1, \quad h_\vartheta = r, \quad h_\alpha = r \sin \vartheta; \\ \text{grad } \varphi &= \mathbf{e}_r \frac{\partial \varphi}{\partial r} + \frac{\mathbf{e}_\vartheta}{r} \frac{\partial \varphi}{\partial \vartheta} + \frac{\mathbf{e}_\alpha}{r \sin \alpha} \frac{\partial \varphi}{\partial \alpha}; \\ \text{div } \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (A_\vartheta \sin \vartheta) + \frac{1}{r \sin \vartheta} \frac{\partial A_\alpha}{\partial \alpha}; \\ (\text{curl } \mathbf{A})_r &= \frac{1}{r \sin \vartheta} \left[\frac{\partial}{\partial \vartheta} (A_\alpha \sin \vartheta) - \frac{\partial A_\vartheta}{\partial \alpha} \right]; \\ (\text{curl } \mathbf{A})_\vartheta &= \frac{1}{r \sin \vartheta} \frac{\partial A_r}{\partial \alpha} - \frac{1}{r} \frac{\partial (r A_\alpha)}{\partial r}; \\ (\text{curl } \mathbf{A})_\alpha &= \frac{1}{r} \frac{\partial (r A_\vartheta)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \vartheta}; \\ \Delta \varphi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial \varphi}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 \varphi}{\partial \alpha^2}. \end{aligned} \right\} \quad (\text{I.18})$$

In the cylindrical system of coordinates we have

$$\left. \begin{aligned} x &= r \cos \alpha, \quad y = r \sin \alpha, \quad z = z; \\ h_r &= 1, \quad h_\alpha = r, \quad h_z = 1; \\ \text{grad } \varphi &= \mathbf{e}_r \frac{\partial \varphi}{\partial r} + \frac{\mathbf{e}_\alpha}{r} \frac{\partial \varphi}{\partial \alpha} + \mathbf{e}_z \frac{\partial \varphi}{\partial z}; \\ \text{div } \mathbf{A} &= \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\alpha}{\partial \alpha} + \frac{\partial A_z}{\partial z}; \\ (\text{curl } \mathbf{A})_r &= \frac{1}{r} \frac{\partial A_z}{\partial \alpha} - \frac{\partial A_\alpha}{\partial z}; \quad (\text{curl } \mathbf{A})_\alpha = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}; \\ (\text{curl } \mathbf{A})_z &= \frac{1}{r} \frac{\partial}{\partial r} (r A_\alpha) - \frac{1}{r} \frac{\partial A_r}{\partial \alpha}; \\ \Delta \varphi &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{\partial^2 \varphi}{\partial z^2}. \end{aligned} \right\} \quad (\text{I.19})$$

The following identities hold for any $\mathbf{A}$ and φ

$$\text{curl grad } \varphi \equiv 0, \quad \text{div curl } \mathbf{A} \equiv 0, \quad \text{div grad } \varphi \equiv \Delta \varphi. \quad (\text{I.20})$$

The following integral theorems may be used to transform volume, surface, and contour integrals.

Gauss-Ostrogradskii theorem:

$$\int_V \text{div } \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S}, \quad (\text{I.21})$$

where V is a given volume and S is a closed surface bounding V .

Stokes' theorem:

$$\oint_l \mathbf{A} \cdot d\mathbf{l} = \int_S \text{curl } \mathbf{A} \cdot d\mathbf{S}, \quad (\text{I.22})$$

where l is a closed contour and S is an arbitrary surface drawn through this contour.

In Eqns. (I.21) and (I.22) the vector $\mathbf{A}$ should be a differentiable function of coordinates.

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36. Write down the cyclic components of the gradient in spherical coordinates (*cf.* Problem 10).

37. Use cartesian, spherical, and cylindrical coordinates to obtain expressions for $\text{div } \mathbf{r}$, $\text{curl } \mathbf{r}$, $\text{grad } (\mathbf{l} \cdot \mathbf{r})$, and $(\mathbf{l} \cdot \nabla) \mathbf{r}$ where $\mathbf{r}$ is the position vector and $\mathbf{l}$ a constant vector.

38. Use only spherical (or cylindrical) coordinates to find $\text{curl } (\boldsymbol{\omega} \times \mathbf{r})$ where $\boldsymbol{\omega}$ is a constant vector parallel to the z -axis.

39. Prove the following identities:

$$\text{grad } (\varphi \psi) = \varphi \text{ grad } \psi + \psi \text{ grad } \varphi;$$

$$\text{div } (\varphi \mathbf{A}) = \varphi \text{ div } \mathbf{A} + \mathbf{A} \cdot \text{grad } \varphi;$$

$$\text{curl } (\varphi \mathbf{A}) = \varphi \text{ curl } \mathbf{A} - \mathbf{A} \times \text{grad } \varphi;$$

$$\text{div } (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{B};$$

$$\text{curl } (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \text{ div } \mathbf{B} - \mathbf{B} \text{ div } \mathbf{A} + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B};$$

$$\text{grad } (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times \text{curl } \mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{A} + (\mathbf{B} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{B}.$$

Hint: Use the operator ∇ and the differentiation and multiplication rules for vectors and not the components along the coordinate axes.

40. Prove the identities

$$\mathbf{C} \cdot \text{grad } (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot (\mathbf{C} \cdot \nabla) \mathbf{B} + \mathbf{B} \cdot (\mathbf{C} \cdot \nabla) \mathbf{A};$$

$$(\mathbf{C} \cdot \nabla) (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times (\mathbf{C} \cdot \nabla) \mathbf{B} - \mathbf{B} \times (\mathbf{C} \cdot \nabla) \mathbf{A};$$

$$(\nabla \cdot \mathbf{A}) \mathbf{B} = (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{B} \text{ div } \mathbf{A};$$

$$(\mathbf{A} \times \mathbf{B}) \cdot \text{curl } \mathbf{C} = \mathbf{B} \cdot (\mathbf{A} \cdot \nabla) \mathbf{C} - \mathbf{A} \cdot (\mathbf{B} \cdot \nabla) \mathbf{C};$$

$$(\mathbf{A} \times \nabla) \times \mathbf{B} = (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A} \times \text{curl } \mathbf{B} - \mathbf{A} \text{ div } \mathbf{B};$$

$$(\nabla \times \mathbf{A}) \times \mathbf{B} = \mathbf{A} \text{ div } \mathbf{B} - (\mathbf{A} \cdot \nabla) \mathbf{B} - \mathbf{A} \times \text{curl } \mathbf{B} - \mathbf{B} \times \text{curl } \mathbf{A}.$$

41. Write down expressions for $\text{grad } \varphi(\mathbf{r})$, $\text{div } \varphi(\mathbf{r}) \mathbf{r}$, $\text{curl } \varphi(\mathbf{r}) \mathbf{r}$, and $(\mathbf{l} \cdot \nabla) \varphi(\mathbf{r}) \mathbf{r}$.

42. Find the function $\varphi(\mathbf{r})$ which satisfies the condition $\text{div } \varphi(\mathbf{r}) \mathbf{r} = 0$.

43. Find the divergence and curl of the following vectors:

$(\mathbf{a} \cdot \mathbf{r})\mathbf{b}$, $(\mathbf{a} \cdot \mathbf{r})\mathbf{r}$, $\mathbf{a} \times \mathbf{r}$, $\varphi(r)(\mathbf{a} \times \mathbf{r})$, $\mathbf{r} \times (\mathbf{a} \times \mathbf{r})$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

44. Evaluate $\text{grad } \mathbf{A}(r) \cdot \mathbf{r}$, $\text{grad } \mathbf{A}(r) \cdot \mathbf{B}(r)$, $\text{div } \varphi(r)\mathbf{A}(r)$, $\text{curl } \varphi(r)\mathbf{A}(r)$, $(\mathbf{l} \cdot \nabla)\varphi(r)\mathbf{A}(r)$.

45. Evaluate $\text{grad } (\mathbf{p} \cdot \mathbf{r}/r^3)$ and $\text{curl } (\mathbf{p} \times \mathbf{r}/r^3)$ where $\mathbf{p}$ is a constant vector, by using the expressions for the gradient and the curl in terms of spherical coordinates. Give a geometrical interpretation of these vectors.

46. Show that $(\mathbf{A} \cdot \nabla)\mathbf{A} = -\mathbf{A} \times \text{curl } \mathbf{A}$ when $\mathbf{A}^2 = \text{const.}$

47. Write down the components of the vector $\Delta\mathbf{A}$ along the axes of the spherical system of coordinates.

Hint: Use the identity $\Delta\mathbf{A} = -\text{curl curl } \mathbf{A} + \text{grad div } \mathbf{A}$.

48. Write down the components of the vector $\Delta\mathbf{A}$ along the axes of the cylindrical system of coordinates.

49. Transform the volume integral $\int (\text{grad } \varphi \cdot \text{curl } \mathbf{A}) dV$ into a surface integral.

50. Evaluate the integrals $\oint \mathbf{r}(\mathbf{a} \cdot \mathbf{n}) dS$, $\oint (\mathbf{a} \cdot \mathbf{r})\mathbf{n} dS$ where $\mathbf{a}$ is a constant vector and $\mathbf{n}$ is a unit normal.

51. Transform the surface integrals $\oint \mathbf{n} \varphi dS$, $\oint (\mathbf{n} \times \mathbf{a}) dS$, $\oint (\mathbf{n} \cdot \mathbf{b})\mathbf{a} dS$ into volume integrals, where $\mathbf{b}$ is a constant vector and $\mathbf{n}$ a unit normal.

52. Use one of the identities established in the preceding problem to derive the law of Archimedes by summing all the forces on the surface elements of a body immersed in a liquid.

53*. Let $f(\mathbf{a}, \mathbf{r})$ satisfy the condition

$$f(c_1\mathbf{a}_1 + c_2\mathbf{a}_2, \mathbf{r}) = c_1f(\mathbf{a}_1, \mathbf{r}) + c_2f(\mathbf{a}_2, \mathbf{r}),$$

where c_1 and c_2 are arbitrary constants, and suppose further that $f(\mathbf{a}, \mathbf{r})$ is a differentiable function of $\mathbf{r}$. Show that if V is an arbitrary volume, S is its bounding surface, and $\mathbf{n}$ is a unit normal to this surface then

$$\oint f(\mathbf{n}, \mathbf{r}) dS = \int f(\nabla, \mathbf{r}) dV,$$

which is the generalised Gauss-Ostrogradskii theorem. The operator ∇ in the integrand $f(\nabla, \mathbf{r})$ operates on $\mathbf{r}$ and lies on the left of all the variables.

Hint: Express $\mathbf{n}$ in terms of the unit vectors of a cartesian system of coordinates and use the Gauss-Ostrogradskii theorem

$$\int \frac{\partial \varphi}{\partial x} dV = \oint \varphi n_x dS.$$

54. Solve Problems 50 and 51 with the aid of the generalised Gauss-Ostrogradskii theorem proved in the preceding problem.

55. Transform the closed line integral $\oint \varphi d\mathbf{l}$ into a surface integral.

56. Transform the closed line integral $\oint u df$ into a surface integral, where u and f are scalar functions of coordinates.

57. Prove the identity

$$\int (\mathbf{A} \cdot \text{curl curl } \mathbf{B} - \mathbf{B} \cdot \text{curl curl } \mathbf{A}) dV = \oint [(\mathbf{B} \times \text{curl } \mathbf{A}) - (\mathbf{A} \times \text{curl } \mathbf{B})] \cdot d\mathbf{S}.$$

58. The vector $\mathbf{A}$ satisfies the condition $\text{div } \mathbf{A} = 0$ within the volume V , while on the surface S which bounds this volume $\mathbf{A}_n = 0$. Show that $\int_V \mathbf{A} dV = 0$.

59*. Show that $\text{div}_R \int \frac{\mathbf{A}(\mathbf{r}) dV}{|\mathbf{R} - \mathbf{r}|} = 0$ where $\mathbf{A}(\mathbf{r})$ is a vector satisfying the conditions of Problem 58.

60. Prove the Gauss-Ostrogradskii theorem

$$\int \frac{\partial T_{ik}}{\partial x_i} dV = \oint T_{ik} dS_i$$

for a three-dimensional tensor of rank 2.

Hint: Start with the Gauss-Ostrogradskii theorem for a vector $\mathbf{A}_i = T_{ik} \mathbf{a}_k$ where $\mathbf{a}$ is an arbitrary constant vector.

61. Find the general form of the solution of Laplace's equation for the scalar functions $f(r)$, $f(\vartheta)$, and $f(\alpha)$ where r , ϑ , and α are the spherical polar coordinates.

62. Repeat Problem 61 for the cylindrical polar coordinates r , α , and z .

63. Show that if the scalar function ψ is a solution of the equation $\Delta \psi + k^2 \psi = 0$ and $\mathbf{a}$ is a constant vector, then the vector functions $\mathbf{L} = \text{grad } \psi$, $\mathbf{M} = \text{curl } (\mathbf{a}\psi)$, $\mathbf{N} = \text{curl } \mathbf{M}$ satisfy the equation $\Delta \mathbf{A} + k^2 \mathbf{A} = 0$.

64*. The equation $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ ($a > b > c$) represents an ellipsoid with semi-axes a , b , and c . The equations

$$\begin{aligned} \frac{x^2}{a^2 + \xi} + \frac{y^2}{b^2 + \xi} + \frac{z^2}{c^2 + \xi} &= 1, & \xi &\geq -c^2, \\ \frac{x^2}{a^2 + \eta} + \frac{y^2}{b^2 + \eta} + \frac{z^2}{c^2 + \eta} &= 1, & -c^2 &\geq \eta \geq -b^2, \\ \frac{x^2}{a^2 + \zeta} + \frac{y^2}{b^2 + \zeta} + \frac{z^2}{c^2 + \zeta} &= 1, & -b^2 &\geq \zeta \geq -a^2, \end{aligned}$$

represent respectively an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets which are confocal with the above

ellipsoid. One of these surfaces, characterised by the numbers ξ, η, ζ , passes through each point in space. The numbers ξ, η, ζ are called the ellipsoidal coordinates of a point x, y, z . Find the transformation formulae relating ξ, η, ζ and x, y, z . Prove the orthogonality of the ellipsoidal system of coordinates and find the coefficients h_i of Eqn. (I.16) and the Laplace operator in ellipsoidal coordinates.

65*. When $a = b > c$, the ellipsoidal system of coordinates (*cf.* preceding problem) degenerates into the so-called oblate spheroidal system. The coordinate ζ thus becomes a constant equal to $-a^2$ and should be replaced by another coordinate.

The azimuthal angle α in the xy -plane is chosen as the latter coordinate. The coordinates ξ, η are then defined by

$$\frac{r^2}{a^2 + \xi} + \frac{z^2}{c^2 + \xi} = 1, \quad \frac{r^2}{a^2 + \eta} + \frac{z^2}{c^2 + \eta} = 1, \quad r^2 = x^2 + y^2,$$

where $\xi \geq -c^2$, $-c^2 \geq \eta \geq -a^2$. The surfaces $\xi = \text{const}$ represent oblate ellipsoids of revolution with the z -axis as the axis of rotation, and the surfaces $\eta = \text{const}$ are hyperboloids of one sheet of revolution which are confocal with them (Fig. 2).

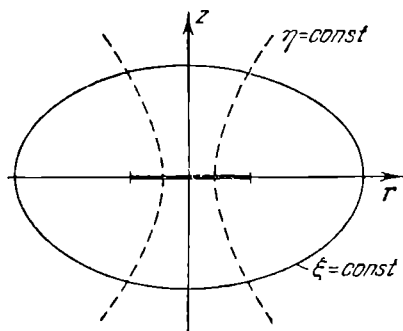


Fig. 2.

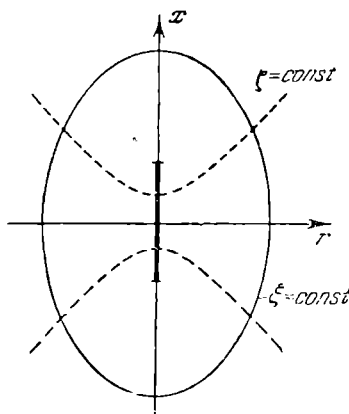


Fig. 3.

Find expressions for r and z in terms of the oblate spheroidal coordinates. Find also the coefficients h_i of Eqn. (I.16) and the Laplace operator in terms of these coordinates.

66*. The prolate spheroidal system of coordinates is obtained from the ellipsoidal system (*cf.* Problem 64) when $a > b = c$. The coordinate η then degenerates into a constant, and should be replaced by the azimuthal angle α measured from the y -axis in

the yz -plane. The coordinates ξ , ζ are defined by

$$\frac{x^2}{a^2 + \xi} + \frac{r^2}{b^2 + \xi} = 1, \quad \frac{x^2}{a^2 + \zeta} + \frac{r^2}{b^2 + \zeta} = 1, \quad r^2 = y^2 + z^2,$$

where $\xi \geq -b^2$, $-b^2 \geq \zeta \geq -a^2$.

The surfaces representing constant values of ξ and ζ are prolate ellipsoids and hyperboloids of two sheets of revolution (Fig. 3). Express x and r in terms of ξ and ζ . Find the coefficients h_i of Eqn. (I.16) and the Laplace operator in terms of the variables ξ , ζ , α .

67. Bispherical coordinates ξ , η , α are related to the cartesian coordinates by the expression

$$x = \frac{a \sin \eta \cos \alpha}{\cosh \xi - \cos \eta}, \quad y = \frac{a \sin \eta \sin \alpha}{\cosh \xi - \cos \eta}, \quad z = \frac{a \sinh \xi}{\cosh \xi - \cos \eta},$$

where a is a constant parameter and $-\infty < \xi < \infty$, $0 < \eta < \pi$, $0 < \alpha < 2\pi$. Show that the coordinate surfaces $\xi = \text{const}$ represent the spheres $x^2 + y^2 + (z - a \coth \xi)^2 = (a/\sinh \xi)^2$, the surfaces $\eta = \text{const}$ represent spindle-shaped surfaces of revolution about the z -axis, whose equation is $[\sqrt{x^2 + y^2} - a \cot \eta]^2 + z^2 = (a/\sin \eta)^2$, and the surfaces $\alpha = \text{const}$ are semi-infinite planes

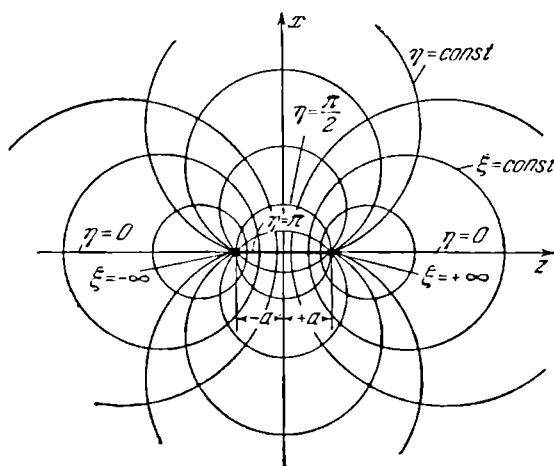


Fig. 4.

diverging from the z -axis (Fig. 4). Show that these coordinate surfaces are orthogonal to each other and find the coefficients h_i of Eqn. (I.16) and the Laplace operator in terms of these coordinates.

68. The toroidal coordinates ρ , ξ , α form an orthogonal system and are related to the cartesian coordinates by the formulae

$$x = \frac{a \sinh \rho \cos \alpha}{\cosh \rho - \cos \xi}, \quad y = \frac{a \sinh \rho \sin \alpha}{\cosh \rho - \cos \xi}, \quad z = \frac{a \sin \xi}{\cosh \rho - \cos \xi},$$

where a is a constant parameter, $-\infty < \rho < \infty$, $-\pi < \xi \leq \pi$, and α is an azimuthal angle lying between 0 and π .

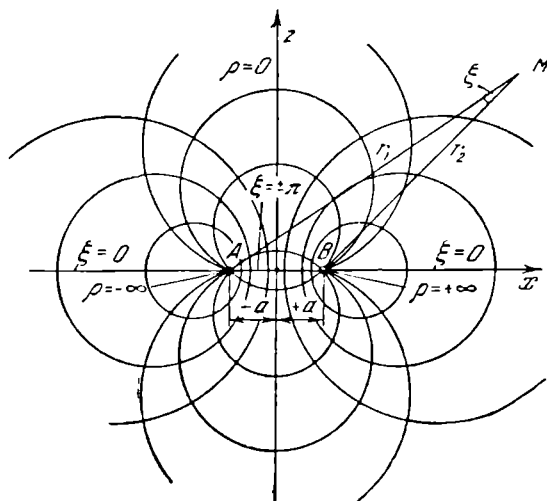


Fig. 5.

Show that $\rho = \ln(r_1/r_2)$ (cf. Fig. 5 showing the planes $\alpha = \text{const}$, $\alpha + \pi = \text{const}$) and that ξ represents the angle between r_1 and r_2 ($\xi > 0$ when $z > 0$; $\xi < 0$ when $z < 0$). What is the form of the coordinate surfaces for ρ and ξ ? Find the coefficients h_i of Eqn. (I.16).

Chapter

2

ELECTROSTATICS IN VACUUM

This chapter contains problems which are concerned with the determination of the potential $\varphi(\mathbf{r})$ and field strength $\mathbf{E}(\mathbf{r})$ from given volume, surface, and linear distributions of charges. The distribution of point charges may be described by a volume density $\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i)$ where q_i is the magnitude of the i -th charge, $\mathbf{r}_i$ is the position vector of the i -th charge, and $\delta(\mathbf{r} - \mathbf{r}_i)$ is the delta-function (*cf.* Appendix I). For a static distribution the electric field satisfies the Maxwell equations

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \quad \operatorname{curl} \mathbf{E} = 0. \quad (\text{II.1})$$

The integral form of the first of these equations is

$$\oint_S \mathbf{E}_n dS = 4\pi q, \quad (\text{II.2})$$

where S is a closed surface and q is the total charge enclosed by this surface. The potential and the electric field strength are related by the expressions

$$\mathbf{E} = -\operatorname{grad} \varphi, \quad \varphi(\mathbf{r}) = \int_{\mathbf{r}}^{\mathbf{r}_0} \mathbf{E} \cdot d\mathbf{r}, \quad \varphi(\mathbf{r}_0) = 0. \quad (\text{II.3})$$

The potential φ satisfies the Poisson equation

$$\Delta\varphi = -4\pi\rho. \quad (\text{II.4})$$

The potential is continuous and finite at all points at which there are no point charges. On a charged surface separating regions 1 and 2 (Fig. 6), $\varphi_1 = \varphi_2$. The normal derivatives of φ are discontinuous across a charged surface so that

$$E_{2n} - E_{1n} = 4\pi\sigma \quad \text{or} \quad \frac{\partial\varphi_1}{\partial n} - \frac{\partial\varphi_2}{\partial n} = 4\pi\sigma. \quad (\text{II.5})$$

where the normal $\mathbf{n}$ is directed from region 1 into region 2.

The change in the potential on passing through an electric double layer of thickness τ is given by

$$\frac{\partial \varphi_1}{\partial n} = \frac{\partial \varphi_2}{\partial n}, \quad \varphi_2 - \varphi_1 = 4\pi\tau \quad (\text{II.6})$$

where the normal $\mathbf{n}$ is directed from the negative to the positive side of the layer.

If the potentials corresponding to charge distributions ρ_1 and ρ_2 are φ_1 and φ_2 then the potential corresponding to the charge distribution $\rho = \rho_1 + \rho_2$ is $\varphi = \varphi_1 + \varphi_2$. This is the so-called superposition principle. An analogous result holds for the electric field $\mathbf{E}$. In particular, the superposition principle may be used to determine the potential due to a complicated charge distribution by integrating over elementary charges:

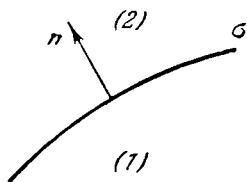


Fig. 6.

$$\varphi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|}. \quad (\text{II.7})$$

In the case of surface or line charge distributions, the volume integral in Eqn. (II.7) must be replaced by the corresponding surface or line integrals, while in the case of a set of point charges it must be replaced by the appropriate summation. The latter remark applies also to all the formulae given below which contain volume integrals over charge distributions.

In the majority of cases, direct evaluation of the integral given by Eqn. (II.7) is difficult. It is, therefore, frequently convenient to expand the integrand into a series in powers of $\mathbf{x}/r$ or $\mathbf{x}'/r$, and then integrate term by term. This expansion may be obtained either in terms of cartesian or spherical polar coordinates.

Cartesian coordinates (Fig. 7). When $r > a$ (a is the maximum distance of the charges from the origin O) we have

$$\begin{aligned} \varphi(x, y, z) = & \frac{q}{r} - p_\alpha \frac{\partial}{\partial x_\alpha} \cdot \frac{1}{r} + \frac{Q_{\alpha\beta}}{2!} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \cdot \frac{1}{r} - \\ & - \frac{Q_{\alpha\beta\gamma}}{3!} \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \cdot \frac{1}{r} \dots \end{aligned} \quad (\text{II.8})$$

The multipole moments q , p_α , $Q_{\alpha\beta}$, ... are given by the following

volume integrals

$$\left. \begin{aligned} q &= \int \rho(\mathbf{r}') dV' && \text{— total charge,} \\ p_\alpha &= \int \rho(\mathbf{r}') x'_\alpha dV' && \text{— dipole moment components,} \\ Q_{\alpha\beta} &= \int \rho(\mathbf{r}') x'_\alpha x'_\beta dV' && \text{— quadrupole moment components.} \end{aligned} \right\} \quad (\text{II.8'})$$

When the coordinate system is rotated, the quantities q , p_α , $Q_{\alpha\beta}$ transform respectively as scalars, vectors, tensors of rank 2, and so on. The second and third terms in the potential given by Eqn. (II.8) may be written in the form

$$\varphi^{(p)} = \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}, \quad (\text{II.9})$$

where $\mathbf{p} = (p_x, p_y, p_z)$ is the dipole moment of the system;

$$\begin{aligned} \varphi^{(Q)} &= \frac{1}{2r^5} [(3x^2 - r^2)Q_{xx} + \\ &+ (3y^2 - r^2)Q_{yy} + (3z^2 - r^2)Q_{zz} + \\ &+ 6xyQ_{xy} + 6xzQ_{xz} + 6yzQ_{yz}]. \end{aligned} \quad (\text{II.9'})$$

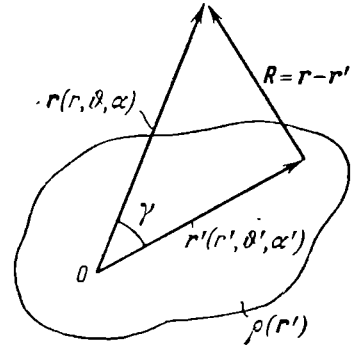


Fig. 7.

Spherical polars. We shall use the expansion for $|\mathbf{r} - \mathbf{r}'|^{-1}$ which is given in Appendix II [Eqn. (15)]. Substituting this expansion into Eqn. (II.7) we have for $r > r'$

$$\varphi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sqrt{\frac{4\pi}{2l+1}} \cdot \frac{Q_{lm} Y_{lm}(\vartheta, \alpha)}{r^{l+1}} \quad (r > r'), \quad (\text{II.10})$$

where Q_{lm} is the multipole moment of order l , m and is given by

$$Q_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int \rho(\mathbf{r}') r'^l Y_{lm}^*(\vartheta', \alpha') dV'. \quad (\text{II.11})$$

If $r' > r$ then r and r' change places in Eqn. (15) of Appendix II and

$$\varphi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sqrt{\frac{4\pi}{2l+1}} r^l Q'_{lm} Y_{lm}(\vartheta, \alpha) \quad (r < r'), \quad (\text{II.12})$$

where

$$Q'_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int \frac{\rho(\mathbf{r}')}{r'^{l+1}} Y_{lm}^*(\vartheta', \alpha') dV'. \quad (\text{II.13})$$

If the point defined by the position vector $\mathbf{r}$ lies inside the charge distribution (cf. Fig. 7), then the region of integration in

Eqn. (II.7) must be divided into two parts by a sphere of radius r , centred on the origin O . The expansion given by Eqn. (15) of Appendix II is then used inside the sphere and the reciprocal expansion is used outside the sphere.

The structure of the electrostatic field can be conveniently represented by lines of force and equipotential surfaces. The lines of force can be defined by the following set of differential equations involving the arbitrary orthogonal coordinates q_1, q_2, q_3 :

$$\frac{h_1 dq_1}{E_1} = \frac{h_2 dq_2}{E_2} = \frac{h_3 dq_3}{E_3}, \quad (\text{II.14})$$

where h_i are the coefficients given by Eqn. (I.16). The equipotential surfaces are defined by $\varphi(\mathbf{r}) = \text{const.}$

Neutral points are defined as those points at which $E = 0$ at finite distances from systems of charges.

The energy of an electrostatic field may be computed from one of the following two formulae:

$$W = \frac{1}{8\pi} \int E^2 dV, \quad W = \frac{1}{2} \int \rho \varphi dV. \quad (\text{II.15})$$

These formulae are equivalent if the charges are localised in a finite region of space and the integration extends over all space.

The energy of interaction between two systems of charges 1 and 2 is given by

$$U = \int \rho_1(\mathbf{r}) \varphi_2(\mathbf{r}) dV = \int \frac{\rho_1(\mathbf{r}_1) \rho_2(\mathbf{r}_2) dV_1 dV_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \quad (\text{II.16})$$

The generalised pondermotive forces may be obtained by differentiating U or W with respect to the corresponding generalised coordinates a_i :

$$F_i = -\frac{\partial U}{\partial a_i} \quad \text{or} \quad F_i = -\frac{\partial W}{\partial a_i}. \quad (\text{II.17})$$

The generalised force is positive if it tends to increase the corresponding coordinate.

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69. An infinite plane plate of thickness a is uniformly charged to a volume density ρ . Find the potential φ and the electric field strength $\mathbf{E}$.

70. An electric charge is distributed throughout space in such a way that its volume density is given by $\rho = \rho_0 \cos \alpha x \cos \beta y \cos \gamma z$, forming an infinite periodic lattice. Find the potential due to this distribution.

71. The plane $z = 0$ is charged to a density which varies in accordance with the periodic law $\sigma = \sigma_0 \sin \alpha x \sin \beta y$, where σ_0 , α , and β are constants. Find the potential due to this charge distribution.

72. An infinitely long cylinder of radius R carries a uniform charge density κ per unit length. Find the potential and the electric field strength for this system.

73. Find the potential and the electric field strength for a uniformly charged infinitely long straight filament.

74. Find the potential and the electric field strength for a uniformly charged line segment of length $2a$, lying between $-a$ and $+a$ along the z -axis, when its total charge is q .

75. Find the form of the equipotential surfaces for the uniformly charged line segment considered in the preceding problem.

76. Find the potential and the electric field strength for a sphere carrying a uniform volume charge distribution.

77. Find the potential and the electric field strength for a sphere of radius R , carrying a uniform surface charge distribution.

78. A sphere of radius R is uniformly charged to a density ρ . The sphere contains an uncharged spherical cavity of radius R_1 . The centres of the two spheres are separated by a distance a , where $a + R_1 < R$. Find the electric field strength inside the cavity.

79. The space between two concentric spheres of radii R_1 and R_2 ($R_1 < R_2$) is charged to a volume density given by $\rho = \alpha/r^2$. Find the total charge, the potential, and the electric field strength. Consider the limiting case $R_2 \rightarrow R_1$.

80. Find the energy W of the electrostatic field for the charge distributions considered in Problems 76, 77, and 79. (Use two methods of computing the energy [cf. Eqn. (II.15)].)

81. Consider a spherically symmetric charge distribution $\rho = \rho(r)$. By dividing the charge distribution into spherical shells, find the potential and the electric field strength in terms of $\rho(r)$ (write down the potential and the field in the form of an integral with respect to r).

82. Use the results of Problem 81 to solve Problems 76 and 79.

83. In the ground state of a hydrogen atom, the electron charge is distributed with a density given by $\rho(\mathbf{r}) = -(e_0/\pi a^3) \times \exp(-2r/a)$ where $a = 0.529 \times 10^{-8}$ cm (Bohr radius) and $e_0 = 4.80 \times 10^{-10}$ e.s.u. (the charge of the electron). Find the potential φ_e and the field strength E_{er} due to the electron charge distribution, and also the total potential φ and the field strength $\mathbf{E}$ in the atom, assuming that the nucleus (proton) is localised at the origin. Give a rough sketch of the spatial variation of φ and E .

Hint: It is convenient to use the method employed in the solution of Problem 81.

84. Assume that the atomic nucleus may be looked upon as a uniformly charged sphere and find the maximum field strength due to this sphere. The radius of the nucleus is given by $R = 1.5 \times 10^{-13} A^{\frac{1}{3}}$ cm and its total charge is Ze_0 , where A is the atomic weight, Z is the atomic number, and e_0 is the elementary charge.

85. Use the result of Problem 81 to solve Problem 77.

86. Two thin coaxial uniformly charged rings of equal radius R lie in parallel planes which are at a distance a from each other. The work which must be done in order to bring a point charge q from infinity to the centres of the two rings is respectively equal to A_1 and A_2 . Find the total charges q_1 and q_2 carried by the rings.

87. Find the potential and the electric field strength along the axis of a thin uniformly charged circular disc of radius R and total charge q . Show that the normal component of the field changes by $4\pi\sigma$ on passing through the surface of the disc. Consider the field at large distances from the disc.

88. A thin circular ring of radius R consists of two uniformly charged semicircular sections carrying charges q and $-q$. Find the potential and field strength along the axis of the ring and in its immediate neighbourhood. Discuss the form of the field at large distances from the ring.

89. Express the potential due to a thin uniformly charged circular ring of radius R , carrying a total charge q , in terms of the complete elliptical integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}}.$$

Hint: Use the substitution $\alpha' = \pi - 2\beta$ when integrating with respect to the azimuth.

90. Use the general formula for the potential due to a thin circular ring (*cf.* Problem 89) to determine the field along the axis of the ring (a) at large distances from the ring, and (b) in the immediate neighbourhood of the ring.

91. A sphere of radius R carries a surface charge $\sigma = \sigma_0 \cos \vartheta$. Find the potential due to this charge distribution by using the multipole expansion in terms of spherical polars.

92. The sources of an electric field are arranged in an axially symmetric distribution in such a way that there are no charges near the axis of symmetry. Express the potential and the electric field strength near the axis of symmetry in terms of the potential and its derivatives along this axis.

93. Find the potential due to a thin uniformly charged circular ring with the aid of the multipole expansion in terms of spherical polars. The total charge q and the radius R are given.

94. Find the potential at large distances for the following systems of charges:

- (a) point charges q , $-2q$, q lying along the z -axis at a distance a from each other (linear quadrupole);
- (b) point charges $\pm q$ at the corners of a square of side a , arranged so that neighbouring charges have opposite signs, and a charge $+q$ placed at the origin; the sides of the square are parallel to the x - and y -axes (planar quadrupole).

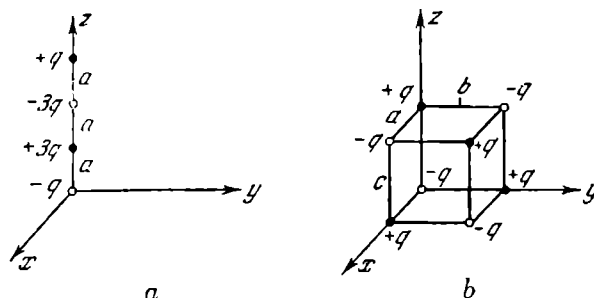


Fig. 8.

95. Find the potential at large distances from the following systems of charges:

- (a) linear octupole (Fig. 8a),
- (b) three-dimensional octupole (Fig. 8b).

96. A point charge q is placed at a point whose spherical coordinates are $r_0, \vartheta_0, \alpha_0$. Find the multipole expansion for the potential due to this charge.

97. An ellipsoid with semi-axes a, b, c is uniformly charged throughout its volume so that the total charge is q . Find the potential at large distances from the ellipsoid (retain terms up to the quadrupole term inclusively). Discuss the special cases of an ellipsoid of revolution with semi-axes $a = b, c$ and of a sphere $a = b = c$. (Atomic nuclei having finite quadrupole moments may be approximately looked upon as ellipsoids of revolution.)

Hint: In integrating inside the ellipsoid use the generalised spherical coordinates $x = ar \sin \vartheta \cos \alpha, y = br \sin \vartheta \sin \alpha, z = cr \cos \vartheta$.

98. Two thin coaxial uniformly charged rings with radii a and b ($a > b$) and charges q and $-q$ respectively, lie in a given plane. Find the potential at a large distance from this system. Compare the result with the potential due to a linear quadrupole (cf. Problem 94).

99*. Show that the charge distribution $\rho = -(\mathbf{p}' \cdot \nabla) \delta(\mathbf{r})$ represents an elementary dipole of moment $\mathbf{p}'$ at the origin. Discuss the result using the δ -function representation (Appendix I).

Hint: Start with the multipole expansion in terms of cartesian coordinates.

100. Show that the charge distribution

$$\rho = q \prod_{i=1}^n (\mathbf{a}_i \cdot \nabla) \delta(\mathbf{r})$$

gives rise to the potential

$$\varphi(\mathbf{r}) = q \prod_{i=1}^n (\mathbf{a}_i \cdot \nabla) \frac{1}{r}.$$

101. Use the results of Problem 94, and the fact that the quadrupole moment is a tensor of rank 2, to determine the potential at a large distance from a linear quadrupole whose orientation is defined by the polar angles γ, β . Indicate a second method of solving this problem.

102. The three-dimensional octupole shown in Fig. 8b' is turned through an angle β about the z -axis. Find the potential at a large distance from it by transforming the components of the octupole moment. Compare with other methods of solution.

103. Find the potential at a large distance from a planar quadrupole which lies on a plane containing the z -axis (Fig. 9).

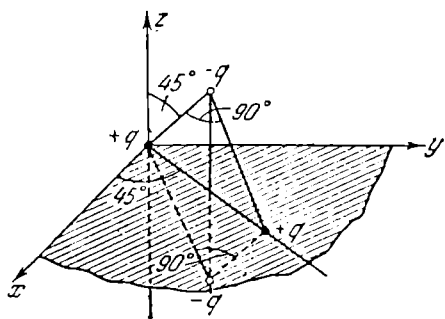


Fig. 9.

Find the components of the quadrupole moment directly, and also by rotating the planar quadrupole discussed in Problem 94(b).

104. A sphere of radius R is uniformly polarised to a dipole moment $\mathbf{P}$ per unit volume. Find the potential due to this sphere.

105. A two-dimensional charge distribution is characterised by a charge density $\rho(\mathbf{r})$ which is independent of the z -coordinate. If $\rho(r) \neq 0$ in a finite region S on the xy -plane, then the potential outside the charge distribution may be expanded into a multipole expansion (two-dimensional multipoles). Find this expansion.

Hint: Use the result of Problem 73, the superposition principle, and the expansion $\ln(1 + u^2 - 2u \cos \varphi) = -2 \sum_{k=1}^{\infty} (\cos k\varphi/k) u^k$, where $|u| < 1$.

106. Expand the potential due to a linear charge κ in terms of two-dimensional multipoles. The charged line is parallel to the z -axis and passes through the point (r_0, α_0) in the xy -plane.

107. Find the potential at a large distance from two parallel linear charges κ and $-\kappa$ at a small distance a from each other (two-dimensional dipole).

108. A disc of radius R carries an electric double layer of strength $\tau = \text{const}$. Find the potential and the field strength along the axis of symmetry perpendicular to the plane of the disc.

109. Find the electric field $\mathbf{E}$ due to an electric double layer of strength $\tau = \text{const}$, lying on the semi-infinite plane $y = 0, x > 0$. Compare with the magnetic field due to an infinite line current flowing along the z -axis. Solve the problem by

- (a) direct integration of contributions due to double layer elements, and
- (b) by determining the electrostatic potential first.

110. Find the equations of the lines of force due to the following system of point charges: $+q$ at $z = a$, $\pm q$ at $z = -a$. Sketch out the lines of force. Are there any neutral points?

Hint: Owing to the symmetry of the problem the lines of force lie in the planes $\alpha = \text{const}$, and E_z and E_r are independent

of α (cylindrical coordinates). The variables in the differential equation for the lines of force [cf. Eqn. (II.14)] may be separated with the aid of the substitution

$$u = \frac{z+a}{r}, \quad v = \frac{z-a}{r}.$$

111. Use the results of the preceding problem to find the equation for the lines of force due to a point dipole at the origin.

112. Find the equation for the lines of force due to a linear quadrupole [cf. Problem 94(a)] and sketch out the lines of force.

113. Show that the flux through a surface S due to a point charge q is equal to $q\Omega$ where Ω is the solid angle subtended by the surface at the point charge.

114. A charge q_1 lies on the axis of symmetry of a circular disc of radius A at a distance a from the plane of the disc. Find the charge q_2 which must be placed at a point located symmetrically relative to the disc in order that the flux through the disc in the direction towards the charge q_1 should be Φ .

115*. Find the equation for the lines of force of a system of n colinear charges $q_1, q_2, \dots, q_n$ placed at the points $z_1, z_2, \dots, z_n$ on the z -axis without integrating the differential equations for the lines of force. Use the theorem of Problem 113 for a tube force generated by rotating a line of force about the axis of symmetry.

116. Use the result of the preceding problem to find the equation for the lines of force due to a system of two point charges (cf. Problem 110) and a linear quadrupole (cf. Problem 112).

117. Uniformly charged filaments carrying charges κ_1 and $-\kappa_2$ per unit length are parallel and lie at a distance h from each other. Find the relation between κ_1 and κ_2 for which the family of equipotential surfaces associated with this system will include cylinders of finite radius and determine the radii and the position of the axes of the cylinders.

118. Point charges q_1 and $-q_2$ lie at a distance h from each other. Show that the equipotential surfaces associated with this system include a sphere of finite radius. Find the coordinates of the centre of the sphere and its radius. Find an expression for the potential on the surface of the sphere if the potential at infinity is zero.

119. Find the charge distribution giving rise to a potential of the form $\varphi(\mathbf{r}) = q \exp(-\alpha r)/r$ where α and q are constants.

120. Find the charge distribution giving rise to the potential $\varphi(r) = (e_0/a)(a/r + 1) \exp(-2r/a)$ where e_0 and a are constants.

121. Find the energy of interaction U between the electron cloud and the nucleus in the hydrogen atom. Assume that the electron charge density is given by $\rho(r) = (e_0/\pi a^3) \exp(-2r/a)$ where e_0 is the charge of the electron and a is a constant (Bohr radius). (cf. Problem 83.)

122. Under certain conditions it may be assumed that the electron clouds due to the two electrons in the helium atom are identical and may be described by the volume charge density $\rho = -(8e_0/\pi a^3) \exp(-4r/a)$ where a is the Bohr radius and e_0 is the charge of the electron. Find the energy of interaction U between the electrons in the helium atom using this approximation (which is in fact the zero-order approximation of perturbation theory).

123. The centres of two spheres carrying charges q_1 and q_2 are at a distance a from each other. The charges form a spherically symmetric distribution. Find the energy of interaction and the force F between the spheres.

124. A soap bubble hanging from the end of an open tube contracts under the action of the surface tension. Assuming that the dielectric strength of air, *i. e.* the field strength at which breakdown occurs, is E_0 , investigate whether the contraction may be prevented by placing a large charge on the bubble. What is the minimum equilibrium radius R of the bubble?

125*. Two thin parallel coaxial rings of radii a and b carry uniformly distributed charges q_1 and q_2 . The distance between the planes of the rings is c . Find the energy of interaction and the force between the rings.

126. Find the force and the couple on an electric dipole of moment $\mathbf{P}$ due to a point charge q .

127. A dipole with a moment $\mathbf{P}_1$ is placed at the origin while another dipole with a moment $\mathbf{P}_2$ is placed at a point whose position vector is $\mathbf{r}$. Find the energy of interaction and the force between the two dipoles. For which orientation of the dipoles does the force become a maximum?

128. A system of charges can be described by a volume density $\rho(\mathbf{r})$ and occupies a finite region in the neighbourhood of a given point O . The system is placed in an external electric field, which in the neighbourhood of this point may be represented by the series

$$\varphi_1(\mathbf{r}) = \sum_{l,m} \sqrt{\frac{4\pi}{2l+1}} a_{lm} r^l Y_{lm}(\vartheta, \alpha).$$

Find the energy of interaction of the system with the external field by expressing it in terms of a_{lm} and the multipole moments Q_{lm} of the system (*cf.* Problem 166).

Chapter

3

ELECTROSTATICS OF CONDUCTORS AND DIELECTRICS

1. Basic concepts and methods of electrostatics

The electrostatic field in a dielectric is characterised by the electric field $\mathbf{E}$ and the induction vector $\mathbf{D}$ which satisfy the following equations:

$$\left. \begin{array}{l} \text{div } \mathbf{D} = 4\pi\rho, \\ \text{curl } \mathbf{E} = 0, \end{array} \right\} \quad \text{or} \quad \left. \begin{array}{l} \oint_S D_n dS = 4\pi q, \\ \oint_l E_t dl = 0, \end{array} \right\} \quad (\text{III.1})$$

where ρ is the density of free charges in the dielectric. The density of bound charges in a dielectric may be expressed in terms of the polarisation $\mathbf{P}$ (electric dipole moment per unit volume, which is produced by the bound charges):

$$\rho_b = -\text{div } \mathbf{P}. \quad (\text{III.2})$$

The polarisation $\mathbf{P}$ can be expressed in terms of $\mathbf{E}$ and $\mathbf{D}$:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}. \quad (\text{III.3})$$

For isotropic dielectrics and sufficiently small fields

$$\mathbf{D} = \epsilon\mathbf{E}, \quad (\text{III.4})$$

where ϵ is the permittivity of the medium. In anisotropic dielectrics, ϵ is a tensor of rank 2, *i.e.*

$$D_i = \epsilon_{ik} E_k. \quad (\text{III.5})$$

The electric field vector may be derived from a scalar potential φ :

$$\mathbf{E} = -\text{grad } \varphi, \quad \varphi(\mathbf{r}) = \int_{\mathbf{r}}^{\mathbf{r}_0} \mathbf{E} \cdot d\mathbf{r}, \quad (\text{III.6})$$

where $\mathbf{r}$ is the position vector of the point of observation and

$\varphi(\mathbf{r}_0) = 0$. The potential satisfies the equation

$$\operatorname{div}(\epsilon \operatorname{grad} \varphi) = -4\pi\rho. \quad (\text{III.7})$$

In those regions where the dielectric is uniform, the latter equation reduces to the Poisson equation

$$\Delta\varphi = -\frac{4\pi\rho}{\epsilon}. \quad (\text{III.8})$$

The boundary conditions at a boundary of separation between two different dielectrics are

$$E_{1\tau} = E_{2\tau}, \quad D_{2n} - D_{1n} = 4\pi\sigma \quad (\text{III.9})$$

or

$$\varphi_1 = \varphi_2, \quad \epsilon_1 \frac{\partial\varphi_1}{\partial n} - \epsilon_2 \frac{\partial\varphi_2}{\partial n} = 4\pi\sigma. \quad (\text{III.10})$$

The boundary conditions given by Eqn. (III.9) hold both for isotropic and anisotropic media. The unit normal $\mathbf{n}$ is directed from the first medium into the second, the unit vector $\boldsymbol{\tau}$ is tangential to the surface, and σ is the surface density of free charges. The surface density of bound charges σ_b on the separation boundary is given by

$$\sigma_b = P_{1n} - P_{2n}. \quad (\text{III.11})$$

The basic problem of electrostatics is to find the electric potential φ . It may be solved in various ways. The main method is to solve the differential equations given by (III.7) and (III.8) subject to the boundary conditions (III.9) or (III.10). It is sometimes possible to choose a set of fictitious point charges for which the potential in a particular region satisfies the same differential equation and the same boundary conditions as the given set of charges (method of images). In many cases it is a simple matter to find the image system (*cf.* Problems 142, 146, 153, and 155).

The field inside a conductor placed in a constant electric field is zero, and hence the boundary conditions on the surface of a conductor are of the form

$$E_\tau = 0, \quad \varphi = \text{const.} \quad (\text{III.12})$$

If a certain region of space is occupied by a dielectric with permittivity ϵ , and the electrostatic field is known in all space,

then as $\epsilon \rightarrow \infty$ the field becomes identical with the field produced when the dielectric is replaced by a conductor of the same geometric form.

The problem of the determination of the electric field due to a given bounded set of charged conductors placed in a dielectric has a unique solution if either the total charge or the potential of each conductor are known. In the first of these two cases, the boundary condition given by (III.12) must be supplemented by

$$q = \oint_S \sigma dS = - \frac{1}{4\pi} \oint_S \epsilon \frac{\partial \varphi}{\partial n} dS, \quad (\text{III.13})$$

where q is the charge on the conductor and the integral is evaluated over the surface of the conductor.

The capacitance C of a capacitor is defined as the ratio of the charge on one of its electrodes to the potential difference between the electrodes:

$$C = \frac{q}{\varphi_1 - \varphi_2}. \quad (\text{III.14})$$

The capacitance of an isolated conductor is defined as the ratio of the charge on the conductor to its potential (it may be assumed that the potential vanishes at infinity).

The energy of an electrostatic field which is localised in a volume V is given by the integral

$$W = \int w dV, \quad (\text{III.15})$$

where $w = (\mathbf{D} \cdot \mathbf{E})/8\pi$ is the energy density and the integral is evaluated in V .

When a dielectric of volume V and permittivity ϵ_2 is introduced into an isotropic dielectric with permittivity ϵ_1 , then the energy of the electrostatic field is changed by an amount given by

$$U = \frac{1}{8\pi} \int_V (\epsilon_1 - \epsilon_2) \mathbf{E}_2 \cdot \mathbf{E}_1 dV, \quad (\text{III.16})$$

where $\mathbf{E}_1$ is the initial field and $\mathbf{E}_2$ the final field (it is assumed that the sources of $\mathbf{E}_1$ are maintained unaltered). The quantity U may be looked upon as the energy of interaction of the dielectric body with the external field $\mathbf{E}_1$.

The force per unit volume of an isotropic dielectric whose permittivity depends only on its mass density τ is given by the

following equation:

$$\mathbf{f} = \rho \mathbf{E} - \frac{1}{8\pi} E^2 \text{grad } \epsilon + \frac{1}{8\pi} \text{grad} \left(E^2 \frac{d\epsilon}{d\tau} \tau \right). \quad (\text{III.17})$$

The body forces acting on free and bound charges in a given volume V may be replaced by the equivalent system of surface stresses applied to the surface S of the volume V and given by

$$\mathbf{F} = \int_V \mathbf{f} dV = \oint_S \mathbf{T}_n dS, \quad (\text{III.18})$$

where $\mathbf{T}_n$ is the surface force per unit area applied to an element having a unit outward normal $\mathbf{n}$.

The surface stresses are described by the stress tensor T_{ik} , and the quantity $\mathbf{T}_n$ in (III.18) is the component of T_{ik} along the outward normal $\mathbf{n}$ to the surface element dS , so that

$$\begin{aligned} (T_n)_i &= T_{ik} n_k, \\ T_{ik} &= \frac{\epsilon}{4\pi} E_i E_k - \frac{1}{8\pi} E^2 \left(\epsilon - \frac{\partial \epsilon}{\partial \tau} \tau \right) \delta_{ik}. \end{aligned} \quad (\text{III.19})$$

The terms containing $(\partial \epsilon / \partial \tau) \tau$ in (III.17) and (III.19) are not, in general, small and represent electrostriction. However, in calculating the resultant of the forces acting on a dielectric this term does not contribute to the final result and may be ignored (*cf.* Problems 140 and 141). The stress tensor (III.19) may then be replaced by the simpler (Maxwell) tensor

$$\mathbf{T}'_n = \frac{\epsilon}{4\pi} \left(E_n \mathbf{E} - \frac{1}{2} n E^2 \right). \quad (\text{III.20})$$

The force per unit area on the surface of a conductor is given by

$$\mathbf{f}_s = \mathbf{T}'_n = \mathbf{n} \frac{\epsilon E^2}{8\pi} = \frac{\sigma \mathbf{E}}{2}. \quad (\text{III.21})$$

In a liquid dielectric which is in equilibrium in an electric field the electric stresses are balanced by the hydrostatic pressure. The equilibrium condition is then given by

$$p\mathbf{n} + \mathbf{T}_n = \text{const}, \quad (\text{III.22})$$

where $p(\tau)$ is the pressure in the liquid and τ is the density. In particular, near the boundary between a liquid and the atmosphere ($\epsilon = 1$) the pressure $p(\tau)$ is greater than the atmospheric pressure

by an amount given by

$$p(\tau) - p_{\text{atm}} = \frac{\tau E^2}{8\pi} \frac{\partial \epsilon}{\partial \tau} - \frac{\epsilon - 1}{8\pi} (\epsilon E_n^2 + E_t^2), \quad (\text{III.23})$$

where $\mathbf{E}$ is the electric field strength in the liquid, and E_n and E_t are the normal and tangential components of $\mathbf{E}$. Equation (III.23) describes the dependence of the pressure in a liquid near its surface on the electric field strength. The pressure inside the liquid (gas) is given by

$$\int_{p_0}^p \frac{dp}{\tau(p)} = \frac{E^2}{8\pi} \frac{\partial \epsilon}{\partial \tau} \quad (\text{III.24})$$

where p_0 is the pressure at the point where $E = 0$. For an incompressible liquid

$$p - p_0 = \frac{\tau E^2}{8\pi} \frac{\partial \epsilon}{\partial \tau}. \quad (\text{III.25})$$

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129. A bound charge q is placed on the planar separation boundary between two homogeneous infinite dielectrics with permittivities ϵ_1 and ϵ_2 . Find the potential ϕ , the field strength $\mathbf{E}$, and the induction $\mathbf{D}$.

130. A point charge q lies on a straight line which is the line of intersection of three planes, the angles between the planes being α_1 , α_2 , and α_3 ($\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$). The space between each pair of planes is filled with homogeneous dielectrics with permittivities ϵ_1 , ϵ_2 , and ϵ_3 . Find the potential ϕ , the electric field strength $\mathbf{E}$, and the induction $\mathbf{D}$.

131. The centre of a conducting sphere carrying a charge q lies on the plane boundary between two infinite homogeneous dielectrics with permittivities ϵ_1 and ϵ_2 . Find the electric field and the charge distribution, σ , on the sphere.

132. The space between the electrodes of a spherical capacitor is partly filled with a dielectric which occupies a region subtending a solid angle Ω at the centre of the spheres. Find the capacitance C assuming that the radii of the electrodes are a and b , and the permittivity is ϵ .

133. The permittivity of a medium filling the space between the electrodes of a spherical capacitor whose radii are a and b

is given by

$$\epsilon(r) = \begin{cases} \epsilon_1 = \text{const} & a \leq r < c, \\ \epsilon_2 = \text{const} & c \leq r \leq b, \end{cases}$$

where $a < c < b$. Find the capacitance, C , of the capacitor, the distribution of bound charges, σ_b , and the total bound charge in the dielectric.

134. A spherical capacitor whose electrodes have radii a and b is filled with a dielectric whose permittivity is given by $\epsilon(r) = \epsilon_0 a^2/r^2$ where r is the distance from the centre. Show that the capacitance of this capacitor is equal to the capacitance of a plane parallel capacitor filled with a homogeneous dielectric with permittivity ϵ_0 , electrode area $4\pi a^2$, and distance between the electrodes equal to $b - a$ (neglect edge effects).

135. A plane parallel capacitor is filled with a dielectric whose permittivity is given by $\epsilon = \epsilon_0(x + a)/a$, where a is the distance between the electrodes, S is the area of the plates, and the x -axis is perpendicular to the plates. Neglecting edge effects, find the capacitance C and the distribution of bound charges when a potential difference V is applied between the planes.

136. Find the force f_0 per unit area of the two electrodes of a plane parallel capacitor when the distance between the plates is a and the potential difference between them is V . Find the new value f of this force when the charged capacitor is separated from the battery and filled either with a liquid dielectric of permittivity ϵ , or when a plate of solid dielectric of the same permittivity and thickness just smaller than a is introduced between the plates so as not to touch the latter. Find the force of attraction f between the plates when the capacitor is first filled with the liquid dielectric, or when the solid dielectric plate is inserted into it, before the capacitor is charged from the battery.

137. The electrodes of a plane parallel capacitor are at a distance h_1 from each other and have the form of rectangles with sides a and b . A plate of permittivity ϵ is placed between the electrodes in such a way that it is parallel to them. The plate is in the form of a parallelepiped of thickness h_2 and base area ab . The plate is not completely inserted into the capacitor so that a length $a - x$ of it remains outside. Find the force F with which the plate is attracted by the capacitor in the following two cases:

- (a) a potential difference V is maintained across the capacitor, and

(b) each of the electrodes carries a constant charge q .
Ignore edge effects.

138*. A plane parallel condenser is placed in an incompressible liquid of permittivity ϵ and density τ so that its plates are vertical. Find the distance h to which the liquid will rise in the capacitor when the distance between the plates is d and the potential difference between them is V .

Hint: Use Eqns. (III.23) and (III.25).

139. Find the direction of the Maxwell stress $\mathbf{T}'_{\mathbf{n}}$ which acts on an area dS whose normal $\mathbf{n}$ makes an angle ϑ with the direction of the electric field $\mathbf{E}$. Find also the magnitude of $\mathbf{T}'_{\mathbf{n}}$ and the electrostrictional stress $\mathbf{T}''_{\mathbf{n}}$.

140. Two identical point charges q are placed in a homogeneous liquid dielectric of permittivity ϵ at a distance a from each other. Use the Maxwell or the total stress tensor to find the force $\mathbf{F}$ acting on each of the charges. What are the components making up the force of interaction $q^2/a^2\epsilon$ between the two charges? For comparison, calculate the forces on: (a) the plane of symmetry perpendicular to the line joining the charges, and (b) the surface of a small sphere centred on one of the charges.

141. An uncharged conducting sphere of radius R and mass m floats in a liquid of permittivity ϵ and density τ , so that three-quarters of its volume are immersed. Find the potential φ_0 at which the sphere has to be maintained in order to ensure that half of its volume will be immersed. Solve the problem by using (a) the Maxwell stress tensor and (b) the total stress tensor including the electrostriction term.

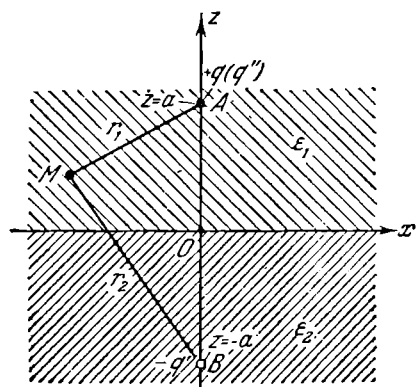


Fig. 10.

142. A point charge q is placed at a point A which is at a distance a from the separation boundary of two semi-infinite homogeneous dielectrics whose permittivities are ϵ_1 and ϵ_2 (Fig. 10). Find the potential φ by the method of images.

Hint: The solution should be sought in the form

$$\begin{aligned}\varphi_1 &= \frac{q}{\epsilon_1 r_1} - \frac{q'}{\epsilon_1 r_2} & z \geq 0, \\ \varphi_2 &= \frac{q''}{\epsilon_2 r_1} & z < 0,\end{aligned}$$

where q' and q'' are the required effective charges at the points B and A respectively.

143. Find the density σ_b of bound surface charges on the plane separation boundary between two homogeneous dielectrics ϵ_1 and ϵ_2 , which is due to a point charge q (*cf.* Problem 142). Investigate what happens when ϵ_2 tends to infinity and give a physical interpretation of the result.

144. Find the force F on the point charge of Problem 142 (image force). Solve the problem by a number of methods, including the Maxwell stress-tensor method. Give a qualitative description of the motion of the charge if it is free to move through the dielectric.

145*. Two semi-infinite homogeneous dielectrics with permittivities ϵ_1 and ϵ_2 are in contact along an infinite plane. Two charges q_1 and q_2 are placed at equal distances from this plane along a straight line perpendicular to it. Find the forces F_1 and F_2 acting on each of the charges. Why are the two forces not equal?

146. A point charge q is placed in a homogeneous dielectric at a distance a from the plane boundary of a semi-infinite conductor. Find the potential φ in the dielectric, the distribution σ of charges induced in the metal, and the force F on the charge q .

147. A point charge q is placed between two earthed intersecting metal planes which are at an angle α_0 to each other. Use the method of images to find the electric field and discuss the special cases $\alpha_0 = 90^\circ$, 60° , and 45° .

148. An electric dipole of moment $\mathbf{p}$ is placed in a homogeneous dielectric near the plane boundary of a semi-infinite conductor. Find the potential energy of interaction of the dipole with the induced charges, and the force $\mathbf{F}$ and couple $\mathbf{N}$ on the dipole.

149*. A homogeneous sphere of radius a and permittivity ϵ_1 is placed in a homogeneous dielectric medium of permittivity ϵ_2 . At a large distance from the sphere the electric field $\mathbf{E}_0$ in the dielectric is uniform. Find the potential φ in all space and sketch out the lines of force for the two cases $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$. Find also the distribution of bound charges.

150. An infinite dielectric was at first homogeneous and uniformly polarised (polarisation vector $\mathbf{P} = \text{const}$). A spherical cavity was then introduced into it. Determine the electric field $\mathbf{E}$ in the cavity in the following two cases:

- (a) the introduction of the cavity does not affect the polarisation in the surrounding dielectric†,
- (b) the change in the field gives rise to the change in the polarisation $\left[\mathbf{P} = \frac{\epsilon - 1}{4\pi} \mathbf{E} \right]$.

151. An uncharged metal sphere of radius R is introduced into a uniform electric field $\mathbf{E}_0$. The permittivity of the surrounding medium is $\epsilon_0 = \text{const.}$ Determine the resulting potential and density of surface charges on the sphere.

152*. Two equal point charges $q_1 = q_2 = q$ are placed at a distance a from each other in a solid dielectric of permittivity ϵ_1 . The charges lie at the centres of small spherical cavities of radius R . Find the forces acting on the charges and compare them with the electric stresses in the plane of symmetry perpendicular to the line joining the charges.

153*. A conducting sphere of radius R is placed in the field of a point charge q which is at a distance a from the centre of the sphere ($a > R$). The system is immersed in a homogeneous dielectric of permittivity ϵ . Find the potential ϕ and the charge distribution σ induced on the sphere in the following two cases:

- (a) the potential of the sphere is maintained at V and the potential at infinity is zero, and
- (b) the charge on the sphere is Q .

Write down the potential in the form of the sum of the potentials due to a number of point-charge images.

Hint: Use the expansion for the solution of Laplace's equation in terms of spherical harmonics (Appendix II) and the expansion for the field due to a point charge which was obtained in Problem 96.

154. A conductor maintained at a potential V contains a spherical cavity of radius R which is filled with a dielectric having a permittivity ϵ . A point charge q is placed at a distance a from the centre of the cavity ($a < R$). Find the field in the cavity and the equivalent system of image charges.

155. An earthed conducting plane has a hemispherical boss of radius a . The centre of the sphere lies on the plane. A point charge q is placed on the axis of symmetry of the system at a distance $b > a$ from the plane. Use the method of images to find the potential ϕ and also the charge q' induced on the boss.

† This occurs in electrets which consist of polar molecules of fixed orientation.

156. A conducting sphere of radius R_1 is placed in a homogeneous dielectric of permittivity ϵ_1 . The sphere contains a spherical cavity of radius R_2 which is filled with a homogeneous dielectric of permittivity ϵ_2 . A point charge q is placed at a distance a from the centre of the cavity ($a < R_2$). Find the potential in all space.

157*. A dielectric sphere of radius R and permittivity ϵ_1 is placed in a homogeneous dielectric ϵ_2 . A point charge q is located at a distance $a > R$ from the centre of the sphere. Find the potential in all space. By passing to the limit, find the field for a conducting sphere.

158. Solve the preceding problem for the case where the point charge q is placed inside the dielectric sphere ϵ_1 ($a < R$). Discuss the special case $a = 0$ (charge at the centre of the sphere).

159*. An insulated metal sphere of radius a is placed inside a hollow metal sphere of radius b . The distance between the centres of the two spheres is c where $c \ll a$, $c \ll b$. The total charge on the inner sphere is q . Find the charge distribution σ on the inner sphere and the force F acting on it. Terms involving second and higher powers of c may be neglected.

160. A spherical capacitor is formed by two non-concentric spheres (*cf.* preceding problem). Calculate the correction ΔC to the capacitance which is due to the departure from exact concentricity on the first non-vanishing approximation.

161. Find the energy of interaction U between a point charge q and an earthed conducting sphere of radius R . The charge is at a distance a from the centre of the sphere and the system is placed in a homogeneous dielectric medium of permittivity ϵ . Find also the force F on the point charge.

162. A point charge q is placed in a dielectric at a distance a from the centre of a conducting insulated sphere of radius R which carries a total charge Q . Find the energy of interaction U between the charge and the sphere and the force F on the point charge.

163. What are the conditions which have to be satisfied by a test charge q (in the sense of its magnitude and position in space) in order that it can be used to investigate the field due to a system of charges situated on conductors and dielectrics, and in particular, the field due to a charged sphere in a homogeneous dielectric?

164*. An electric dipole $\mathbf{p}$ is placed in a homogeneous dielectric at a distance r from the centre of an earthed conducting

sphere of radius R . Find the system of images which is equivalent to the induced charges. Find also the energy of interaction between the dipole and the sphere, and the force F and couple N on the dipole. Discuss the limiting case $r \rightarrow R$ ($r > R$).

165. An electric dipole $\mathbf{p}$ is placed at the centre of a spherical cavity of radius R in a conductor. Find the distribution of charges σ induced on the surface of the cavity. Determine the field $\mathbf{E}'$ due to these charges inside the cavity.

166*. The potential at a point O in a homogeneous dielectric with permittivity ϵ may be written down in the form

$$\varphi_1 = \sum_{l, m} \sqrt{\frac{4\pi}{2l+1}} a_{lm} r^l Y_{lm}(\vartheta, \alpha).$$

The dielectric is then disturbed so that it is no longer homogeneous and neutral in the neighbourhood of O (e.g. an inclusion in the form of a conductor or a dielectric with permittivity $\epsilon_1 \neq \epsilon$ is placed in the region). As a result, the potential outside the disturbed region becomes $\varphi = \varphi_1 + \varphi_2$ where

$$\varphi_2 = \sum_{l, m} \sqrt{\frac{4\pi}{2l+1}} \epsilon^{-1} r^{-(l+1)} Q_{lm} Y_{lm}(\vartheta, \alpha)$$

The potential φ_2 is due to free and bound charges in the region of the disturbance (the factor involving ϵ is introduced for convenience). Find the potential energy of interaction U between the inclusion and the external field φ_1 .

Hint: Consider the electric stresses acting on a surface enclosing the inclusion. Use the result of Problem 128.

167. Find the energy of interaction U_0 of the inclusion considered in the preceding problem with the external field in the case where, owing to the rapid convergence of the series, terms beyond $l = 1$ may be neglected (slowly varying external field). Give the result in a vector form. Find the force $\mathbf{F}$ and the couple $\mathbf{N}$ on the inclusion using this approximation.

168. Show that an uncharged dielectric body of permittivity ϵ_0 which is placed in a dielectric of permittivity ϵ is attracted towards regions of higher electric field strength if $\epsilon_0 > \epsilon$ and *vice versa*.

Hint: Use Eqn. (III.16).

169. In general, the components of a dipole moment $\mathbf{p}$ induced in a dielectric body placed in an external uniform field $\mathbf{E}$ may be written down in the form $p_i = \beta_{ik} E_k$ where β_{ik} is the

symmetric polarisability tensor. What orientation will the body tend to assume in the external uniform field? The body is uncharged and the tensor β_{ik} is of constant sign [$\beta_{ik} x_i x_k > 0$ where x_i ($i = 1, 2, 3$) is an arbitrary vector].

170. A cylinder made from a dielectric with permittivity ϵ_1 is placed in a homogeneous liquid dielectric with permittivity ϵ_2 . What will be the orientation of the cylinder when the system is placed in a uniform external field? What would be the orientation of a thin disc in the liquid dielectric?

171. Find the force F on a dielectric sphere due to a point charge q (cf. the conditions of Problem 157). Consider the limiting case of a conducting sphere. Solve the problem by two methods: the method of Problem 166 and by means of Eqn.(III.16).

172. An electrostatic field is produced by two conducting cylinders with parallel axes, radii R_1 and R_2 , and charges $\pm\kappa$ per unit length. The distance between the axes is $a > R_1 + R_2$. Find the capacitance C of the system per unit length [$C = \kappa/(\varphi_1 - \varphi_2)$ where φ_1 and φ_2 are the potentials of the cylinders].

Hint: Use the result of Problem 117.

173. The axes of two identical conducting cylinders of radius R are placed at a distance a from each other. The cylinders carry a charge of $\pm\kappa$ per unit length. Find the distribution of charges σ on the surfaces of the two cylinders.

174. A capacitor consists of two cylindrical conducting surfaces with radii R_1 and $R_2 > R_1$. The distance between the axes is $a < R_2 - R_1$. Find the capacitance C of the system.

175. Determine the potential φ due to a point charge in a homogeneous anisotropic medium characterised by a permittivity tensor ϵ_{ik} .

176. A plane parallel plate made from an anisotropic homogeneous dielectric having a permittivity tensor ϵ_{ik} is placed in a vacuum. A uniform electric field $\mathbf{E}_0$ is present outside the plate. Using the boundary conditions for the field vector determine the field $\mathbf{E}$ inside the plate.

177. Find the capacitance C of a plane parallel capacitor with electrode area equal to S when the distance between the plates is a and the space between them is filled with an anisotropic dielectric with a permittivity ϵ_{ik} . Neglect edge effects.

178. Find the change in the direction of the lines of force associated with an electric field $\mathbf{E}$ on passing from vacuum into an anisotropic dielectric. Use the results of Problem 176.

2. Coefficients of potential and capacitance

The potentials V_i of a system of n conductors are homogeneous linear functions of the charges q_k on the conductors:

$$V_l = \sum_{k=1}^n s_{lk} q_k \quad (l = 1, 2, 3, \dots, n). \quad (\text{III.26})$$

The quantities s_{ik} are called the coefficients of potential. They depend on the mutual disposition, form, and geometrical dimensions of the conductors, and also on the permittivity of the surrounding medium. The matrix $\hat{s}$ is symmetric:

$$s_{ik} = s_{ki}. \quad (\text{III.27})$$

The quantity s_{ik} is the potential of the i -th conductor when the k -th conductor carries a charge $q_k = 1$ while the remaining conductors remain uncharged. All the s_{ik} are positive quantities.

It is clear that the charges on the conductors are homogeneous linear functions of their potentials so that

$$q_i = \sum_{k=1}^n c_{ik} V_k \quad (i = 1, 2, 3, \dots, n). \quad (\text{III.28})$$

The quantities c_{ik} are called the coefficients of capacitance. Moreover, $c_{ii} > 0$ (self-capacitances) and $c_{ik} = c_{ki} < 0$ when $i \neq k$ (mutual capacitances).

The quantity c_{ik} is the charge on the i -th conductor when all the conductors other than the k -th conductor are earthed while the k -th conductor is maintained at a potential $V_k = 1$. The matrices s_{ik} and c_{ik} are reciprocal to each other.

In the case of an isolated conductor the only finite coefficient of capacitance is c_{11} and this is referred to simply as the capacitance of the conductor. The capacitance of a capacitor (III.14) may be expressed in terms of the coefficients of capacitance of its electrodes (cf. Problem 180). The energy of a system of conductors is given by

$$W = \frac{1}{2} \sum_{l, k} c_{lk} V_l V_k = \frac{1}{2} \sum_{l, k} s_{lk} q_l q_k. \quad (\text{III.29})$$

The generalised force F_Q corresponding to a generalised

coordinate a is given by

$$F_a = -\frac{1}{2} \sum_{i, k} \frac{\partial s_{ik}}{\partial a} q_i q_k = +\frac{1}{2} \sum_{i, k} \frac{\partial c_{ik}}{\partial a} V_i V_k. \quad (\text{III.30})$$

Green's reciprocity theorem is useful in the solution of electrostatic problems. This theorem reads as follows: If total charges $q_1, q_2, q_3, \dots, q_n$ on n conductors produce potentials $V_1, V_2, V_3, \dots, V_n$ and if charges $q'_1, q'_2, q'_3, \dots, q'_n$ produce potentials $V'_1, V'_2, V'_3, \dots, V'_n$, then

$$\sum_{i=1}^n q_i V'_i = \sum_{i=1}^n q'_i V_i. \quad (\text{III.31})$$

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179. Prove Green's reciprocity theorem [Eqn. (III.31)].

180. A given system consists of two conductors at a large distance from all other conductors. The conductor 1 is placed inside the hollow conductor 2. Express the capacitances C and C' of the capacitor and the isolated conductor forming this system in terms of its coefficients of capacitance.

181. Express the coefficients of potential s_{ik} in terms of the coefficients of capacitance c_{ik} in the case of a system of two conductors.

182. The capacitances of two isolated conductors are c_1 and c_2 . The conductors are in a vacuum at a distance r from each other, where r is large compared with the linear dimensions of the conductors. Show that the coefficients of capacitance of the system are given by

$$c_{11} = C_1 \left(1 + \frac{C_1 C_2}{r^2} \right), \quad c_{12} = -\frac{C_1 C_2}{r}, \quad c_{22} = C_2 \left(1 + \frac{C_1 C_2}{r^2} \right).$$

Hint: Determine the coefficients of potential to within terms of the order of $1/r$.

183. The coefficients of capacitance of a system of two conductors are c_{11} , c_{22} , and $c_{12} = c_{21}$. Find the capacitance C of a capacitor formed by the two conductors.

184. Four identical conducting spheres are placed at the corners of a square. Sphere 1 carries a charge q . It is then connected by means of a thin wire in turn to spheres 2, 3, and 4

(cyclic numeration) until equilibrium is established in each case. Find the charge on each of the conductors at the end of the operation. The coefficients of potential may be looked upon as given.

185. Three identical spheres of radius a are placed at the corners of an equilateral triangle with sides $b \gg a$. To start with all the spheres carry equal charges q . They are then earthed in turn until equilibrium is established. What is the charge of each sphere at the end of the process?

186. The self-capacitance of two conductors at potentials V_1 and V_2 is C_1 and C_2 respectively. The distance between the conductors is much larger than their linear dimensions. Find the force between them.

187. A closed conducting surface at a potential V_1 encloses a conductor at a potential V_0 . At the same time, the potential at a point P in the space between the two surfaces is V_P . The conductors are then earthed and a charge q is placed at the point P . Find the charges induced on the conductors.

188. Show that the geometrical locus of points from which a unit point charge induces in an earthed conductor a charge of equal magnitude is an equipotential surface in the field of the conductor in the absence of the point charge.

189. Two conductors with self-capacitances c_{11} and c_{22} and mutual capacitances c_{12} form a part of a system of insulated conductors and are connected by a thin wire. Find the self-capacitance of the sub-system of joined conductors and the coefficients of mutual capacitance between them and the remaining conductors.

190. Two identical spherical capacitors with inner and outer radii a and b are insulated and placed at a large distance from each other. The inner spheres are given charges q and q_1 , after which the outer spheres are joined by a wire. Find an approximate expression for the change in the energy of the system.

191. The thin outer electrode of a spherical capacitor is earthed and a small aperture is cut in it. An insulated wire is then passed through this aperture so that the inner sphere becomes connected to a third conductor which lies at a large distance from the capacitor. The self-capacitance of the third conductor is C and the total charge carried by the inner sphere and the distant conductor is q . Find the force F on the third conductor assuming that the radius of the outer sphere is b and that of the inner sphere is a .

192*. A conductor is charged by placing it in contact with the plate of an electrophorus. The conductor is then removed and

the plate is recharged to a charge Q . On first contact the charge given to the conductor is q . What is the charge received by the conductor after a very large number of contacts?

3. Special methods of electrostatics

This section contains electrostatic problems which are mathematically more difficult. The various methods which may be used to solve electrostatic problems are discussed in several books, some of which are listed at the end of the present book. In this section only some of these methods are illustrated, namely, the method of curvilinear coordinates (in the case of elliptical surfaces and the surfaces of two spheres), and the methods of images, integral transformations, and inversion. These methods are explained in the solutions of the various problems (for example, Problems 193, 195, 205, 209, 211, and 215). We shall confine our attention here to a brief account of the method of inversion.

The inversion transformation is defined as the transformation of space in which each point is transformed into a conjugate point relative to some suitably chosen sphere of inversion of radius R . If the spherical coordinates (with origin at the centre of the sphere of inversion) of the original point are r , ϑ , and α , then the spherical coordinates of the inverse point are $r' = R^2/r$, ϑ , and α . In vector form,

$$\mathbf{r}' = \frac{R^2 \mathbf{r}}{r^2} \quad \text{or} \quad \mathbf{r} = \frac{R^2 \mathbf{r}'}{r'^2}. \quad (\text{III.32})$$

The inversion transformation is conformal. Inversion transforms a sphere into a sphere. If, in particular, the centre of inversion lies on the sphere to be transformed, then the latter becomes a plane, and *vice versa*.

Laplace's equation is invariant with respect to the inversion transformation, *i.e.* if a function $\varphi(\mathbf{r})$ is a solution of Laplace's equation in the original space, then

$$\varphi'(\mathbf{r}') = \frac{r}{R} \varphi(\mathbf{r}) = \frac{R}{r'} \varphi\left(\frac{R^2}{r'^2} \mathbf{r}'\right) \quad (\text{III.33})$$

is a solution of the equation in the inverted space.

The basic problem which may be solved by the method of inversion may be formulated as follows: it is required to find the field due to a system of earthed conductors and point charges q_i

placed at points with position vectors $\mathbf{r}_i$. The potential at infinity is $V = \text{const.}$ In order to solve the problem let us use the method of inversion to transform the surfaces of the conductors into a simpler form.

The point charges q_i are then replaced by

$$q'_i = \frac{R}{r_i} q_i, \quad (\text{III.34})$$

which lie at the points

$$\mathbf{r}'_i = R^2 \frac{\mathbf{r}_i}{r_i^2}.$$

Moreover, a charge

$$q_0 = -RV \quad (\text{III.35})$$

appears at the point $\mathbf{r}' = 0$. The electrostatic problem, *i.e.* the determination of the potential $\varphi'(\mathbf{r}')$, is then solved in the inverted system. The potential $\varphi(\mathbf{r})$ may then be obtained by means of the reciprocal transformation.

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193*. A conducting ellipsoid carrying a charge q and having semi-axes a , b , and c is placed in a homogeneous dielectric of permittivity ϵ . Find the potential φ and the capacitance C of the ellipsoid and also the surface density of charge σ .

Hint: Use the ellipsoidal coordinates (*cf.* Problem 64). The potential should be sought in the form $\varphi(\xi)$.

194. Using the results of the preceding problem find the potentials and the capacitances of a prolate and an oblate ellipsoid of revolution. Consider the special cases of a thin long rod and a thin disc. The capacitance C and the potential φ of the prolate ellipsoid should also be found using the result of Problem 75.

195*. A conducting ellipsoid carrying a charge q is placed in a vacuum in a uniform external field $\mathbf{E}_0$ which is parallel to one of the axes of the ellipsoid. Find the potential φ of the resultant electric field.

Hint: Use the ellipsoidal coordinates of Problem 64. The boundary conditions on the surface of the ellipsoid ($\xi = 0$) can only be satisfied if the dependence of the potential φ' due to the

induced charges on the coordinates η , ξ is the same as for the external field: $\varphi' = \varphi_0(\xi, \eta, \xi) \cdot F(\xi)$.

196. The field strength in a plane parallel capacitor is E_0 . The earthed electrode carries a boss in the form of one-half of a prolate ellipsoid of revolution whose axis of symmetry is perpendicular to the plates. The distance between the plates is large compared with the linear dimensions of the boss. Find the potential at a general point between the plates, and the ratio of the maximum value $E_{\max}$ of the field to the field E_0 .

197. An uncharged conducting ellipsoid is placed in a uniform external field $\mathbf{E}_0$ which is arbitrarily oriented relative to the axes of the ellipsoid. Find the total potential φ . Discuss the potential at a large distance from the ellipsoid by expressing it in terms of the depolarisation coefficients

$$n^{(x)} = \frac{abc}{2} \int_0^\infty \frac{ds}{(s+a^2)R_s}, \quad n^{(y)} = \frac{abc}{2} \int_0^\infty \frac{ds}{(s+b^2)R_s},$$

$$n^{(z)} = \frac{abc}{2} \int_0^\infty \frac{ds}{(s+c^2)R_s} \quad (R_s = \sqrt{(s+a^2)(s+b^2)(s+c^2)}).$$

198. Find expressions for the depolarisation coefficients introduced in the preceding problem for a prolate ellipsoid of revolution ($a > b = c$). Discuss the special cases of a very elongated ellipsoid (rod) and an almost spherical ellipsoid.

199. Find the depolarisation coefficients for an oblate conducting ellipsoid ($a = b > c$). Discuss the special case of a disc.

200*. A dielectric ellipsoid with semi-axes a , b , and c is placed in a uniform external field $\mathbf{E}_0$. The permittivity of the ellipsoid and of the homogeneous dielectric which surrounds it is ϵ_1 and ϵ_2 respectively. Find the potential φ of the resultant electric field (use the hint given in Problem 195). Find the field strength $\mathbf{E}$ inside the ellipsoid and also the potential φ_2 at large distances from the ellipsoid by expressing it in terms of the polarisability components along the principal axes.

201. An ellipsoid of revolution having a permittivity ϵ_1 is placed in a uniform external field $\mathbf{E}_0$ in a homogeneous dielectric with permittivity ϵ_2 . Find the energy U of the ellipsoid in this field and the couple N acting on it. Discuss also the case of a conducting ellipsoid of revolution.

202*. Show that when a conducting liquid sphere is given a sufficiently large charge it becomes unstable. Find the critical charge q_c assuming that the radius of the drop is R and the surface tension is α .

Hint: Compare the energy of a spherical drop with the energy of a deformed drop in the shape of a prolate ellipsoid of revolution. The surface area of this ellipsoid is given by

$$S = 2\pi b^2 + \frac{2\pi b a^2}{\sqrt{a^2 - b^2}} \cos^{-1} \frac{b}{a} \quad (a > b = c).$$

203*. A uniform electric field $\mathbf{E}_0$ is in the half-space $z < 0$ is bounded by an earthed conducting plane $z = 0$ which carries a circular aperture of radius a . Find the potential φ in all space. Consider the special case of large distances from the aperture ($z > 0$).

Hint: Use the oblate spheroidal coordinates (cf. Problem 65) with $c = 0$. The solution should be sought in the form $\varphi = -E_0 z F(\xi)$.

204. Find the distribution of charges σ on the conducting plane of the preceding problem.

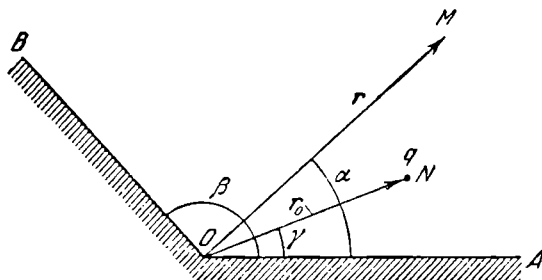


Fig. 11.

205*. A point charge q is placed between two earthed conducting half-planes OA and OB which are at an angle β to each other. The point charge is at $N(\mathbf{r}_0)$ in Fig. 11. The cylindrical coordinates of the point charge are $(r_0, \gamma, 0)$, the z -axis lies along the line of intersection of the two planes, and the azimuthal angle α is measured from the plane OA. Show that the potential $\varphi(r, \alpha, z)$ may be written in the form

$$\varphi(r, \alpha, z) = \int_0^\infty \varphi_k(r, \alpha) \cos kz \, dk,$$

where

$$\varphi_k(r, \alpha) = \frac{8q}{\beta} \begin{cases} \sum_{n=1}^{\infty} K_{\frac{n\pi}{\beta}}(kr_0) I_{\frac{n\pi}{\beta}}(kr) \sin \frac{n\pi\gamma}{\beta} \sin \frac{n\pi\alpha}{\beta} & r < r_0, \\ \sum_{n=1}^{\infty} I_{\frac{n\pi}{\beta}}(kr_0) K_{\frac{n\pi}{\beta}}(kr) \sin \frac{n\pi\gamma}{\beta} \sin \frac{n\pi\alpha}{\beta} & r > r_0, \end{cases}$$

and I_ν and K_ν are the cylinder functions.

Hint: Use Eqn. (11) of Appendix I and the results of Appendix III.

206. Show that the potential due to a point charge placed between the two planes of the preceding problem may be expressed in the form

$$\varphi(r, \alpha, z) = \frac{q}{\beta \sqrt{2rr_0}} \int_{\eta}^{\infty} \left[\frac{\sinh \frac{\pi\zeta}{\beta}}{\cosh \frac{\pi\zeta}{\beta} - \cos \frac{\pi(\alpha - \gamma)}{\beta}} - \frac{\sinh \frac{\pi\zeta}{\beta}}{\cosh \frac{\pi\zeta}{\beta} - \cos \frac{\pi(\alpha + \gamma)}{\beta}} \right] \cdot \frac{d\zeta}{\sqrt{\cosh \zeta - \cosh \eta}},$$

where

$$\cosh \eta = \frac{r_0^2 + r^2 + z^2}{2rr_0}, \quad \eta > 0.$$

Hint: Use the following formulae

$$\int_0^{\infty} K_\nu(kr) I_\nu(kr_0) \cos kz \, dk = \frac{1}{2\sqrt{2rr_0}} \int_{\eta}^{\infty} \frac{e^{-\xi\nu} d\xi}{\sqrt{\cosh \xi - \cosh \eta}}$$

and

$$\sum_{n=1}^{\infty} p^n \cos nx = \frac{1}{2} \left(\frac{1-p^2}{1-2p \cos x + p^2} - 1 \right).$$

207. Find the potential φ due to a charge q having cylindrical coordinates $r_0, \gamma, z = 0$.

Hint: Use the result of Problem 206. In order to evaluate the integral use the substitution $\cosh \frac{1}{2}\zeta = \cosh \frac{1}{2}\eta \cosh u$ where $0 < u < \infty$.

208. Find the distribution of surface charge σ near the line of intersection of the two conducting planes of Fig. 11. Assume that the system lies in the field of an arbitrary charge distribution.

Hint: Consider the case of a single charge near the system; use the result of Problem 205, the expansion given by Eqn. (6) of Appendix III, and the formula

$$\int_0^{\infty} K_{\nu}(k\rho) k^{\nu} \cos kz \, dk = 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) \frac{\rho^{\nu}}{(\rho^2 + z^2)^{\nu+1/2}}.$$

209*. A point charge q is placed at a distance a from a homogeneous plane parallel dielectric plate of thickness c . Find the electric field using the fact that both the product $J_0(kr_1) \exp(\pm kz)$ and the integral $\int_0^{\infty} A(k) J_0(kr_1) \exp(\pm kz) dk$ satisfy the Laplace equation [r_1, z are the cylindrical coordinates of the point, J_0 is the zero-order Bessel function and $A(k)$ is an arbitrary function of k].

Hint: Use the result

$$\frac{1}{\sqrt{r_1^2 + z^2}} = \int_0^{\infty} e^{-k|z|} J_0(kr_1) dk,$$

which is derived in Appendix III.

210. A plane parallel dielectric plate of thickness $a/2$ and permittivity ϵ is placed between the plates of a plane parallel capacitor which are at a distance a from each other. The dielectric is in contact with one of the plates, both of which are earthed. A charge q which may be looked upon as a point charge is then placed on the surface of the dielectric. Find the potential φ in the capacitor and discuss its form in the neighbourhood of the charge. Use the method of images.

211*. The radii of the electrodes of a non-concentric spherical capacitor are a_1 and a_2 and the distance between their centres is b ($a_1 + b < a_2$); the outer sphere is earthed and the inner sphere is maintained at a potential V . Find the potential inside the capacitor. Find also the capacitance C of the spherical capacitor.

Hint: Use bispherical coordinates (*cf.* Problem 67). Substitute $\varphi = (2 \cosh \xi - 2 \cos \eta)^{\frac{1}{2}} \psi$, carry out the separation of variables in the equation for ψ , and use the results of Appendix II including Eqn. (16) of that appendix.

212. Find the capacitance of a non-concentric spherical capacitor assuming that the distance between the centres is small ($b \ll a_1, a_2$) to within terms of the order of b^2 by using the results of the preceding problem (*cf.* Problem 160).

213. The distance between the centres of two conducting spheres of radii a_1 and a_2 is b ($b > a_1 + a_2$). Find the coefficients of capacitance c_{ik} of the system using bispherical coordinates.

214. The two conducting spheres of the preceding problem are at a large distance from each other ($b \gg a_1, a_2$). Find the coefficients of capacitance c_{ik} to within terms of the order of b^{-4} .

215*. Two conducting spheres of equal radii a touch each other. Find the capacitance C of the system by the method of inversion. Find also the potential due to the system when the spheres carry a total charge q .

Hint: Use the results of Problem 210.

216. Use the method of inversion to determine the field of an earthed sphere of radius R with a point charge q placed at a distance $a > R$ from its centre (cf. Problem 153).

Hint: The potential of a uniformly charged sphere in the absence of the point charge may be regarded as known.

217*. The surface of a conductor is formed by two spheres of radii R_1 and R_2 which intersect on a circle of radius a . Find the capacitance C of the conductor using the solution of Problem 206 (two intersecting planes) and the method of inversion.

Hint: The surface of the conductor is described in terms of toroidal coordinates (cf. Problem 68) by the equations

$$\xi = \xi_1 = \text{const}, \quad \zeta = \zeta_2 = \text{const} \quad \left(\sin \xi_1 = \pm \frac{a}{R_1}, \quad \sin \xi_2 = \pm \frac{a}{R_2} \right).$$

It is sufficient to consider the transformation of the coordinates in the plane perpendicular to the line of intersection of the two planes and passing through the centre of inversion, which should be taken to lie on the line of intersection of the two spheres. In order to determine the charge q on the conductor at a given potential use the fact that the field at large distances from the conductor is of the form $\varphi = (q/r) - V$ where $-V$ is the potential at infinity.

218. Find the capacitance C of the following conductors:

- (a) a hollow spherical segment of radius R which subtends an angle 2θ at the centre;
- (b) a hemisphere of radius R .

219. A conductor is formed by two spheres each of radius a whose surfaces intersect at an angle $\pi/3$. Find the capacitance C of the conductor.

Chapter

4

STEADY CURRENTS

The distribution of steady currents in a conducting medium with specific conductivity $\kappa(\mathbf{r})$ may be described with the aid of a current density $\mathbf{j}(\mathbf{r})$ which satisfies the equation

$$\operatorname{div} \mathbf{j} = 0. \quad (\text{IV.1})$$

This equation is a consequence of the law of conservation of charge. The current density in a medium is proportional to the sum of the electric field $\mathbf{E}$ and the external field $\mathbf{E}_e$ which includes non-electrical effects:

$$\mathbf{j} = \kappa(\mathbf{E} + \mathbf{E}_e). \quad (\text{IV.2})$$

This expression is equivalent to Ohm's law.

The electric field $\mathbf{E}$ and the current distribution $\mathbf{j}$ in a conductor can be conveniently described by a scalar potential φ (just as in electrostatics). The potential is defined by $\mathbf{E} = -\operatorname{grad} \varphi$. It follows from this definition and from Eqns. (IV.1) and (IV.2) that the basic differential equation for φ is

$$\operatorname{div} (\kappa \operatorname{grad} \varphi) = \operatorname{div} \kappa \mathbf{E}_e. \quad (\text{IV.3})$$

On surfaces across which κ or $\mathbf{j}_e = \kappa \mathbf{E}_e$ are discontinuous, Eqn. (IV.3) is replaced by the boundary conditions

$$\kappa_2 E_{2n} - \kappa_1 E_{1n} = j_{e1n} - j_{e2n}, \quad (\text{IV.4})$$

$$\varphi_1 = \varphi_2. \quad (\text{IV.5})$$

On the surfaces of insulators ($\kappa = 0$) the condition given by (IV.4) becomes

$$j_n = 0 \quad \text{or} \quad \kappa E_n + j_{en} = 0. \quad (\text{IV.6})$$

If the medium consists of a number of homogeneous regions and contains no external sources of e.m.f. then within each such region

$$\Delta \varphi_k = 0, \quad (\text{IV.7})$$

They are independent of the potentials V_k and the currents $\mathcal{I}_k$, and are determined exclusively by the geometry of the electrodes and the form of conductivity function κ . The coefficients of resistance are analogous to the coefficients of potential in the electrostatic problem (*cf.* Chapter III Section 2).

The distribution of currents in quasi-linear conductors which is often encountered in practice may be determined with the aid of Kirchhoff's laws. A convenient method of considering complicated circuits involving quasi-linear conductors is the method of circulating currents.

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220. An accumulator with a low internal resistance and e.m.f. $\mathcal{E}$ is unable to supply a given instrument with a current i for a long period of time. In order to extend the life of the accumulator the instrument and the accumulator are connected in parallel and then joined through a resistance R to d.c. mains. The voltage V of the mains is subject to fluctuations which are such that $V_1 \leq V \leq V_2$ where $V_1 > V_2 > \mathcal{E}$. The resistance R is chosen so that when $V = V_1$ no current is drawn from the accumulator. Find the current drawn from the accumulator when $V = V_2$.

221. Find the parameters of the coil of a moving coil galvanometer which gives a maximum deflection for a given external e.m.f. and external resistance R (series connection). The deflection of the galvanometer is proportional to the number of turns n in the coil and the current $\mathcal{I}$ in the circuit. Since the volume occupied by the coil is limited, it may be assumed that nS is approximately constant where S is the cross-section of the wire from which the coil is made.

222. A square net made of a wire of uniform cross-section consists of n^2 identical square cells. The resistance of one side of each of these cells is r . The current enters at one of the corners of the net and leaves at the opposite corner. Find the resistance R of the entire net for $n = 2, 3$, and 4.

Hint: In order to reduce the number of circulating currents make use of the symmetry of the circuit.

223*. A telegraph line (Fig. 12) is suspended from n insulators at $A_1, A_2, \dots, A_n$ (the ground forms the return lead). The sections $AA_1, A_1A_2, \dots, A_nA_{n+1}$ have the same resistance R and the insulators have an infinite resistance when dry. When

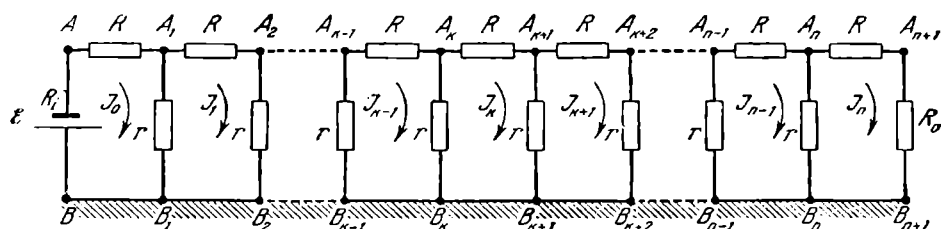


Fig. 12.

the insulators are damp there is leakage to earth through them and the resistance of each of the insulators is then r . At one end of the line there is a battery with an internal resistance R_i and e.m.f. $\mathcal{E}$ while the final element is connected to earth through the resistance R_Q . Find the current in each of the sections of the line and also the current flowing through the load R_Q . By how much should the e.m.f. of the battery be increased in order that the current through the load should be the same both in the case of dry and damp insulation? Consider the special case of $R_Q = 0$.

Hint: Consider circulating currents in the circuits formed by the section $A_{k-1}A_k$ of the line and the leakage paths for the insulators A_{k-1} and A_k . The solution of the resulting differential equation is the hyperbolic cosine.

224*. An underground cable has a constant resistance ρ per unit length. The insulation of the cable is imperfect and such that the leakage conductivity per unit length is constant and equal to $\frac{1}{\rho'}$ with the earth acting as the return lead. Find the differential equation which describes the distribution of steady currents in the cable. Find the relation between the current $\mathcal{J}(x)$ in the cable and the potential difference between the central conductor and the earth.

Hint: Use Eqn. (1) of the solution of Problem 223.

225*. A battery of e.m.f. $\mathcal{E}$ and internal resistance R_i is connected between one end of an underground cable and the earth. The length of the cable is a and its resistance per unit length and conductivity per unit length are ρ and $\frac{1}{\rho'}$ respectively. The other end of the cable is connected to earth through a resistance R_Q . Find the current distribution $\mathcal{J}(x)$ along the cable. In particular, consider the special case $R_i = R_Q = 0$. Check the result by finding the solution for the case where there is no leakage.

Hint: Use the differential equations obtained in Problem 224, or Eqn. (7) of the solution of Problem 223.

226. Two plane parallel conducting plates are placed in contact with the two electrodes of a plane parallel capacitor. The thickness of the two plates is h_1 and h_2 and their conductivities and permittivities are respectively κ_1 , κ_2 and ϵ_1 , ϵ_2 . The plates of the capacitor are made of a material whose conductivity is much greater than κ_1 and κ_2 and a constant potential difference V is maintained across them. Find the electric field E , the induction D , and the current density j in the plate, and also the densities of free and bound charges on all three separation boundaries.

227. Find the law of refraction for current lines across a smooth separation boundary between two media with conductivities κ_1 and κ_2 respectively.

228*. A constant current $\mathcal{I}$ flows through an infinitely long straight conductor of radius a and conductivity κ . The conductor is surrounded by a thick cylindrical envelope which is coaxial with it and serves as the return lead. The inner radius of the envelope is b and the outer radius $c \rightarrow \infty$. Find the electric and magnetic fields in all space and determine the distribution of surface charges. Assume that the permittivity of the medium between the conductors is ϵ .

229. Three conductors of circular cross-section and the same radius r are connected in series forming a closed ring. The lengths of the conductors are l_0 , l_1 , $l_2 \gg r$ and their conductivities are κ_0 , κ_1 , κ_2 . An external e.m.f. $\mathcal{E}_e$, which is independent of time, is uniformly distributed throughout the first of the three conductors. Find the electric field E_0 and the distribution of charges inside the ring.

230. Find the energy flux γ through the surfaces of the three conductors considered in Problem 229. Hence deduce Joule's law.

231. The current distribution in a three-dimensional conductor with conductivity κ is such that the electric field strength and, therefore, the current density are constant on an equipotential surface. Show that the resistance of the conductor is given by the same formula as the resistance of a quasi-linear conductor with a variable cross-section. Note that these conditions are analogous to the conditions under which the electrostatic Gauss theorem may be used in the corresponding electrostatic problem.

232. Using the result of the preceding problem find the resistance R for the following conductors:

- (a) a spherical capacitor with plate radii a and b ($a < b$), filled with a homogeneous medium of conductivity κ ;

- (b) a similar capacitor filled with two homogeneous layers with conductivities κ_1 and κ_2 , which are separated by a boundary of radius c (the layer with conductivity κ_1 is in contact with the inner electrode).
- (c) a cylindrical capacitor with plate radii a and b ($a < b$), and length l which is filled with a medium of conductivity κ (neglect edge effects).

233. A particular system is earthed with the aid of a perfectly conducting sphere of radius a . One half of the sphere is in contact with the ground (ground conductivity $\kappa_1 = \text{const}$). The layer of earth which is in immediate contact with the sphere has an artificially increased conductivity κ_2 . Find the resistance R to earth of the earthing device assuming that the outer radius of the layer next to the sphere is b .

234*. A system of perfect conductors (electrodes) is placed in a medium with conductivity $\kappa(\mathbf{r})$ and permittivity $\epsilon(\mathbf{r})$. The medium is such that $\kappa(\mathbf{r})/\epsilon(\mathbf{r}) = \text{const}$ at all points. Find the relation between the coefficients of potential s_{ik} and the coefficients of resistance R_{ik} of the system of conductors. What is the relation between the charges q_k on the electrodes and the currents $\mathcal{J}_k$ leaving them?

235. A capacitor of arbitrary form is filled with a homogeneous dielectric of permittivity ϵ . Find its capacitance C if it is known that when it is filled with a homogeneous conductor having a conductivity κ its d.c. resistance is R .

236. A system of electrodes is characterised by the coefficients of resistance R_{ik} . Find the amount of heat Q liberated per unit time in the space between the electrodes given that the currents $\mathcal{J}_k$ leaving the electrodes are known.

237. Two perfectly conducting spheres of radii a and b are placed in a homogeneous medium with conductivity κ and dielectric permittivity ϵ . The distance between the centres of the spheres is l and a current $\mathcal{J}$ enters one of the spheres and is taken out through the other. Find a resistance $R = \frac{V_a - V_b}{\mathcal{J}}$

between the spheres, where V_a and V_b are the potentials of the two spheres and $\mathcal{J}$ is the current leaving the sphere of radius a .

Hint: Express R_{ik} in terms of the coefficients of capacitance c_{ik} of the system (cf. Problem 213).

238. The ends of a given circuit are earthed by means of two perfectly conducting spheres (radii a_1 and a_2). One half of

each of the spheres is in contact with the earth which acts as the second lead. The distance between the spheres is $l \gg a_1, a_2$ and the conductivity of the ground is κ . Find the resistance R between the spheres.

239. Solve the preceding problem in the case where the two spheres are replaced by identical ellipsoids of revolution volume V and eccentricity e_0 . The axes of revolution of the ellipsoids are perpendicular to the earth's surface and their centres lie on this surface. Which of the two systems is a more efficient earth-ing device (*i. e.* ensures lower resistance)?

240*. A plane electrode $x = 0$ is capable of emitting an unlimited number of particles of charge e and mass m when an electric field is applied to it. The particles leave the plate with zero initial velocity and are accelerated towards another plane electrode which is parallel to the first electrode and is at a distance a from it. The potential difference between the electrodes is φ_0 . The emission from the first electrode continues until the field due to the space charge produced between the plates compensates the external field so that $(\partial\varphi/\partial x)_{x=0} = 0$ on the surface of the first electrode. Find the steady-state current j between the electrodes as a function of the potential difference φ_0 between the electrodes.

Hint: The potential in the space between the electrodes obeys the Poisson equation $\Delta\varphi = -4\pi\rho$, $\rho = j/v$ where v is the velocity of the particles at the particular point.

Chapter

5

MAGNETOSTATICS

In the case of a constant magnetic field Maxwell's equations are of the form

$$\text{curl } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}, \quad \text{div } \mathbf{B} = 0, \quad (\text{V.1})$$

where $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the magnetic field strength, and $\mathbf{j}$ is the current density. The quantity c is equal to the ratio of the electromagnetic to electrostatic units of charge ($c = 3 \times 10^{10} \text{ cm sec}^{-1}$). In isotropic diamagnetics and paramagnetics $\mathbf{B}$ and $\mathbf{H}$ are related by

$$\mathbf{B} = \mu \mathbf{H}, \quad (\text{V.2})$$

where μ is the magnetic permeability of the material (a scalar). In the case of anisotropic materials the permeability is a tensor of rank 2. The density of molecular currents in a material placed in a magnetic field may be expressed in terms of the magnetisation vector $\mathbf{M}$ (magnetic moment per unit volume):

$$\mathbf{j}_{\text{mol}} = c \text{ curl } \mathbf{M}. \quad (\text{V.3})$$

The three vectors $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{H}$ are related by

$$\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}. \quad (\text{V.4})$$

The main methods which may be used to determine the magnetic field in a non-ferromagnetic medium are as follows:

(a) The use of the Biot-Savart law. The magnetic field due to a current element $\mathcal{I} d\mathbf{l}$ in a vacuum or a homogeneous medium is given by

$$d\mathbf{H} = \frac{\mathcal{I}}{cr^3} (d\mathbf{l} \times \mathbf{r}). \quad (\text{V.5})$$

According to the superposition principle, the total field at a given point may be obtained by integrating Eqn. (V.5) with respect to all the current elements.

(b) Direct integration of Eqns. (V.1) and (V.2) subject to the boundary conditions

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0, \quad \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \frac{4\pi}{c} \mathbf{i}, \quad (\text{V.6})$$

where $\mathbf{i}$ is the surface current density and the normal $\mathbf{n}$ is drawn from the first region into the second. If the current distribution is axially symmetric then the following integral form of the first of the two equations in Eqn. (V.1) is useful:

$$\oint H_l dl = \frac{4\pi}{c} \mathcal{J}. \quad (\text{V.7})$$

The closed integral in this expression is evaluated over an arbitrary closed path and $\mathcal{J}$ is the total current flowing through an arbitrary surface drawn through this path.

(c) The vector potential method. The vector potential $\mathbf{A}$ is defined by

$$\mathbf{B} = \text{curl } \mathbf{A} \quad (\text{V.8})$$

with the subsidiary condition

$$\text{div } \mathbf{A} = 0. \quad (\text{V.9})$$

In all regions in which the magnetic material is homogeneous, $\mathbf{A}$ satisfies the equation

$$\Delta \mathbf{A} = - \frac{4\pi\mu}{c} \mathbf{j}. \quad (\text{V.10})$$

The boundary conditions for the vector potential may be deduced from the boundary conditions for $\mathbf{B}$ and $\mathbf{H}$ [Eqn. (V.6)].

The vector potential due to a given current distribution in a homogeneous medium with a given permeability μ may be written down in the form of the volume integral

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{c} \int \frac{\mathbf{j}(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|}. \quad (\text{V.11})$$

The corresponding expression for a linear current may be obtained by making the following substitution $\mathbf{j}dV' \rightarrow \mathcal{J}d\mathbf{l}'$. At large distances from a given current distribution Eqn. (V.11) assumes the simpler form

$$\mathbf{A} = \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad (\text{V.12})$$

where the magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{m} = \frac{1}{2c} \int (\mathbf{r}' \times \mathbf{j}) dV' \quad (\text{V.13})$$

and it is assumed that $\mu = 1$.

(d) The scalar potential method. In all regions of space where $\mathbf{j} = 0$, we have $\text{curl } \mathbf{H} = 0$, and hence

$$\mathbf{H} = -\text{grad } \psi, \quad (\text{V.14})$$

where ψ is a scalar potential which, for $\mu = \text{const}$, satisfies the Laplace equation. However, the scalar potential defined in this way will not in general be a single-valued point function. The scalar potential is used in Problems 254, 255, etc.

The energy of the magnetic field localised in a volume V is given by the following integral

$$W = \frac{1}{8\pi} \int (\mathbf{H} \cdot \mathbf{B}) dV, \quad (\text{V.15})$$

which is evaluated over the volume V . If a given system of currents has finite dimensions then its total energy may be calculated with the aid of the formula

$$W = \frac{1}{2c} \int (\mathbf{A} \cdot \mathbf{j}) dV, \quad (\text{V.16})$$

in which the integration is carried out over the volume occupied by the currents.

The magnetic energy of a quasi-linear conductor carrying a current $\mathcal{I}$ may be expressed in terms of the self-inductance L of the conductor:

$$W = \frac{1}{2c^2} L \mathcal{I}^2. \quad (\text{V.17})$$

The self-inductance may also be written down in the form of a double integral over the volume occupied by the conductor:

$$L = \frac{1}{\mathcal{I}^2} \int \int \frac{\mathbf{j}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV dV'. \quad (\text{V.18})$$

The energy of interaction between two current-carrying conductors is given by

$$W_{12} = \frac{1}{4\pi} \int (\mathbf{H}_1 \cdot \mathbf{B}_2) dV = \frac{1}{c} \int (\mathbf{j}_1 \cdot \mathbf{A}_2) dV_1 \quad (W_{12} = W_{21}), \quad (\text{V.19})$$

where the first integral is evaluated over all space and the second over the volume of one of the conductors. In the case of quasi-linear currents the energy may be expressed in terms of the coefficient of mutual inductance L_{12} :

$$W_{12} = \frac{1}{c^2} L_{12} \mathcal{I}_1 \mathcal{I}_2, \quad (\text{V.20})$$

which may also be re-written in the form

$$W_{12} = \frac{1}{c} \mathcal{I}_1 \Phi_{12}, \quad (\text{V.21})$$

where Φ_{12} is the magnetic flux due to the first conductor which is intercepted by the second and is given by

$$\Phi_{12} = \int \mathbf{B}_2 \cdot d\mathbf{S}_1 = \oint \mathbf{A}_2 \cdot d\mathbf{l}_1 = \frac{1}{c} L_{12} \mathcal{I}_2. \quad (\text{V.22})$$

The mutual inductance may be obtained from the expression for the energy (V.20), from the expression for the flux (V.22), or, in the case of linear currents, from the formula

$$L_{12} = \mu \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r_{12}}. \quad (\text{V.23})$$

The generalised forces F_i acting between two fixed currents may be obtained by differentiating the energy of interaction W_{12} (or the quantity $U_{12} = -W_{12}$ which is known as the potential function) with respect to the corresponding generalised coordinates:

$$F_i = \frac{\partial W_{12}}{\partial q_i} = - \frac{\partial U_{12}}{\partial q_i}. \quad (\text{V.24})$$

The forces may also be obtained from Ampère's formula

$$d\mathbf{F} = \frac{\mathcal{I}}{c} (d\mathbf{l} \times \mathbf{B}), \quad (\text{V.25})$$

where $d\mathbf{l}$ is a circuit element carrying a current $\mathcal{I}$ and $d\mathbf{F}$ is the force on this element due to the external magnetic induction $\mathbf{B}$.

The forces which act on currents and magnetic materials may be calculated with the aid of the Maxwell stress tensor

$$T_{lm} = \frac{\mu}{4\pi} \left(H_l H_m - \frac{1}{2} H^2 \delta_{lm} \right) \quad (\text{V.26})$$

which is analogous to the stress tensor used in Chapter III for the electric field.

In all regions of space which are occupied by ferromagnetics, $\mathbf{B}$ and $\mathbf{H}$ are not linearly related. The relation between $\mathbf{B}$ and $\mathbf{H}$ is then not even single-valued (hysteresis) and the solution of magnetostatic problems becomes exceedingly difficult. However, in discussing permanent magnets it is frequently assumed that the relation between $\mathbf{B}$ and $\mathbf{H}$ is linear as before:

$$\mathbf{B} = \mu \mathbf{H} + 4\pi \mathbf{M}_0, \quad (\text{V. 27})$$

where $\mathbf{M}_0$ is a "constant" magnetisation, *i.e.* independent of $\mathbf{H}$, which is looked upon as a given function of the coordinates. Ferromagnetics obeying Eqn. (V.27) are called idealised ferromagnetics.

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* 241. A conductor of radius a is surrounded by a thin conducting shell of radius b . The two conductors are coaxial and carry equal but opposite currents $\mathcal{I}$. Determine the magnetic field $\mathbf{H}$ due to this system at all points in space. Solve the problem by two methods, namely, by integrating the differential form of Maxwell's equations and by using the integral form of these equations which is given by Eqn. (V.7).

✓* 242. Determine the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ due to a constant current $\mathcal{I}$ flowing through an infinite cylindrical conductor of radius a . Assume that the permeability of the conductor and the surrounding material is μ_0 and μ respectively. Solve the problem by the simplest method, *i.e.* with the aid of the integral form of Maxwell's equations, and also by introducing the vector potential $\mathbf{A}$.

* 243. Solve the preceding problem for a hollow cylindrical conductor of inner and outer radii a and b respectively.

* 244. A straight and infinitely long strip of width a carries a current $\mathcal{I}$ which is uniformly distributed across the width of the strip. Find the magnetic field $\mathbf{H}$ and verify the results by considering the limiting case of large distances from the strip.

* 245. Two thin infinitely long parallel plates carry equal and opposite currents $\mathcal{I}$. The width of each plate is a and the distance between them b . Find the force per unit length of each plate.

✓ 246. Find the vector potential $\mathbf{A}$ of the magnetic field $\mathbf{H}$ due to two parallel line currents flowing in opposite directions. The distance between the two currents is $2a$.

247. Determine the magnetic field $\mathbf{H}$ due to two parallel planes carrying equal surface current densities $i = \text{const.}$ Consider the two cases when the currents flow in the same and in opposite directions.

✕ 248. Determine the magnetic field $\mathbf{H}$ in a cylindrical cavity inside an infinitely long cylindrical conductor. The radii of the cavity and of the conductor are respectively a and b , and the distance between their parallel axes is d ($b > a + d$). Assume that the current $\mathcal{I}$ is uniformly distributed across the cross-section of the conductor.

Hint: Use the principle of superposition of fields.

249*. Find the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{H}$ due to a thin ring of radius a which carries a current $\mathcal{I}$. The surrounding medium is uniform and has a magnetic permeability μ . Express the results in terms of elliptical integrals.

Hint: Use the method employed in the solution of Problem 89.

250*. Show that if a given magnetic field is axially symmetric and may be represented by a vector potential with components $A_\alpha(r, z)$, $A_r = A_z = 0$ (cylindrical coordinates), then the equation for the lines of magnetic induction is

$$r A_\alpha(r, z) = \text{const.}$$

Hint: Consider the flux of magnetic induction inside the tube produced by rotating one of the lines of induction about the axis of symmetry (cf. solution of Problem 115).

251. Express the magnetic field $\mathbf{H}$ and the vector potential $\mathbf{A}$ of an axially symmetric field outside its sources in terms of the magnetic field strength $H(z)$ along the axis of symmetry.

252. Determine the magnetic field $\mathbf{H}$ along the axis of a solenoid in the form of a closely wound cylindrical coil. The height of the cylinder is h , radius a , number of turns per unit length n , and the current in the coil $\mathcal{I}$.

✓ 253*. A sphere of radius a is uniformly charged over its surface to a total charge e and is rotating about one of its diameters with a constant angular velocity ω . Find the magnetic field inside and outside the sphere. Express the magnetic field $\mathbf{H}$ outside the sphere in terms of the magnetic moment $\mathbf{m}$ of the sphere.

✓ 254. Find the scalar potential ψ of the magnetic field due to an infinitely long straight wire carrying a current $\mathcal{I}$. Calculate the components of the magnetic field.

✓255*. Find the scalar potential ψ of the magnetic field due to a closed linear circuit carrying a current. Solve the problem by integrating the Laplace equation and also by using the following well-known expression for the vector potential:

$$\mathbf{A} = \frac{\mathcal{I}}{c} \oint \frac{d\mathbf{l}}{r}.$$

256. Find the force $\mathbf{F}$ and the couple $\mathbf{N}$ on a closed thin conductor carrying a current and placed in a uniform magnetic field $\mathbf{H}$. The conductor is of an arbitrary form. Solve the problem in two ways, namely, by direct integration of the forces and moments of forces on current elements, and also by using the potential function. Express the result in terms of the magnetic moment $\mathbf{m}$.

257. Find the potential function U for two small current loops with magnetic moments $\mathbf{m}_1$ and $\mathbf{m}_2$. Find the force $\mathbf{F}$ and the couple $\mathbf{N}$ on the two currents. Consider the special case $\mathbf{m}_1 \parallel \mathbf{m}_2$.

258. Show that the forces between two small current loops are such that the corresponding magnetic moments tend to set themselves parallel to each other and to the line joining their centres.

259. Find the potential function u_{21} (per unit length) for two parallel infinitely long straight line currents $\mathcal{I}_1$ and $\mathcal{I}_2$, and the force f per unit length between them.

260. A square frame carrying a current $\mathcal{I}_2$ is placed so that two of its parallel sides are parallel to a long straight wire carrying a current $\mathcal{I}_1$ (Fig. 13). The length of each side of the frame is a . Find the force $\mathbf{F}$ and the couple $\mathbf{N}$ on the frame.

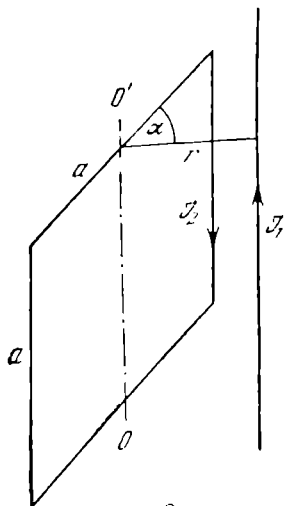


Fig. 13.

261. A frame carrying a current $\mathcal{I}_2$ consists of the arc of a circle which subtends an angle $2(\pi - \varphi)$. The ends of the arc are joined by a cord as shown in Fig. 14. A straight wire

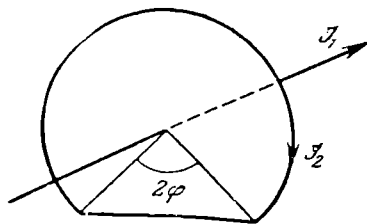


Fig. 14.

carrying a current $\mathcal{I}_1$ passes through the centre of the circle and is perpendicular to the plane of the frame. Find the couple N on the frame.

262. A thin conducting cylindrical shell of radius b contains a coaxial wire of radius a and magnetic permeability μ_0 . The space between the wire and the shell is filled with material of permeability μ . Find the self-inductance per unit length of the line.

263. A line consists of two thin coaxial cylindrical shells of radii a and b ($a < b$). The space between the two shells is filled with a material of permeability μ . Find the self-inductance per unit length of the line.

264. A long straight conductor and a ring of radius a lie in the same plane. The distance between the wire and the centre of the ring is b . Find the mutual inductance L_{12} and the force F between the two conductors if the current through the wire is $\mathcal{I}_1$ and the current through the ring is $\mathcal{I}_2$.

265*. Two thin rings of radii a and b are placed so that their planes are perpendicular to the line joining their centres. Assuming that the distance between the centres is l , find the mutual inductance L_{12} and express the result in terms of elliptical integrals. Consider the limiting case $l \gg a, b$.

266. Find the force F between the two circular currents in the preceding problem.

267. Find the self-inductance per unit length of an infinite cylindrical solenoid with a closely wound coil and arbitrary cross-section (not necessarily circular). Assume that the cross-sectional area is S and the number of turns per unit length is n .

268. Find the self-inductance L of a solenoid of finite length h and radius a ($h \gg a$) to within terms of the order of a/h . Replace the current in the coil by the equivalent surface current.

269. Find the self-inductance L of a toroidal solenoid. The radius of the toroid is b , the total number of turns is N , and the cross-section of the toroid is circular (radius a). Determine the self-inductance per unit length of the toroid in the limiting case $b \rightarrow \infty$ ($N/b = \text{const}$). Solve the problem for a toroidal solenoid of rectangular cross-section (sides a and h).

270. Determine the self-inductance per unit length of a "line" consisting of two straight wires of radii a and b which are at a distance h from each other. The currents in the two wires are equal and opposite.

271*. Show that the self-inductance of a thin closed conductor of circular cross-section is approximately given by†

$$L = \frac{\mu_0 l}{2} + L',$$

where μ_0 is the magnetic permeability of the conductor, l is its length, and L' is the mutual inductance of two linear circuits, one of which lies along the axis AMB of the quasi-linear conductor and the other along the line CND along which an arbitrary open surface S bounded by the axis cuts the surface of the conductor (Fig. 15).

272. Determine the self-inductance L of a ring of wire having a radius b . The radius of the wire is $a \ll b$.

Hint: Use the formula given in the preceding problem.

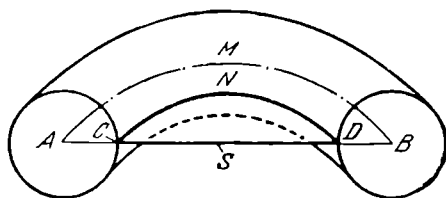


Fig. 15.

273. Determine the mutual inductance L_{12} of two thin parallel wires of length a placed at a distance l from each other and lying along the parallel sides of a rectangle. (The mutual inductance has no direct physical meaning since the currents in these wires are not closed. However, the inductance may be used to find the inductance of a closed circuit with parallel straight sections; cf. Problems 274 and 275.)

274. Determine the coefficient of mutual inductance L_{12} of two identical squares of side a placed at a distance l from each other and coinciding with the opposite faces of a rectangular parallelepiped. Find the force F between them assuming that $\mu = 1$ in all space.

275. Determine the self-inductance L of a square of side b which is made of a thin wire of radius $a \ll b$. The magnetic permeability in the surrounding space and of the material of the wire is μ and $\mu_0 = 1$ respectively.

Hint: Use the formulae obtained in the solution of Problems 271 and 273.

† The two terms in this formula may be called the internal and external self-inductance respectively, since they represent the magnetic energy stored inside and outside the conductor.

276. Determine the magnetic moment $\mathbf{m}$ of a charged sphere rotating about one of its diameters. Consider the cases of uniform volume and surface charge distributions.

277*. The current density associated with the spin magnetic moment of the electron in the hydrogen atom is given by $\mathbf{j} = c \operatorname{curl} [\rho(\mathbf{r})\mathbf{a}]$ where $\mathbf{a}$ is a constant vector, c is the ratio of the electromagnetic to the electrostatic units of charge, and ρ is the volume density of charge in the atom, which depends only on the radial distance r and is zero at infinity. Show that the magnetic field at the origin is $\frac{8\pi}{3}\rho(0)\mathbf{a}$.

Hint: Use the results of Problem 32.

278. Reduce the determination of the magnetic field due to given currents in a non-uniform, non-ferromagnetic medium to the corresponding problem in electrostatics. To do this, the magnetic field should be written in the form $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}'$ where $\mathbf{H}_0$ is the "primary" field which would be produced by the same current distribution in empty space and $\mathbf{H}'$ is the field due to the presence of the magnetic material. Introduce the scalar potential ψ for $\mathbf{H}'$ and obtain an equation for ψ and the corresponding boundary conditions.

279. A current-carrying circuit lies on the plane separation boundary between two media with magnetic permeabilities μ_1 and μ_2 . Determine the magnetic field strength $\mathbf{H}$ in all space assuming that the field produced by the circuit in the absence of the media is known.

280. An infinite straight wire carrying a current $\mathcal{I}$ is parallel to the plane separation boundary between two media with magnetic permeabilities μ_1 and μ_2 . The distance from the wire to the separation boundary is a . Determine the magnetic field.

Hint: Use the method of images as in electrostatic problems (Chapter III).

281. A sphere of radius a and magnetic permeability μ is placed in a uniform magnetic field $\mathbf{H}_0$. Determine the resulting magnetic field $\mathbf{H}$, the induced magnetic moment $\mathbf{m}$, and the current density $\mathbf{j}_{\text{mol}}$ which is equivalent to the magnetisation of the sphere.

282*. An anisotropic non-ferromagnetic sphere is placed in a uniform magnetic field. Find the resulting field $\mathbf{H}$ and the moment $\mathbf{N}$ of forces acting on the sphere.

283. An infinitely long hollow cylindrical shell with internal and external diameters a and b respectively, is placed in an external uniform magnetic field $\mathbf{H}_0$ which is perpendicular to its axis. The magnetic permeability of the cylinder and of the surrounding medium is μ_1 and μ_2 respectively. Find the field strength H in the cavity. Consider in particular the case $\mu_1 \gg \mu_2$.

284. A hollow sphere of internal and external radii a and b respectively, is placed in an external uniform magnetic field $\mathbf{H}_0$. The magnetic permeability of the sphere and of the surrounding medium is μ_1 and μ_2 respectively. Find the field H in the cavity. Consider in particular the case $\mu_1 \gg \mu_2$.

285. An infinitely long straight wire of radius a and magnetic permeability μ_1 is placed in an external uniform magnetic field $\mathbf{H}_0$ in a medium of magnetic permeability μ_2 . The field is transverse to the wire and the wire carries a constant current $\mathcal{I}$. Find the resulting magnetic field inside and outside the wire.

286. Determine the scalar and vector potentials ψ and $\mathbf{A}$ in a bounded region in which the magnetisation is known to be $\mathbf{M}(\mathbf{r})$. Show by direct calculation that the vectors $\mathbf{B} = \text{curl } \mathbf{A}$ and $\mathbf{H} = -\text{grad } \psi$ are related by $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$.

287. A body of arbitrary form is uniformly magnetised. Show that the scalar potential due to this body may be written in the form $\psi = -\mathbf{M} \cdot \text{grad } \varphi$ where $\mathbf{M}$ is the magnetisation and φ is the electrostatic potential due to a uniformly charged body (with density $\rho = 1$) of the same form and dimensions.

288. A straight wire carrying a current $\mathcal{I}$ is parallel to the axis of an infinitely long circular cylinder and at a distance b from it. The radius of the cylinder is a where $a < b$ and the magnetic permeability of the cylinder is μ . Find the force f per unit length of the wire.

Hint: Use the method of images.

289. A straight wire carrying a current $\mathcal{I}$ lies inside an infinitely long cylindrical cavity cut in a uniform magnetic medium. The wire is parallel to the axis of the cylinder and lies at a distance b from it. The radius of the cylinder is a and the magnetic permeability of the medium is μ . Find the force f per unit length of the wire.

290. A current $\mathcal{I}$ flows through a straight wire lying along the z -axis. Three half-planes at angles α_1 , α_2 , and α_3 to each other ($\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$) intersect along the z -axis and the spaces between them are filled with magnetic media having permeabilities

μ_1 , μ_2 , μ_3 respectively. Determine the magnetic field H_i ($i = 1, 2, 3$) in the three regions.

291*. Find the magnetic field due to a uniformly magnetised permanent magnet of spherical form. The magnetic permeability of the sphere and of the external medium is μ_1 and μ_2 respectively.

292*. Find the field due to an infinitely long uniformly magnetised cylinder of radius a . The magnetisation vector $\mathbf{M}_0$ is perpendicular to the axis of the cylinder and the magnetic permeability of the cylinder and of the surrounding medium is μ_1 and μ_2 respectively.

293. A uniformly magnetised sphere (idealised ferromagnetic) is placed in an external uniform magnetic field $\mathbf{H}_0$. Find the resulting magnetic field and the couple $\mathbf{N}$ acting on the sphere. The magnetic permeability of the sphere is μ_1 and of the surrounding medium is $\mu_2 = 1$.

294. A small permanent magnet having a magnetic moment $\mathbf{m}$ is placed in a vacuum near the plane boundary of a medium of magnetic permeability μ . Find the force F and the couple N acting on the permanent magnet.

Hint: Use the method of images.

295. An ellipsoid with permeability μ is placed in a uniform magnetic field $\mathbf{H}_0$. Find the internal field and the magnetic moment of the ellipsoid.

296*. An ellipsoid made from an anisotropic material of permeability μ_{ik} is placed in a uniform external magnetic field $\mathbf{H}_0$. Find the internal magnetic field $\mathbf{H}_1$ in the ellipsoid.

Chapter

6

ELECTRICAL AND MAGNETIC PROPERTIES OF MATTER

1. Polarisation of matter in a constant field

In general, the polarisation vector $\mathbf{P}$ (electric dipole moment per unit volume) is a non-linear function of the electric field $\mathbf{E}$. However, in isotropic dielectrics and for sufficiently low fields, the polarisation is proportional to the field strength:

$$\mathbf{P} = \alpha \mathbf{E}. \quad (\text{VI.1})$$

The dielectric susceptibility α is determined by the properties of the dielectric and is in general a function of temperature. The relation between α and the permittivity ϵ is

$$\epsilon = 1 + 4\pi\alpha. \quad (\text{VI.2})$$

It is found that in a constant field $\alpha > 0$, $\epsilon > 1$. In anisotropic dielectrics ϵ and α are tensors of rank 2.

In a sufficiently rarified medium (gas) the polarisability α is proportional to the number of particles N per unit volume:

$$\alpha = N\beta, \quad \epsilon = 1 + 4\pi N\beta, \quad (\text{VI.3})$$

where β is the average polarisability per molecule. This result is obtained if it is assumed that the field $\mathcal{E}$ acting on a molecule is equal to the average field $\mathbf{E}$. In the case of a dense material the difference between the two fields must be taken into account. Thus, it is found that for non-polar substances, *i. e.* substances whose molecules do not have a permanent dipole moment in the absence of the field,

$$\mathcal{E} = \mathbf{E} + \frac{4\pi}{3} \mathbf{P}. \quad (\text{VI.4})$$

When the latter equation holds, Eqns. (VI.3) are replaced by

$$\alpha = \frac{N\beta}{1 - \frac{4\pi}{3} N\beta}, \quad \frac{\epsilon - 1}{\epsilon + 2} = \frac{4\pi}{3} N\beta. \quad (\text{VI.4}')$$

Eqns. (VI.4) and (VI.4') are known as the Lorenz-Lorentz relations.

The magnetisation vector $\mathbf{M}$ (magnetic dipole moment per unit volume) is frequently proportional to the magnetic field strength:

$$\mathbf{M} = \chi \mathbf{H} = \frac{\mu - 1}{4\pi} \mathbf{H}. \quad (\text{VI.5})$$

The magnetic permeability μ and the magnetic susceptibility χ are determined by the properties of the medium and are functions of temperature. In distinction to the dielectric susceptibility, the magnetic susceptibility may be either positive or negative. Materials with $\chi > 0$ are known as paramagnetics while those with $\chi < 0$ are called diamagnetics. The relation between $\mathbf{M}$ and $\mathbf{H}$ for ferromagnetics is non-linear and not single-valued.

In solving the problems given in this section, it will be necessary to use Boltzmann's formula in addition to the equations of mechanics and electrodynamics. This formula gives the distribution of non-interacting particles in an external field:

$$dN(q_i) = C e^{-\frac{U(q_i)}{kT}} dV, \quad (\text{VI.6})$$

where $U(q_i)$ is the potential energy of the i -th particle in the external field, q_i are the generalised coordinates characterising the position and the orientation of the particles, dV is a volume element in the generalised space, $dN(q_i)$ is the number of particles in dV , $k = 1.38 \times 10^{-16}$ erg deg $^{-1}$ is Boltzmann's constant, T is the absolute temperature, and C is a normalising constant determined from the normalisation condition

$$\int dN(q_i) = N_0, \quad (\text{VI.7})$$

where N_0 is the total number of particles in the system and the integral is evaluated over the volume occupied by the system.

When the polar angles ϑ , α defining the orientation of a molecule are chosen as the generalised coordinates, the volume element may be written $dV = \sin \vartheta d\vartheta d\alpha$.

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297. The charge density in the electron cloud of a hydrogen atom is described by the function $\rho(r) = -(e_0/\pi a_0^3) \exp(-2r/a_0)$ where e_0 is the electronic charge and a_0 is a constant. Calculate the polarisability β of the hydrogen atom in a weak external field,

assuming that the distortion of the electron cloud may be neglected. Repeat your calculation on the assumption that the electron cloud is distributed uniformly in a sphere of radius a_0 .

298. An atom with a spherically symmetric charge distribution is placed in a uniform magnetic field $\mathbf{H}$. Show that the additional field around the nucleus which is due to the diamagnetic current is given by

$$\Delta\mathbf{H} = -\frac{e\mathbf{H}}{3mc^2} \nabla^2 \varphi(0),$$

where $\varphi(0)$ is the electrostatic potential due to the atomic electrons and e/m is the charge to mass ratio for an electron.

299*. A molecule consists of two atoms at a distance a from each other. The atoms are spherically symmetric and their polarisabilities are β' and β'' . Find the polarisability tensor for the molecule assuming that the atomic radii are small compared with a . Discuss in particular the case $\beta' = \beta''$.

300. Starting from the law of conservation of energy, show that the polarisability tensor for a molecule in a constant field is symmetric.

301. A dielectric consists of identical molecules having zero dipole moments in the absence of an external field. The polarisability tensor β_{ik} for each molecule is known. Find the polarisability α of the dielectric. Consider the two cases where (a) all the molecules are oriented in the same direction, and (b) the orientation of the molecules is random. Use the Lorenz-Lorentz formula to allow for the difference between the field acting on the molecule and the average field. [Note that case (a) may occur for crystalline and amorphous solids and case (b) for gases, liquids, and solids. However, in distinction to a gas, a solid is a system of strongly interacting particles and therefore the concept of isolated molecules may have no physical significance for solids.]

302*. If the polarisability of a molecule is a function of its orientation, then the energy of interaction between the molecule and an external field will also depend on the orientation. Hence, in addition to the distorting polarisation mechanism, there is also a directional mechanism which will come into play even when the molecule is non-polar. This will give rise to a temperature dependence of the permittivity of a substance consisting of randomly oriented non-polar molecules. Investigate this effect

in the case of a diatomic gas in a weak electric field, and calculate the polarisability of the dielectric assuming that the longitudinal and transverse polarisabilities are β_1 and β_2 respectively.

303. Two gas molecules have dipole moments p_1 and p_2 and lie at a distance R from each other. Owing to collisions with other molecules, their orientation will change and the probability that they will assume a given mutual orientation will be given by Boltzmann's formula (VI.6) with U equal to the energy of interaction between the two dipoles. Assuming that $U \ll kT$, show that the magnitude of U averaged over the Boltzmann distribution is $U(R) = -2p_1^2 p_2^2 / 3kTR^6$.

Hint: In averaging over the directions of the dipoles, use the formulae obtained in Problem 32. These formulae should also be used in the next problem.

304. A molecule having an electric dipole moment p interacts with a non-polar molecule having a polarisability β . Show that the energy of interaction averaged over all possible orientations is $U(R) = -\beta p^2 / R^6$ where R is the distance between the molecules (*cf.* hint to the preceding problem).

305*. In general, a dielectric placed in a constant electric field will possess higher order moments in addition to the dipole moment represented by the polarisation vector $\mathbf{P}$. Find the volume and surface charge densities equivalent to a quadrupole polarisation Q_{ik} (Q_{ik} are the components of the quadrupole moment per unit volume of the dielectric).

306. The permittivity of polar substances, for which the Lorenz-Lorentz formula does not hold, may be calculated by the following approximate method due to Onsager. Consider a small spherical cavity containing a single molecule and suppose that the cavity is surrounded by a homogeneous dielectric of permittivity ϵ . The cavity is evacuated and the field within it is equal to the effective field acting on the molecule. This field is determined by solving the macroscopic equations of electrostatics. Use this method to find the relation between the permittivity ϵ of the medium and the polarisability β of its molecules.

307*. Consider a system consisting of particles with charge e and mass m , each of which moves at a certain fixed distance a from a given (its own) centre. The system is in a magnetic field in a state of statistical equilibrium. Show that the total magnetic susceptibility of the system is zero.

308*. An ionised gas consists of ions (charge Ze , mean concentration N_0) and electrons (charge e , mean concentration n_0). The gas as a whole is electrically neutral ($ZN_0 = n_0$) and is in a state of statistical equilibrium at a temperature T . Assuming that the gas may be described in terms of classical statistics and that the energy of interaction between the particles is small compared with the thermal energy kT , find the charge density in the neighbourhood of an ion.

309. An infinite conducting plate bounded by the planes $x = h$, $x = -h$ is placed in a constant uniform electric field E_0 which is at right angles to the plate. The plate as a whole is electrically neutral, the mean concentration of free charges is N_0 and the permittivity is ϵ . Assuming that the change in the concentration under the action of the applied field is small ($|N - N_0| \ll N_0$), find the field distribution inside the plate and determine the thickness of the layer in which the "surface" charge is concentrated.

310. A layer of an electrolyte lies between two infinite plane electrodes $x = h$, $x = -h$ between which a potential difference $2\phi_0$ is maintained. The electrolyte consists of two types of ions having charges $+e$ and $-e$, and the mean concentration in the absence of the external field is N_0 . Assuming that the permittivity of the electrolyte is ϵ , determine the potential at any point between the electrodes.

Hint: Use the method employed in the solution of Problem 308.

2. Polarisation of matter in a variable field

In the case of a variable electromagnetic field, the electric induction at a time t depends on the values of the field at all the preceding instants of time:

$$\mathbf{D}(t) = \mathbf{E}(t) + \int_{-\infty}^t f(t-u) \mathbf{E}(u) du, \quad (\text{VI.8})$$

where $f(t-u)$ is a function determined by the properties of the medium. A similar formula will of course hold for the magnetic vectors. The direct proportionality between the induction and the field strength will only hold for the Fourier components of

these vectors, *i.e.* for fields which are sinusoidal functions of time:

$$\mathbf{D}_\omega = \epsilon(\omega) \mathbf{E}_\omega, \quad \mathbf{B}_\omega = \mu(\omega) \mathbf{H}_\omega. \quad (\text{VI.9})$$

The permittivity ϵ and the magnetic permeability μ are then functions of the frequency of the field. In order to calculate the frequency dependence, some assumptions must be made about the microscopic structure of matter. A rigorous theory of the polarisation of matter can only be developed on the basis of quantum mechanics, since classical mechanics and electrodynamics are unable to explain the structure of matter. However, the classical oscillator model of an atom does yield a number of important qualitative predictions about the behaviour of matter in a variable field. According to this model the force on an atomic electron is

$$\mathbf{F} = -k\mathbf{r}, \quad (\text{VI.10})$$

where $\mathbf{r}$ is the distance of the electron from the nucleus and k is a constant. In order to account for the dissipation of electromagnetic energy, it is also necessary to introduce a frictional force which acts on the electron and is proportional to its velocity:

$$\mathbf{F}_{\text{fr}} = -\eta \dot{\mathbf{r}}. \quad (\text{VI.11})$$

The permittivity deduced for this oscillator model is given by

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}, \quad (\text{VI.12})$$

where $\omega_p^2 = 4\pi e^2 N/m$, $\omega_0 = k/m$, $\gamma = \eta/m$, N is the number of atoms per unit volume and e and m is the charge and mass of the electron respectively.

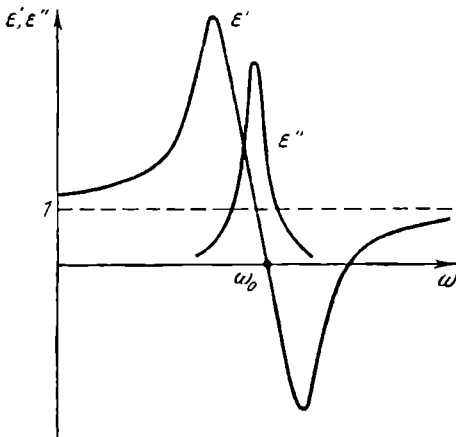


Fig. 16

Fig. 16 shows the dependence of the real (ϵ') and imaginary (ϵ'') parts of ϵ as functions of the frequency ω . The imaginary part is responsible for the absorption of electromagnetic energy and becomes appreciable only near the proper frequency ω_0 of the oscillators. In this region ϵ' decreases with increasing ω (anomalous dispersion). In the remaining regions ϵ' increases with increasing ω (normal dispersion).

According to quantum theory, (VI.12) must be replaced by the somewhat similar formula

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega}, \quad (\text{VI.13})$$

where ω_p , f_i , ω_i , γ_i are constants.

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311. An artificial dielectric consists of identical perfectly conducting metal spheres of radius a which are randomly distributed in a vacuum. The average number of spheres per unit volume is N . An electromagnetic wave is propagated through the dielectric. Neglecting the difference between the field acting on each sphere and the average field, determine the permittivity ϵ and the magnetic permeability of the dielectric. Under what conditions can the dielectric be looked upon as a continuous medium?

Hint: The electric and magnetic polarisabilities of perfectly conducting spheres were calculated in Problems 151 and 377.

312*. Determine the permittivity of a conducting sphere assuming that the ions are fixed and the effect of bound electrons may be neglected. Energy dissipation should be allowed for by introducing the "frictional" force $-\eta\dot{\mathbf{r}}$ acting on the electrons whose concentration is N . Find the relation between the coefficient η and the conductivity.

313*. A gaseous dielectric in a state of statistical equilibrium at a temperature T consists of molecules for which the principal values of the polarisability tensor are $\beta^{(1)} = \beta$, $\beta^{(2)} = \beta^{(3)} = \beta'$ where β and β' are functions of frequency ω . The dielectric is acted upon by a constant uniform electric field $\mathbf{E}_0$ and the concentration of the molecules is N . Find the permittivity tensor for the dielectric assuming that $\mathbf{E}(t) = \mathcal{E} \exp(-i\omega t)$ where $\mathcal{E} \ll E_0$.

314. A gaseous dielectric consists of polar molecules whose electric dipole moment in the absence of an external field is p_0 . The principal values of the polarisability tensor of a molecule in a variable field are $\beta^{(1)} = \beta$, $\beta^{(2)} = \beta^{(3)} = \beta'$, where the x_1 -axis is parallel to p_0 . The dielectric is placed in an electric field having a constant component $\mathbf{E}_0$ and an alternating component

$\mathbf{E}(t) = \mathbf{E} \exp(-i\omega t)$. Neglecting the orienting effect of the alternating field, and also the orienting effect associated with the anisotropy of the molecular polarisation in the constant field, find the permittivity tensor for the alternating field.

315*. A system of charges (molecule) is placed in a sinusoidal electromagnetic field. Show that if there is no dissipation of electromagnetic energy in the system, then the polarisability tensor is Hermitian, *i.e.* $\beta_{ik} = \beta_{ki}^*$.

316. Show that if a tensor β_{ik} is Hermitian, then in a suitably chosen coordinate system it may be written in the form $\beta_{ik} = \beta^{(i)}\delta_{ik} + ie_{ikl}g_l$, where e_{ikl} is the skew-symmetric unit tensor of rank 3 (*cf.* Problem 24), and $\mathbf{g}$ is a real vector (gyration vector). Note that media in which the gyration vector is finite are called gyrotropic. The propagation of electromagnetic waves in gyrotropic media is considered in Section 3 of Chapter VIII.

317. Find the polarisability β_{ik} of an atom in the field of a plane monochromatic wave and a weak constant external magnetic field $\mathbf{H}_0$. Use the oscillator model and the method of successive approximations. Neglect the effect of the magnetic component of the wave and losses of electromagnetic energy. Determine also the gyration vector $\mathbf{g}$.

318. Use the oscillator model of an atom to find the permittivity tensor $\varepsilon_{ik}(\omega)$ for a dielectric containing N atoms per unit volume, when the dielectric is placed in an arbitrary constant magnetic field $\mathbf{H}_0$. The dissipation of electromagnetic energy and the effects of the magnetic component of the plane wave may be neglected. When will the exact solution become identical with the approximate solution of the preceding problem?

Hint: In integrating the equations of motion of the electron use the cyclic coordinates:

$$x_{\pm 1} = \mp \frac{1}{\sqrt{2}}(x \pm iy), \quad x_0 = z.$$

319*. Find the permittivity tensor for a plasma in an external constant magnetic field H , given that the average electron density is N . Positive ions may be regarded as fixed and energy losses should be taken into account by introducing a "frictional force" $-\eta\dot{\mathbf{r}}$.

320. Suppose that in the plasma considered in the preceding problem there is also a constant electric field $\mathbf{E}$. Use an approximation which is linear in H_0 to find the relation between

the current density $\mathbf{j}$ and the electric field $\mathbf{E}$. Find also the electrical conductivity tensor.

Hint: Solve the equations of motion by the method of successive approximations.

321*. Find the permittivity of an ionised gas placed in a constant magnetic field, taking into account the motion of positive ions but assuming that the ion mass is much greater than the electron mass. Discuss the dependence of the permittivity on the frequency ω and compare the result with the case where the ions may be considered as fixed. Assume that the concentration of ions and electrons is N .

Hint: The frictional force on an electron is $-\eta(\dot{\mathbf{r}} - \dot{\mathbf{R}})$, the frictional force on an ion is $-\eta(\dot{\mathbf{R}} - \dot{\mathbf{r}})$, and $\mathbf{r}$ and $\mathbf{R}$ are the position vectors of the electron and ion respectively.

322. Suppose that in an infinite homogeneous medium there is only one special direction (e.g. the direction of an external field). Suppose further that T_{ik} is a tensor parameter of this medium. e.g. the permittivity or the magnetic permeability. It is clear that the components of the tensor T_{ik} should be invariant with respect to any rotation of the coordinate system about the special direction. Derive the limitations which are imposed by this invariance condition on the form of the tensor T_{ik} .

323. In some cases the function $f(t)$ which determines the relation between $\mathbf{D}$ and $\mathbf{E}$ [cf. Eqn. (VI.8)] is of the form $f(t) = f_0 \exp(-t/\tau)$ where f_0 and τ are constants. Show that

$$\varepsilon(\omega) = 1 + \frac{\varepsilon_0 - 1}{1 - i\omega\tau},$$

where ε_0 is the static value of the permittivity.

324*. Assuming that the polarisation of a particular medium can only arise after an electric field has been applied to it, show that the real and imaginary parts of the permittivity $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$ are given by

$$\varepsilon'(\omega) = 1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon''(\omega')}{\omega' - \omega} d\omega', \quad \varepsilon''(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon'(\omega') - 1}{\omega' - \omega} d\omega'.$$

These are the Kramers-Krönig dispersion relations. The symbol $\int$ denotes the principal value of the integral. Note that in the case of metals for which $\varepsilon(\omega)$ may have a singularity at $\omega = 0$,

the second of these formulae is of the form

$$\epsilon''(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon'(\omega') - 1}{\omega' - \omega} d\omega' + \frac{4\pi\sigma}{\omega},$$

where σ is the static conductivity of the metal.

Hint: Consider the polarisation $\mathbf{P}(t)$ due to a field $\mathbf{E}(t) = \mathbf{E}_0\delta(t)$. Use Eqn. (17) of Appendix I.

325. Use the Kramers-Krönig dispersion relations (*cf.* Problem 324) to determine the real part $\epsilon'(\omega)$ of the permittivity, assuming that the imaginary part is given by

$$\epsilon''(\omega) = \frac{(\epsilon_0 - 1)\omega\tau}{1 + \omega^2\tau^2},$$

where ϵ_0 and τ are constants.

3. Ferromagnetic resonance

Classical theories are unable to supply a self-consistent explanation of ferromagnetism. Ferromagnetism is largely due to intrinsic (spin) magnetic moments of electrons and the specific interaction forces between them, which are all of quantum-mechanical origin. However, some ferromagnetic phenomena may be discussed on the basis of a semi-classical theory. Among these phenomena is ferromagnetic resonance.

Ferromagnetic resonance is established under the following conditions: a constant magnetic field acting on the magnetic moment of an atom or a single electron gives rise to the Larmor precession of the moment about the direction of the field. This motion is eventually damped out due to the conversion of the Larmor precession energy into thermal energy. If the external field is sufficiently large, then all the elementary magnetic moments become parallel to the external field. The ferromagnetic is then referred to as saturated and, correspondingly, the magnetic moment per unit volume is called the saturation magnetisation. In this section we shall always assume that the ferromagnetic is in fact magnetised to saturation. If in addition to the constant field there is also an alternating magnetic field, which is perpendicular to the constant field, then the alternating field will tend to maintain the precessional motion and when its frequency becomes equal to the precession frequency, ferromagnetic resonance will set in.

The motion of the magnetisation vector in a ferromagnetic is described by the Landau–Lifshits equation which may be obtained as follows. The couple acting on a magnetic moment $\mathbf{m}$ of a particle (atom or single electron) placed in a magnetic field $\mathbf{H}_{\text{eff}}$ is $\mathbf{N} = \mathbf{m} \times \mathbf{H}_{\text{eff}}$. Since the rate of change of the angular momentum is equal to the sum of the moments of the external forces we have

$$\frac{d\mathbf{K}}{dt} = \mathbf{N} = \mathbf{m} \times \mathbf{H}_{\text{eff}}. \quad (\text{VI.14})$$

According to quantum mechanics, the relation between the magnetic moment and the angular momentum of an electron is

$$\mathbf{m} = -\gamma \mathbf{K}, \quad \gamma = \frac{e_0}{mc},$$

where e_0 and m is the charge and mass of the electron and c is the velocity of light *in vacuo*. Using this relation and taking the average of both sides of Eqn. (VI.14) over an infinitely small volume, we obtain the Landau–Lifshits equation

$$\frac{d\mathbf{M}}{dt} = -\gamma (\mathbf{M} \times \mathbf{H}_{\text{eff}}), \quad (\text{VI.15})$$

where $\mathbf{M}$ is the magnetisation vector and $\mathbf{H}_{\text{eff}}$ is the average magnetic field acting on each elementary magnetic moment. In an infinite isotropic medium magnetised to saturation

$$\mathbf{H}_{\text{eff}} = \mathbf{H} - \lambda \mathbf{M} + q \nabla^2 \mathbf{M}, \quad (\text{VI.16})$$

where $\mathbf{H}$ is the average magnetic field in the medium and λ , q are constants. The second term in this expression is the so-called Weiss molecular field which does not contribute to Eqn. (VI.15) because $\mathbf{M} \times \mathbf{M} = 0$. The third term is only important when $\mathbf{M}$ is a very rapid function of the coordinates. We shall not consider such changes in $\mathbf{M}$ in this section and will therefore assume that $\mathbf{H}_{\text{eff}} = \mathbf{H}$.

In order that Eqn. (VI.15) should describe electromagnetic energy losses in the medium as well as the other effects described above, an additional dissipative term must be added to it. It is usual to assume that $\mathbf{H}_{\text{eff}}$ includes a “frictional field” $-p d\mathbf{M}/dt$. Eqn. (VI.15) then becomes

$$\frac{d\mathbf{M}}{dt} = -\gamma \mathbf{M} \times \left(\mathbf{H} - p \frac{d\mathbf{M}}{dt} \right), \quad (\text{VI.17})$$

where p is the loss parameter. If the losses are small and the total magnetic field is the sum of a constant field $\mathbf{H}_0$ and a variable field $\mathbf{h}(t)$, where $|\mathbf{h}| \ll H_0$, then Eqn. (VI.17) may be simplified to read

$$\frac{d\mathbf{M}}{dt} = -\gamma(\mathbf{M} \times \mathbf{H}) + \omega_r(\chi_0\mathbf{H} - \mathbf{M}), \quad (\text{VI.18})$$

where $\chi_0 = M_0/H_0$, $\omega_r = p\gamma^2 M_0^2/\chi_0$, and $M_0 = |\mathbf{M}|$ is the saturation magnetisation. The Landau-Lifshits equation is the starting point in the solution of problems on ferromagnetic resonance. In recent years ferromagnetics with very low conductivity (ferro-dielectrics, ferrites) have found extensive application in ultra-high frequency electronics. The propagation of electromagnetic waves in ferrites is discussed in Chapters VIII and IX.

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326. Find the law of motion of the magnetisation vector $\mathbf{M}$ in an infinite ferrite medium magnetised to saturation. Assume that there are no losses and that the magnetic field $\mathbf{H}$ in the medium is constant and uniform.

327. Solve the preceding problem in the case of finite losses using the Landau-Lifshits equation in the form given by Eqn. (VI.18). Assume that the angle between $\mathbf{M}$ and $\mathbf{H}$ is small and that $\omega_r \ll \omega_0 = \gamma H_0$.

328*. Suppose that in an infinite ferromagnetic medium there is a constant uniform magnetic field $\mathbf{H}_0$ and a high frequency field $\mathbf{h} \exp(-i\omega t)$ where $\mathbf{h} = \text{const.}$ Assuming that $h \ll H_0$ and neglecting energy losses, find the forced oscillations of the magnetisation vector $\mathbf{M}$ using an approximation which is linear in h . (The natural oscillations, *i.e.* the Larmor precession under the action of the constant field $\mathbf{H}_0$, will be damped out as the result of losses which are present in all real systems.)

329. Use the result of the preceding problem to find the magnetic susceptibility and permeability tensors χ_{ik} and μ_{ik} for a high-frequency field. Find the components of the tensor μ_{ik} as functions of the constant magnetic field H_0 for $M_0 = 160$ gauss and $\nu = \omega/2\pi = 9375 \text{ Mc sec}^{-1}$ ($\lambda = 3.2 \text{ cm}$). Investigate the resonance character of the variation in these quantities and determine H_0 at resonance.

330*. The magnetic field in an infinite ferrite medium magnetised to saturation consists of a constant component $H_0 = H_z$ and a circularly polarised alternating component $H_x = h \cos \omega t$, $H_y = h \sin \omega t$, $h = \text{const.}$ Find the exact solution of the Landau-Lifshits equation corresponding to the forced precession of the vector $\mathbf{M}$ at the frequency ω of the external field. The energy dissipation may be ignored.

331. Solve Problem 328 with allowance for energy losses. Use the Landau-Lifshits equation in the form given by Eqn. (VI.18).

332. Use the result of the preceding problem to find the magnetic permeability tensor μ_{ik} for a high-frequency field. Obtain expressions for the real and imaginary parts of the components of this tensor. Find the dependence of both parts of the components of the permeability tensor on a constant magnetic field for $M_0 = 160$ gauss, $\nu = \omega/2\pi = 9375$ Mc sec $^{-1}$, and $\omega_r = 3 \times 10^9$ sec $^{-1}$. Determine the resonance value of the constant field, *i.e.* the value at which the imaginary parts of the components of the permeability tensor are a maximum.

333. Determine the half-width ΔH_0 of the resonance curve for the imaginary parts of the components of the magnetic permeability tensor, assuming that $\omega_r \ll \omega$. The half-width is defined as the distance between the ordinates $\mu'' = \mu_{\text{res}}$ and $\mu'' = \frac{1}{2}\mu_{\text{res}}$.

334*. Find the Larmor precession frequency ω_k in an ellipsoidal ferromagnetic specimen in which energy losses may be neglected. The specimen is in an external uniform magnetic field $\mathbf{H}_0$ which is parallel to one of the axes of the ellipsoid. Assume that the departure of the magnetisation vector $\mathbf{M}$ from the equilibrium position is small.

Hint: The Landau-Lifshits equation will now include the internal field $\mathbf{H}_1$. This field will differ from the external field $\mathbf{H}_0$ owing to the demagnetising effect which is a consequence of the form of the body:

$$\mathbf{H}_1 = \mathbf{H}_0 - \mathbf{H}_{\text{dm}}, \quad H_{k \text{ dm}} = 4\pi N_{kl} M_l,$$

where N_{kl} is the demagnetisation tensor (*cf.* Problem 295).

335. Solve the preceding problem with allowance for energy losses. Retain only those terms which are linear in ω_r .

336*. Discuss forced oscillations in a small ellipsoidal specimen in the presence of energy losses. Determine the components of the magnetic susceptibility tensor for a high-frequency

field, assuming that the amplitude h of the field is small compared with the constant field H_0 .

337. In some ferromagnetic media (antiferromagnetics), the resultant magnetisation $\mathbf{M}$ consists of two parts, so that $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ where $\mathbf{M}_1$ and $\mathbf{M}_2$ are due to ions lying at different sites of the crystal lattice and forming two magnetic sub-lattices. In the equilibrium state $\mathbf{M}_1$ and $\mathbf{M}_2$ are antiparallel so that $M = |M_1 - M_2|$. When an external magnetic field is introduced, each of the vectors experiences the molecular Weiss field [cf. Eqn. (VI.16)] and the two vectors cease to be antiparallel. Find the natural precession frequencies assuming that $\lambda |M_1 - M_2| \gg H_0$ where H_0 is the external field and λ is the molecular Weiss constant. Assume that the departures of the vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ from their equilibrium positions are small.

Chapter 7

QUASI-STATIONARY ELECTROMAGNETIC FIELDS

1. Quasi-stationary phenomena in linear conductors

If the period of an alternating electromagnetic field is much greater than the time taken by the electromagnetic wave to travel through a given system, *i.e.*

$$T \gg \frac{l}{c}, \quad \omega \ll \frac{c}{l}, \quad (\text{VII.1})$$

where c is the velocity of light and l is a characteristic linear dimension of the system, then the velocity of propagation of electromagnetic disturbances through the system may be assumed to be infinite. The corresponding approximation is known as the quasi-stationary approximation. The quasi-stationary approximation occasionally gives good results even when this condition is not satisfied, *e.g.* in the theory of long transmission lines.

On the quasi-stationary approximation, the current in a closed circuit consisting of a source of e.m.f. $\mathcal{E}(t)$, a capacitance C , an inductance L , and a resistance R , satisfies the following differential equations

$$\mathcal{I} = \frac{dq}{dt}, \quad \frac{1}{c^2} L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = \mathcal{E}(t), \quad (\text{VII.2})$$

where q is the charge on one of the capacitor plates. When the e.m.f. is of the form $\mathcal{E}(t) = \mathcal{E}_0 \exp(-i\omega t)$, then in steady state the current is proportional to the e.m.f. so that

$$\mathcal{I} = \frac{\mathcal{E}}{Z}, \quad Z = R + i \left(\frac{1}{\omega C} - \frac{\omega L}{c^2} \right). \quad (\text{VII.3})$$

The quantity Z is called the complex impedance of the circuit.

In the case of a circuit consisting of a number of branches, the differential equations giving the currents in the various branches may be set up with the aid of Kirchhoff's laws.

The e.m.f. induced in a linear conductor due to a change in the magnetic flux intercepted by it is given by Faraday's law

$$\mathcal{E}_{\text{ind}} = -\frac{1}{c} \cdot \frac{d\Phi}{dt}, \quad (\text{VII.4})$$

where Φ is the flux of magnetic induction through the circuit. The change in flux may be due to a change in the magnetic field or to a deformation of the circuit. If there are a number of inductively coupled circuits then the total flux of magnetic induction through the i -th circuit Φ_i is given by

$$\Phi_i = \frac{1}{c} \sum_k L_{ik} \mathcal{J}_k, \quad (\text{VII.5})$$

where $\mathcal{J}_k$ is the current in the k -th circuit, L_{ik} ($i \neq k$) is the mutual inductance between the i -th and the k -th circuits, and $L_{ii} \equiv L_i$ is the self-inductance of the i -th circuit (*cf.* the introduction to Chapter V).

The generalised force acting on a conductor carrying a current in a quasi-stationary field is given by

$$F_i = \left(\frac{\partial W}{\partial q_i} \right)_{\mathcal{J}}, \quad (\text{VII.6})$$

where W is the magnetic energy of the system, q_i is the generalised coordinate, and the differentiation is carried out at constant current in the conductors. The magnetic energy can be expressed in terms of the currents and the coefficients of inductance by means of the same formulae as in the static case [*cf.* Eqns. (V.17) and (V.20)].

The time average of the product of two harmonic functions of time may be calculated with the aid of the formulae

$$\overline{a^2(t)} = \frac{1}{2} |a|^2, \quad \overline{a(t)b(t)} = \frac{1}{2} \text{Re}(ab^*). \quad (\text{VII.7})$$

For example, the average dissipation of heat in a circuit may be calculated from the following formulae

$$Q = \frac{1}{2} \text{Re}(\mathcal{E} \mathcal{J}^*) = \frac{1}{2} |\mathcal{J}|^2 \text{Re} Z. \quad (\text{VII.8})$$

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338. A circular loop of wire of radius a is placed in a magnetic field H_0 and rotates with an angular velocity ω about a diameter which is perpendicular to H_0 . Find the current in the loop $\mathcal{J}(t)$, the retarding couple $N(t)$, and the average power $\bar{P}$ which is required to maintain the rotation.

339. A plane LCR circuit of area S rotates with an angular velocity ω in a constant magnetic field H_0 about an axis lying in the plane of the circuit and perpendicular to H_0 . Determine the average retarding couple $\bar{N}$ on the circuit.

340. The current through one of two inductively coupled circuits is $\mathcal{J}(t) = \mathcal{J}_0 \exp(-i\omega t)$. The inductance and resistance of the two circuits are given. Express the average generalised force on the circuits in terms of the derivative of the mutual inductance with respect to the generalised coordinate q_i .

341. Two identical circuits each have a resistance R and inductance L , and one of them includes a source of e.m.f. $\mathcal{E}(t) = \mathcal{E}_0 \exp(-i\omega t)$. The mutual inductance between the circuits is L_{12} . Determine the average force $\bar{F}$ on each of the circuits. Express the result in terms of the derivative of the mutual inductance with respect to the corresponding coordinate.

342. Determine the natural frequencies of electrical oscillations in the two circuits shown in Fig. 17 when the coupling between the two circuits is achieved through

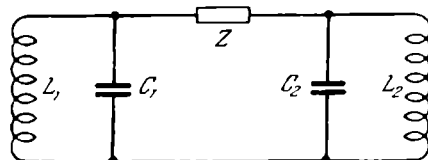


Fig. 17.

a capacitance C ($Z = \frac{-i}{\omega C}$).

Hint: Set up a system of algebraic equations for the currents and put the determinant of the system equal to zero.

343. Solve the preceding problem in the case when the coupling between the two circuits is through an inductance ($Z = i\omega L/c^2$).

344. Find the natural frequencies of oscillation $\omega_{1,2}$ in two inductively coupled circuits with capacitances C_1, C_2 , inductances L_1, L_2 , and mutual inductance L_{12} .

345. Two circuits are coupled to each other through a resistance R (cf. Fig. 17 with $Z = R$). Find the natural frequencies of oscillation of the two circuits when the coupling is small, *i.e.* R is large.

346. A circuit consisting of an inductance L_1 , a capacitance C_1 , and a resistance R_1 , includes an external source of e.m.f. $\mathcal{E}(t) = \mathcal{E}_0 \exp(-i\omega t)$. The circuit is inductively coupled to another circuit whose parameters are L_2, C_2, R_2 . The mutual inductance between the two circuits is L_{12} . Find the currents $\mathcal{J}_1$ and $\mathcal{J}_2$ in the two circuits. Consider in particular the case where the second circuit is purely inductive ($R_2 = 0, C_2 = \infty$) and determine

the frequency at which the current $\mathcal{I}_2$ is a maximum.

347. Find the complex impedance Z of the two-terminal network shown in Fig. 18.

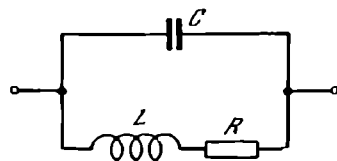


Fig. 18.

348. A capacitor is filled with a dielectric whose permittivity is given by $\epsilon = 1 - \omega_p^2 / \omega(\omega + i\gamma)$ (ionised gas; cf. Problem 312). In the absence of the medium the capacitance is C_0 . Show that the complex impedance of such a capacitor is equal to the impedance of the two-terminal network shown in Fig. 18 when the parameters L , C , and R of this circuit are suitably chosen. Find L , C , and R .

349. Determine the average energy $\bar{W}$ stored per unit time in the capacitor described in the preceding problem. Find also the heat loss Q per unit time and express both quantities in terms of the potential difference between the capacitor plates $U = U_0 \exp(-i\omega t)$.

350. A capacitor is filled with a medium having a permittivity $\epsilon = 1 + \frac{\omega_p^2}{\omega_0^2 - i\gamma\omega - \omega^2}$ [dielectric with losses; cf. Eqn. (VI.12)]. In the absence of the dielectric the capacitance is C_0 . Find the

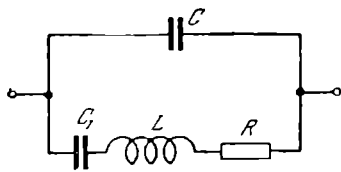


Fig. 19.

parameters C , C_1 , L , and R of the two-terminal network shown in Fig. 19 when its a.c. impedance is equal to the impedance of the above capacitor.

351. Find the average energy $\bar{W}$ stored in the capacitor considered in Problem 350 per unit time and the average heat

loss Q per unit time. Assume that the potential difference between the capacitor plates is $U = U_0 \exp(-i\omega t)$.

352. An oscillatory circuit consists of a capacitance C and an inductance L . At a certain instant of time a battery of constant e.m.f. $\mathcal{E}$ and an internal resistance R is connected across plates of the capacitor. Find the current flowing through the inductance as a function of time. Investigate the dependence of this current on L , C , and R .

353. A rectangular voltage pulse $U_1(t) = U_0$ at $0 \leq t \leq T$ and $U_1(t) = 0$ at $t < 0$, $t > T$ is applied to a circuit consisting of

a resistance R and a capacitance C connected in series. Find the voltage across the resistance.

354. A rectangular pulse $U_1(t) = U_0$ is applied across a circuit consisting of a resistance R and an inductance L connected in series. Find the voltage $U_2(t)$ across the inductance.

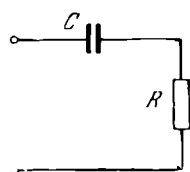


Fig. 20.

355. A circuit consists of a plane parallel capacitor having a capacitance C and a resistance R (Fig. 20). It is required to produce a field between the capacitor plates which will increase linearly from zero to E_0 during a time T and then decrease linearly to zero over an equal interval of time. Determine the form of the pulse

which must be applied across the input of this circuit in order to produce the required variation.

356. An e.m.f. $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t + \varphi_0)$ is connected at $t = 0$ across a circuit consisting of a resistance R and an inductance L . Find the current in the circuit and determine the phase angle at which there will be no transients.

357*. An artificial transmission line consists of N identical sections ($N \gg 1$) and is open at each end (Fig. 21). Find the natural frequencies of the system.

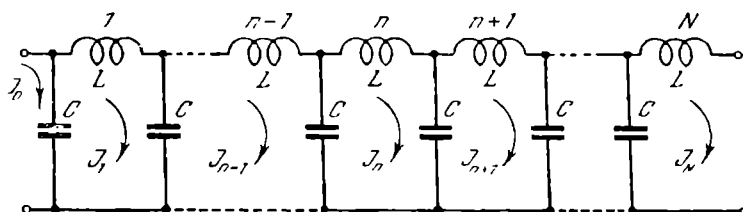


Fig. 21.

358. Assuming that the total number of natural frequencies of a long artificial transmission line (cf. Problem 357) is large, find the number of oscillations per unit frequency interval.

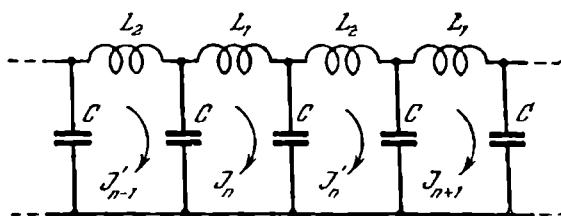


Fig. 22.

359*. A long artificial transmission line consisting of $2N$ alternately identical sections (Fig. 22) is open at each end. Investigate the spectrum of natural frequencies of the system.

360*. A long artificial transmission line consists of N identical sections (Fig. 23) whose impedances are given by

$$Z_1 = -i\left(\frac{\omega}{c^2}L_1 - \frac{1}{\omega C_1}\right), \quad Z_2 = -i\left(\frac{\omega}{c^2}L_2 - \frac{1}{\omega C_2}\right).$$

A potential difference U_1 is applied across one end of the line while the other end remains on open circuit. Find the potential difference U_2 between the points a and b .

Hint: The solution should be sought in the form of a difference equation involving the current $\mathcal{J}_n$ in the n -th section with $\mathcal{J}_n = \text{const} \cdot q^n$.

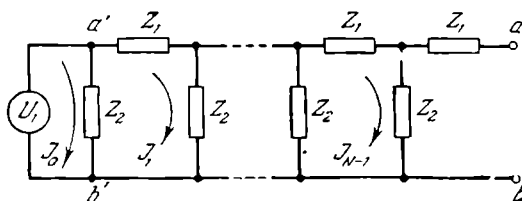


Fig. 23.

361. Use the results of the preceding problem to investigate the transmission coefficient $K = U_2/U_1$ as a function of frequency for $N \gg 1$. Find the frequency interval in which K is appreciably different from zero.

362. From a consideration of a long artificial transmission line with lumped parameters (Problem 357) obtain a differential equation for the current in a long line with uniformly distributed parameters by taking the appropriate limit.

363. A long lossless transmission line with continuously distributed parameters and length l is on open circuit at each end. Determine the spectrum of natural oscillations of the system and compare it with the spectrum of the lumped-parameter line considered in Problem 357.

364*. An e.m.f. connected to a closed circuit gives rise to a current $\mathcal{J}(t) = \mathcal{J}_0 \exp(-i\omega t)$ in the circuit. Find the general expression for the complex impedance of the circuit without neglecting the delay in the system.

365. Find the correction to the inductance and resistance $R_r(\omega)$ of a circular circuit of radius a on the first non-vanishing

approximation (*cf.* preceding problem). Show that $R_r(\omega)$ is the coefficient of proportionality between the average energy emitted per unit time and the r.m.s. current in the circuit.

2. Eddy currents and skin effect

If a conductor placed in an external magnetic field satisfies the conditions given by Eqn. (VII.1), then at each instant of time the field near the conductor must satisfy the following equations of magnetostatics:

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{H} = 0, \quad (\text{VII.9})$$

and the equation

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}. \quad (\text{VII.10})$$

When the conductivity σ of a conductor is sufficiently high ($\sigma/\omega \gg \gg \epsilon'$ where ϵ' is the real part of the permittivity), the field inside the conductor is described by Eqn. (VII.10) and

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \sigma \mathbf{E}. \quad (\text{VII.11})$$

From Eqns. (VII.10) and (VII.11) one can obtain second-order equations for the vectors $\mathbf{E}$, $\mathbf{H}$. In the case of a homogeneous medium these equations are of the form

$$\Delta \mathbf{H} = \frac{4\pi\mu\sigma}{c^2} \frac{\partial \mathbf{H}}{\partial t}, \quad \Delta \mathbf{E} = \frac{4\pi\mu\sigma}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \quad (\text{VII.12})$$

On the boundary of separation between two conductors, or a conductor and a dielectric, the field components must satisfy the boundary conditions

$$B_{1n} = B_{2n}, \quad H_{1\tau} = H_{2\tau}, \quad E_{1\tau} = E_{2\tau}. \quad (\text{VII.13})$$

The quantity $\delta = c/\sqrt{2\pi\mu\sigma\omega}$ (the thickness of the skin layer) characterises the depth of penetration of the field into the conductor (ω is the frequency of the field). In the case of a strong skin effect it may be assumed that the field does not penetrate into the conductor, in which case $\mathbf{H} = 0$ inside the conductor, while outside the conductor and at its surface the relation between the surface current $\mathbf{i}$ and the field is

$$\mathbf{H} = \frac{4\pi}{c} \mathbf{i}. \quad (\text{VII.14})$$

Owing to the appearance of eddy currents, a conductor placed in a magnetic field assumes a magnetic moment even if its permeability is $\mu = 1$. This magnetic moment may be conveniently characterised by the magnetic polarisability tensor β_{ik} which is defined by

$$m_i = \beta_{ik} H_{0k}, \quad (\text{VII.15})$$

where $\mathbf{m}$ is the magnetic moment of the body and $\mathbf{H}_0$ is a periodic external magnetic field. The tensor β_{ik} is symmetric ($\beta_{ik} = \beta_{ki}$) and its components are in general complex and depend on the frequency.

The time average of the heat liberated inside a conductor may be calculated from

$$Q = \int (\overline{\mathbf{j} \cdot \mathbf{E}}) dV = \int \overline{\mathbf{j} \cdot \mathbf{E}} dV \quad (\text{VII.16})$$

or

$$Q = -\frac{c}{4\pi} \oint (\overline{\mathbf{E} \times \mathbf{H}}) \cdot d\mathbf{S}. \quad (\text{VII.17})$$

In the first of these formulae the integral is evaluated over the volume of the conductor, while in the second formula it is evaluated over its surface. The heat Q can also be expressed in terms of the imaginary part of the magnetic polarisability tensor ($\beta_{ik} = \beta'_{ik} + i\beta''_{ik}$)

$$Q = \frac{\omega}{2} \beta''_{ik} \text{Re} (H_{0i} H_{0k}^*). \quad (\text{VII.18})$$

The latter formula will only hold for a field which is a harmonic function of time.

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366. A coil in which a current $\mathcal{I}_0 \exp(-i\omega t)$ is flowing is wound on a wide flat plate having a conductivity σ , magnetic permeability μ , and thickness $2h$. The number of turns per unit length is n and the thickness of the coil is very small. Neglecting edge effects, determine the real part of the amplitude of the magnetic field inside the plate. Investigate the limiting cases of a strong ($\delta \ll h$) and weak ($\delta \gg h$) skin effect.

367*. An infinitely long metal cylinder with conductivity σ and magnetic permeability μ is placed so that its axis coincides

with the axis of an infinite solenoid of circular cross-section, which carries a current $\mathcal{J} = \mathcal{J}_0 \exp(-i\omega t)$. Find the magnetic and electric field in all space, and also the current distribution j in the cylinder, assuming that the radius of the cylinder a , the radius of the solenoid b , and the number of turns per unit length, n , are known.

368. A conducting cylinder is placed in the uniform magnetic field $H = H_0 \exp(-i\omega t)$ which is parallel to its axis. Using the results of the preceding problem, investigate the current distribution inside the cylinder in the limit of low and high frequencies.

369. Calculate the amount of heat Q liberated per unit time per unit length of the cylinder considered in Problem 367. Investigate the limit of low and high frequencies.

370. Find the magnetic polarisability β (per unit length) of a cylinder placed in an alternating magnetic field which is parallel to its axis. Assume that the magnetic permeability is $\mu = 1$ and consider the limit of high and low frequencies.

371*. A metal cylinder is placed in an external uniform magnetic field $\mathbf{H} = \mathbf{H}_0 \exp(-i\omega t)$ which is perpendicular to its axis. Assuming that the magnetic permeability is $\mu = 1$, find the resultant field and the current density $\mathbf{j}$ in the cylinder.

Hint: Express $\mathbf{E}$, $\mathbf{H}$ in terms of the vector potential $\mathbf{A}$ and integrate the differential equation for $\mathbf{A}$.

372. Find the energy dissipation per unit length of a perfectly conducting circular cylinder placed in an alternating magnetic field of frequency ω which is perpendicular to the axis of the cylinder.

373*. An infinitely long cylinder of radius a and conductivity σ is placed in a circularly polarised magnetic field $\mathbf{H}_0(t) = (\mathbf{H}_{01} + i\mathbf{H}_{02}) \exp(-i\omega t)$ which is perpendicular to its axis. $\mathbf{H}_{01}$ and $\mathbf{H}_{02}$ are mutually perpendicular vectors of equal length $H_{01} = H_{02} = H_0$ [the vector $\mathbf{H}_0(t)$ describes a circle of radius H_0 in the plane perpendicular to the axis of the cylinder]. Find the average couple $\bar{\mathbf{N}}$ per unit length of the cylinder ($\mu = 1$).

374. An infinitely long cylinder rotates about its axis with an angular velocity ω in a constant uniform magnetic field $\mathbf{H}_0$ which is perpendicular to the axis. Find the retarding couple $\bar{\mathbf{N}}$ per unit length of the cylinder.

375*. An infinitely long metal cylinder of radius a , conductivity σ , and permeability μ is placed in a constant and uniform magnetic field H_0 which is parallel to its axis. The external field is then switched off and is maintained at zero. Find the magnetic

field inside the cylinder as a function of time.

376. A metal sphere of radius a , conductivity σ , and magnetic permeability μ is placed in a uniform alternating magnetic field $H_0(t) = H_0 \exp(-i\omega t)$. Assuming that the frequency is low, find the distribution of eddy currents in the sphere and the average power Q absorbed by it. Use the first-order approximation.

377. A metal sphere is placed in a uniform magnetic field which varies with a frequency ω . Find the resultant field $\mathbf{H}$ and the average power Q absorbed by the sphere at high frequencies.

Hint: In calculating the field outside the sphere, assume that the field inside it is zero, *i.e.* neglect the depth of penetration δ in comparison with the radius a . In determining the field inside the sphere assume that its surface is plane.

378*. A conducting ellipsoid is placed in a uniform alternating magnetic field. Determine the magnetic polarisability of the ellipsoid in the case of a strong skin effect, *i.e.* assume that the depth of penetration of the field into the conductor can be neglected. Consider the limiting cases of a thin circular disc and a long thin rod.

379*. A sphere of radius a and conductivity σ is placed in a uniform magnetic field $H(t) = H_0 \exp(-i\omega t)$. Find the resultant magnetic field and the distribution of eddy currents in the sphere in the general case of an arbitrary frequency. Verify that in the limiting cases of a strong and weak skin effect the general solutions become identical with those obtained in Problems 376 and 377 respectively (assume for simplicity that $\mu = 1$).

380. Find the average power, Q , absorbed by a conducting sphere in a uniform alternating magnetic field of arbitrary frequency.

381. Find the active resistance R of a thin cylindrical conductor in the presence of the skin effect, assuming that the permeability is $\mu = 1$. Investigate the limit of low and high frequencies.

382. A layer of another metal is deposited on the surface of a cylindrical conductor of radius a and conductivity σ_1 . The thickness of the layer is h and its conductivity is σ_2 ($h \ll a$). Find the active resistance R of the composite conductor to alternating currents, assuming that the thickness of the skin layer is small compared with a ($\mu = 1$).

383. An infinitely long hollow cylinder of internal radius a and wall thickness h ($h \ll a$) is placed in a uniform magnetic field

$H_0(t) = H_0 \exp(-i\omega t)$, which is parallel to its axis. Find the amplitude H' of the magnetic field in the cavity and investigate its dependence on ω .

Hint: Since $h \ll a$, the field inside the shell may be determined on the assumption that the shell is plane.

384. An alternating current $\mathcal{J} = \mathcal{J}_0 \exp(-i\omega t)$ flows along a hollow cylindrical conductor of average radius a , conductivity σ , magnetic permeability μ , and thickness $h \ll a$. Find the current distribution across the conductor and its active resistance R per unit length. Find also the condition which must be satisfied in order to ensure that the resistance of a hollow conductor is very nearly equal to the resistance of a solid conductor of equal external radius.

Hint: Neglect the curvature of the surface of the conductor.

385*. A linear current $\mathcal{J}$ flows inside a metal tube at a distance l from its axis. The radius of the tube is a , the wall thickness is $h \ll a$, and the conductivity of the walls is σ ($\mu = 1$). Both the current $\mathcal{J}$ and the distance l are arbitrary functions of time, but are such that at all times $l \ll a$. Assuming that Eqn. (VII.1) is satisfied, determine the force f per unit length on the current $\mathcal{J}$ due to eddy currents induced in the cylindrical shell in the case of a weak skin effect ($h \ll \delta$).

386*. Solve the preceding problem in the case of a strong skin effect ($h \gg \delta$).

Chapter

8

PROPAGATION OF ELECTROMAGNETIC WAVES

1. Plane waves in a homogeneous medium.
Reflection and refraction.
Wave packets

In the absence of charges and currents, the electromagnetic field vectors in a dielectric satisfy the equations

$$\text{curl } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (\text{VIII.1})$$

$$\text{curl } \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad (\text{VIII.2})$$

$$\text{div } \mathbf{D} = 0, \quad (\text{VIII.3})$$

$$\text{div } \mathbf{B} = 0. \quad (\text{VIII.4})$$

In a non-dispersive medium the field vectors are related by

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad (\text{VIII.5})$$

where ϵ is the permittivity and μ is the magnetic permeability.

If the losses of electromagnetic energy are negligible then ϵ and μ are real. In the case of a homogeneous medium Eqns. (VIII.1)–(VIII.5) may be combined to yield the following second-order equation for $\mathbf{E}$ and $\mathbf{H}$:

$$\Delta \mathbf{E} - \frac{1}{v_\varphi^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad \Delta \mathbf{H} - \frac{1}{v_\varphi^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0, \quad (\text{VIII.6})$$

where $v_\varphi = c/\sqrt{\epsilon\mu}$ can be identified with the phase velocity of the waves described by Eqn. (VIII.6).

In general, Eqn. (VIII.5) will only hold for monochromatic field components and the permittivity and permeability will be complex functions of frequency (dispersion). The imaginary parts of the permittivity and permeability determine the dissipation of electromagnetic energy in the medium.

In a conducting medium in which the field is varying slowly, so that the relation between the current and the field is of the form

$\mathbf{j} = \sigma \mathbf{E}$, where σ is the static conductivity, Eqn. (VIII.2) must be replaced by

$$\text{curl } \mathbf{H} = \frac{4\pi}{c} \sigma \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}. \quad (\text{VIII.7})$$

The latter relation will again assume the form of Eqn. (VIII.2) if one introduces the complex permittivity, which at low frequencies is given by

$$\epsilon(\omega) = \epsilon' + i \frac{4\pi\sigma}{\omega}, \quad (\text{VIII.8})$$

where ϵ' and σ are the static values of the permittivity and conductivity. At high frequencies the permittivity of a conducting medium is also a complex function of frequency.

In the case of good conductors (metals) the second term in Eqn. (VIII.8) is very large and hence at low frequencies

$$\epsilon(\omega) = i \frac{4\pi\sigma}{\omega}. \quad (\text{VIII.9})$$

If the frequency is such that the depth of penetration of the field into the metal is much smaller than the radius of curvature of its surface, and also much less than the wavelength in the surrounding medium, then the tangential components of $\mathbf{E}$ and $\mathbf{H}$ near the surface of the conductor are related by

$$\mathbf{E}_\tau = \zeta (\mathbf{H}_\tau \times \mathbf{n}), \quad (\text{VIII.10})$$

where $\mathbf{n}$ is an inward unit normal and ζ is the surface impedance which depends on frequency and the properties of the metal, and is defined by

$$\zeta = \sqrt{\frac{\mu}{\epsilon}}. \quad (\text{VIII.11})$$

Eqn. (VIII.10) will hold provided $|\zeta| \ll 1$. It may be used as an approximate boundary condition for the determination of the field outside a conductor.

If the field is a simple periodic function of time, the medium is homogeneous, and $\mu = 1$, then the electric field will satisfy the equation

$$\Delta \mathbf{E} + \frac{\epsilon \omega^2}{c^2} \mathbf{E} - \text{grad div } \mathbf{E} = 0, \quad (\text{VIII.12})$$

and the magnetic field $\mathbf{H}$ will be given in terms of $\mathbf{E}$ by Eqn. (VIII.1).

A plane monochromatic wave propagating in the direction of the wave vector $\mathbf{k}$, where $k = 2\pi/\lambda$ and λ is the wavelength, is described by

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}. \quad (\text{VIII.13})$$

The wave amplitude $\mathbf{E}_0 = \mathbf{E}' + i\mathbf{E}''$ is in general a complex vector. Moreover, $\mathbf{E}_0 \perp \mathbf{k}$, *i.e.* the waves are transverse. Depending on the magnitude and direction of the real vectors $\mathbf{E}'$ and $\mathbf{E}''$, the wave will be plane, circularly, or elliptically polarised.

In addition to monochromatic waves it is often necessary to deal with "almost monochromatic" waves or groups of waves which are superpositions of monochromatic waves with frequencies contained within a small range $\Delta\omega$. At a given point in space, such waves may be described by functions of the form $\mathbf{E} = \mathbf{E}_0(t) \exp(-i\omega t)$ where ω is a mean frequency within the range $\Delta\omega$ and $\mathbf{E}_0(t)$ is a function which varies much more slowly than $\exp(-i\omega t)$. Such waves are partially polarised and may be characterised by the polarisation tensor

$$I_{ik} = \overline{E_{0i} E_{0k}^*}, \quad (\text{VIII.14})$$

where the bar indicates an average with respect to time. The polarisation tensor is Hermitian, *i.e.* $I_{ik} = I_{ki}^*$. It may be written down in the form

$$I_{ik} = I_1 e_i^{(1)} e_k^{(1)*} + I_2 e_i^{(2)} e_k^{(2)*}, \quad (\text{VIII.15})$$

where I_1 and I_2 are positive quantities and $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are complex vectors which are orthogonal and are normalised so that $\mathbf{e}^{(i)} \cdot \mathbf{e}^{(k)} = \delta_{ik}$. The quantities I_i and $\mathbf{e}^{(i)}$ are defined by

$$I_{ik} e_k = I e_i. \quad (\text{VIII.16})$$

It follows from this relation that a partially polarised wave may be looked upon as an incoherent superposition of two elliptically polarised waves (incoherent waves are defined as waves whose phases vary randomly). The form and orientation of the polarisation ellipses are described by the vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$. The ellipses corresponding to the two component waves are similar, and their corresponding axes are perpendicular. The quantities I_1 and I_2 represent the intensities of the two waves.

The ratio

$$\rho = \frac{I_2}{I_1} \quad (I_2 \leq I_1) \quad (\text{VIII.17})$$

is called the degree of depolarisation. For a completely polarised wave $\rho = 0$, and for an unpolarised wave $\rho = 1$, $I_1 = I_2 = I$ and the polarisation tensor is of the form $I_{ik} = I\delta_{ik}$.

When a plane wave falls on a plane separation boundary between two media, the angles θ_0 , θ_1 , and θ_2 , which are the angles of incidence, reflection, and refraction (Fig. 24), are related by

$$\theta_1 = \theta_0, \quad \frac{\sin \theta_2}{\sin \theta_0} = \frac{n_1}{n_2}, \quad n_i = \sqrt{\epsilon_i}, \quad (\text{VIII.18})$$

where $n_{1,2}$ are the refractive indices of the first and second media (we assume that $\mu_1 = \mu_2 = 1$).

The amplitudes of the reflected (E_1 , H_1) and refracted (E_2 , H_2) waves can be expressed in terms of the amplitude (E_0 , H_0) of the incident wave. The corresponding relationships are known as Fresnel's formulae:

(a) If E_0 is normal to the plane of incidence, then

$$E_1 = \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)} E_0, \quad E_2 = \frac{2 \cos \theta_0 \sin \theta_2}{\sin(\theta_2 + \theta_0)} E_0. \quad (\text{VIII.19})$$

(b) If H_0 is normal to the plane of incidence, then

$$H_1 = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} H_0, \quad H_2 = \frac{\sin 2\theta_0}{\sin(\theta_0 + \theta_2) \cos(\theta_0 - \theta_2)} H_0. \quad (\text{VIII.20})$$

The angle θ_2 is given in terms of the permittivities of the two media by Eqn. (VIII.18). Eqns. (VIII.18)–(VIII.20) retain their form even when ϵ_2 is complex, but then the angle θ_2 will also be complex and will not have a simple geometrical interpretation. The treatment of a complex θ_2 is discussed in Problem 404.

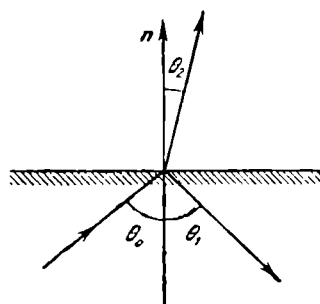


Fig. 24.

The reflection coefficient R is defined as the ratio of the time average of the reflected energy flux to the time average of the incident energy flux.

The superposition of plane monochromatic waves with different wave vectors and frequencies is often referred to as a wave group, or wave packet. A wave packet is described by

$$\Psi(\mathbf{r}, t) = \int \psi(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d k_x d k_y d k_z. \quad (\text{VIII.21})$$

where $\Psi(\mathbf{r}, t)$ is a cartesian component of $\mathbf{E}$ or $\mathbf{H}$. The function $\psi(\mathbf{k})$ is called the amplitude function. The maximum of the amplitude of a wave packet moves through space with the group velocity $v_g = d\omega/dk$.

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387. Two plane monochromatic linearly polarised waves of the same frequency propagate along the z -axis. The first wave is polarised along the x -axis and has an amplitude a , and the second is polarised along the y -axis and has an amplitude b . The phase of the second wave leads the phase of the first wave by χ . Find the polarisation of the resultant wave.

388. In the preceding problem, consider the dependence of the polarisation on the phase difference χ when $a = b$.

389. Two monochromatic waves of equal frequency are circularly polarised in opposite directions. They are in phase and propagate in the same direction. Their amplitudes are a and b . Determine the type of polarisation for different values of the ratio a/b (a and b may be chosen to be real).

390*. An electromagnetic wave is formed by the superposition of two incoherent "almost monochromatic" waves of the same intensity I and with approximately equal frequencies and wave vectors. Both waves are linearly polarised and the directions of polarisation are defined in the plane perpendicular to the wave vector by the unit vectors $\mathbf{e}^1(1, 0)$ and $\mathbf{e}^2(\cos \vartheta, \sin \vartheta)$. Find the polarisation tensor I_{ik} of the resultant wave and its degree of depolarisation.

391. Solve the preceding problem in the case where the two intensities are unequal ($I_1 \neq I_2$) and the two directions of polarisation are at an angle of $\pi/4$.

392. The Hermitian polarisation tensor of an electromagnetic wave is given by

$$I_{ik} = \frac{1}{2} I \left(\delta_{ik} + \sum_{l=1}^3 \xi_l \hat{\tau}_{ik}^{(l)} \right) = \frac{1}{2} I \begin{pmatrix} 1 + \xi_1 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1 - \xi_1 \end{pmatrix},$$

where I is the total intensity of the wave, ξ_i are real parameters which are such that $\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \leq 1$ (Stokes' parameters), and $\hat{\tau}^{(l)}$ are the matrices

$$\hat{\tau}^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\tau}^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\tau}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Explain the physical significance of the parameters ξ_i . To do this,

express the degree of depolarisation ρ in terms of ξ_i and determine the polarisation of the two component waves into which a partially polarised wave may be resolved under the following three conditions:

- a) $\xi_1 \neq 0, \xi_2 = \xi_3 = 0$;
- b) $\xi_2 \neq 0, \xi_1 = \xi_3 = 0$;
- c) $\xi_3 \neq 0, \xi_1 = \xi_2 = 0$.

393. The electric field $\mathbf{E}$ of an electromagnetic wave in which the real and imaginary components of the complex wave vector $\mathbf{k}$ are in different directions, is linearly polarised. Determine the mutual disposition of the vectors $\mathbf{E}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathbf{k}'$, and $\mathbf{k}''$, where $\mathcal{H}_1$, $\mathcal{H}_2$ are the real and imaginary parts of the complex amplitude $\mathbf{H}_0$ and $\mathbf{k}'$, $\mathbf{k}''$ are the real and imaginary parts of the wave vector $\mathbf{k}$. Find the locus of the end point of the vector $\mathbf{H}$ at a given point in space.

Solve the same problem in the case where the vector $\mathbf{H}$ is linearly polarised.

394. A circularly polarised monochromatic wave falls obliquely on the plane boundary of a dielectric. Determine the polarisation of the reflected and refracted waves.

395*. A beam of almost monochromatic unpolarised light is incident on the plane boundary of a dielectric. Find the polarisation tensors $I_{ik}^{(1)}$, $I_{ik}^{(2)}$ and the depolarisation coefficients ρ_1 , ρ_2 of the reflected and refracted light.

396. An unpolarised, almost monochromatic beam of light is incident on the plane boundary of separation between two dielectrics. Determine the coefficient of reflection R and the depolarisation coefficients $\rho_{1,2}$ of the reflected and refracted light, if the angle of incidence is equal to Brewster's angle.

397. Derive the Fresnel formulae for the case where an electromagnetic wave is incident on the plane boundary of a conducting medium with a small surface impedance ζ .

398. Find the reflection coefficient R of a metal surface with a small surface impedance $\zeta = \zeta' + i\zeta''$. Find the angle of incidence θ_0 at which the reflection coefficient is a minimum.

399. A linearly polarised wave is incident on the plane boundary of a conducting medium with a low surface impedance ζ . Determine the polarisation of the reflected wave if the glancing angle of incidence is equal to the angle Φ_0 determined in the solution of the preceding problem.

400. A linearly polarised plane wave is incident at an angle θ_0 on the surface of a metal. The direction of the electric vector is at an angle $\pi/4$ to the plane of incidence. The experimentally determined ratio of the transverse and longitudinal (with respect to the plane of incidence) components of the reflected wave is found to be $E_{\parallel 1}/E_{\perp 1} = \tan \rho$ and the phase difference between the components δ is such that

$$\frac{E_{\parallel 1}}{E_{\perp 1}} = \tan \rho e^{i\delta}.$$

Express the real part of the refractive index n' and the absorption coefficient n'' in terms of ρ , δ , and θ_0 where $n' + in'' = \zeta^{-1}$, if ζ is the surface impedance and $|n'^2 - n''^2| \gg \sin^2 \theta_0$.

401. Find the reflection coefficient R of the plane surface of a conductor at normal incidence, in the limiting case of low conductivity [cf. Eqn. (VIII.8)].

402*. Show that a linearly polarised wave will in general become elliptically polarised after total reflection from the surface of a dielectric. Under what conditions will the polarisation become circular?

403. Investigate the transport of energy in the case of total internal reflection. Find the energy flux along the separation boundary, and also in the perpendicular direction, in the medium from which the reflection takes place. Determine the lines of the Poynting vector γ .

404*. An electromagnetic wave is incident obliquely from a dielectric on the plane boundary of a conducting medium. Find the direction of propagation, the attenuation, and the phase velocity v_φ in the conducting medium.

405*. A dielectric layer of permittivity ϵ_2 is bounded by the planes $z = 0$ and $z = a$ and lies between dielectric media with permittivities ϵ_1 and ϵ_3 ($\mu_1 = \mu_2 = \mu_3 = 1$). An electromagnetic wave falls normally on the surface of the layer from the region $z < 0$. Find the thickness of the layer corresponding to minimum reflection, and the ratio of ϵ_1 , ϵ_2 , and ϵ_3 for which there will be no reflection.

406*. A plane wave falls normally from a vacuum on the boundary of a dielectric. Investigate the effect of the "sharpness" of the boundary on the reflection coefficient, assuming that the permittivity is given by

$$\epsilon(z) = \epsilon - \frac{\Delta\epsilon}{e^{z/a} + 1}, \quad \epsilon = 1 + \Delta\epsilon,$$

where ϵ and $\Delta\epsilon$ are constants. Investigate the special cases of large and small a .

Hint: Use the substitutions $\xi = -\exp(-z/a)$ and $E(\xi) = \xi^{-ik_0 a} \psi(\xi)$ in the differential equation for $E(z)$ [cf. Eqn. (VIII.12)] where $\psi(\xi)$ satisfies the hypergeometric equation.

407*. In the absence of absorption, the permittivity of a plasma is given by (cf. Problem 312)

$$\epsilon = 1 - \frac{4\pi e^2 N}{m\omega^2}.$$

Discuss the propagation of electromagnetic waves in a plasma whose concentration is described by the linear function $N(z) = N_0 z$. Consider the case where a plane monochromatic wave is incident normally on a non-homogeneous layer of plasma (this may be of importance in the propagation of radio waves in the ionosphere).

Hint: Solve the equation for $E(z)$ by expanding the required function into a Fourier integral.

408. Construct a one-dimensional wave packet Ψ at time $t = 0$ in the case where the amplitude is given by the Gaussian function $a(k) = a_0 \exp[-(k - k_0)^2/(\Delta k)^2]$ and a_0 , k_0 and Δk are constants. Find the relation between the width of the packet Δx and the range of wave numbers Δk which contribute significantly to the wave packet.

409. A wave packet Ψ is formed by the superposition of plane waves of different frequency. The amplitude function is given by the Gaussian curve $a(\omega) = a_0 \exp[-(\omega - \omega_0)^2/(\Delta\omega)^2]$ where a_0 , ω_0 , and $\Delta\omega$ are constants. Find the amplitude of the packet as a function of time at the point $x = 0$. Determine the relation between the length of the wave packet Δt and the frequency range $\Delta\omega$.

410. A given object is illuminated by light of wavelength λ and is viewed through a microscope. Find the minimum possible linear dimension $\Delta x_{\min}$ of the object which is allowed by the condition $\Delta x \Delta k \geq 1$.

411. The position of an object is determined by means of radar. What is the limiting accuracy with which this determination may be carried out if the wavelength is λ and the distance to the object is l ?

412. Investigate the form and the motion of a wave packet obtained as a result of the superposition of plane waves with equal

amplitudes a_0 and wave vectors lying in the range $|\mathbf{k}_0 - \mathbf{k}| \leq q$, where $\mathbf{k}_0$ and q are constants. Replace the actual law of dispersion $\omega(\mathbf{k})$ by the approximate relation $\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + \left. \frac{d\omega}{d\mathbf{k}} \right|_0 \cdot (\mathbf{k} - \mathbf{k}_0)$.

413*. Investigate the spreading of a one-dimensional wave packet in a dispersive medium. Assume that the amplitude function is given by the Gaussian curve $a(k) = a_0 \exp[-\alpha(k - k_0)^2]$, and retain all terms up to and including the second-order term in the expansion of ω in terms of k .

414. Find the phase velocity v_ϕ and the group velocity v_g for a medium whose permittivity is given by

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}.$$

Consider only the cases of large and small frequencies (compared with ω_0) and assume that $\mu = 1$.

415. Determine the rate of transport of energy by a one-dimensional wave packet in a dispersive medium. Show that this velocity is equal to a group velocity v_g .

Hint: The rate of transport of energy v is given by the relation $\bar{\gamma} = v\bar{w}$, where

$$\bar{w} = \frac{1}{16\pi} \left[\frac{d(\omega\epsilon)}{d\omega} EE^* + \frac{d(\omega\mu)}{d\omega} HH^* \right]$$

is the average energy density in the dispersive medium and $\bar{\gamma}$ is the average energy flux.

2. Scattering of electromagnetic waves by macroscopic bodies. Diffraction

The diffraction of electromagnetic waves by conducting or dielectric bodies is described by the solutions of Maxwell's equations subject to appropriate boundary conditions. An exact solution is only possible in a very limited number of cases (cf. Problems 418 and 423). However, an approximate solution may frequently be obtained.

If the linear dimensions of the body are small in comparison with the wavelength, then the electromagnetic field near the body may be looked upon as uniform. A body placed in a uniform periodic field will assume electric and magnetic moments which will be the same functions of time as the external field. The

scattered wave appears as a result of the emission of radiation by these alternating moments. The scattering of electromagnetic waves by bodies of small linear dimensions may therefore be reduced to the determination of the dipole moments induced in them. The radiation fields are given in terms of the dipole moments by Eqns. (XII.17) and (XII.20).

The effective differential cross-section for scattering into a solid angle $d\Omega$ is defined by

$$d\sigma_s = \frac{dI(\theta, \alpha)}{\bar{\gamma}_0}, \quad (\text{VIII.22})$$

where $dI = \bar{\gamma} dS = \bar{\gamma} r^2 d\Omega$ is the time average of the intensity emitted into the solid angle $d\Omega$ and $\bar{\gamma}$ and $\bar{\gamma}_0$ are the average energy flux densities of the scattered and incident radiation. The energy flux density is described by the Poynting vector

$$\gamma = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}). \quad (\text{VIII.23})$$

The effective absorption cross-section is equal to the ratio of the average energy absorbed by a body per unit time to the average energy flux density in the incident wave:

$$\sigma_a = \frac{Q}{\bar{\gamma}_0}. \quad (\text{VIII.24})$$

In the opposite limiting case, *i. e.* when the wavelength is much smaller than the linear dimensions of the body, one can use the methods of geometrical optics. In the case of diffraction of an electromagnetic wave by an aperture in an infinite opaque screen, the amplitude of the diffracted field is, on the geometrical optics approximation, given by

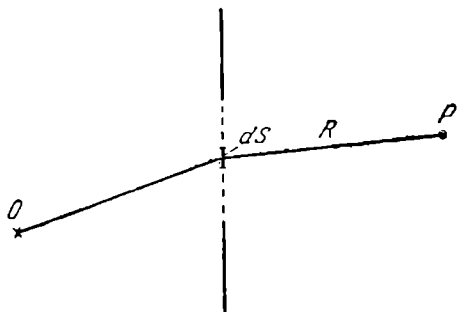


Fig 25.

$$u_P = \frac{k}{2\pi i} \int \frac{u}{R} e^{ikR} dS_n. \quad (\text{VIII.25})$$

This formula may be derived from Huygens' principle. The quantity u_P is the field at a point P in front of the screen (Fig. 25), u is the field at an element of area dS of the aperture (this field is assumed to be the same as in the absence of the screen,

i.e. undistorted by the screen), dS_n are the components of the area vector $d\mathbf{S}$ in the direction of the ray reaching dS from the source of light O , R is the distance between dS and the point of observation P , and k is the modulus of the wave vector.

The source of light O and the point of observation P may be either at finite or infinite distances from the screen. When both O and P , or only one of them, are at a finite distance from the screen, the phenomenon is known as Fresnel diffraction. When both O and P are at very large distances from the screen, then the rays of light reaching the aperture and the rays leaving it may be regarded as parallel, and this case is known as Fraunhofer diffraction. Eqn. (VIII.25) may then be written in the approximate form

$$u_P = \frac{u_0 e^{ikR_0}}{2\pi i R_0} \int e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}} dS_n, \quad (\text{VIII.26})$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the wave vectors of the incident and diffracted radiation, R_0 is the distance from the aperture to the point of observation, and u_0 is the field amplitude within the aperture. The intensity of the diffracted wave is proportional to the square of the modulus of the amplitude $|u_P|^2$.

Two screens are defined as complementary if the apertures in one correspond to identical opaque regions in the other, and *vice versa*. The Babinet principle holds for such screens. Suppose that u_1 and u_2 are the wave fields at a given point due to two complementary screens, and u is the wave field in the absence of the two screens. We then have

$$u_1 + u_2 = u. \quad (\text{VIII.27})$$

Eqns. (VIII.25) and (VIII.26) take no account of the polarisation of the waves, *i.e.* the amplitude u is looked upon as a scalar. The diffraction formula which takes into account the vector nature of the electromagnetic field is

$$\mathbf{E}_P = -\frac{ik}{4\pi R} \int \left\{ \mathbf{n}_0 \times \mathbf{H} - \mathbf{n} [\mathbf{n} \cdot (\mathbf{n}_0 \times \mathbf{H})] + \mathbf{n} \times (\mathbf{n}_0 \times \mathbf{E}) \right\} e^{ikr} dS, \quad (\text{VIII.28})$$

where $\mathbf{E}$ and $\mathbf{H}$ are the fields within the aperture, $\mathbf{E}_P$ is the electric field at a large distance from the screen (in the so-called wave zone), $\mathbf{n}$ is a unit vector in the direction of propagation of the diffracted wave, $\mathbf{n}_0$ is a unit normal to the plane of the aperture and is drawn towards the point of observation, r is the distance of

the element of area dS from the point of observation, and R is the distance of the point of observation from the origin of coordinates, which is taken to lie in the plane of the aperture. The magnetic field in the wave zone is given in terms of the electric field by the usual formula $\mathbf{H}_P = \mathbf{n} \times \mathbf{E}_P$.

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416*. A plane monochromatic wave is incident on an infinitely long, circular, perfectly conducting cylinder of radius a , in a direction at right angles to its axis. The cylinder lies in a vacuum, and the vector $\mathbf{E}_0$ of the incident wave is parallel to the axis of the cylinder. Determine the resultant field, the distribution of currents on the surface of the cylinder, and the total current $\mathcal{I}$ flowing along the cylinder.

417. Find the differential scattering cross-section $d\sigma_s$ for the cylinder considered in Problem 416. Find also the total scattering cross-section σ_s .

418*. A plane monochromatic wave is incident on a perfectly conducting, circular cylinder so that its magnetic vector $\mathbf{H}_0 = \mathcal{H}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ is parallel, and the wave vector $\mathbf{k}$ is perpendicular, to the axis of the cylinder. The cylinder lies in a vacuum. Find the resultant electromagnetic field. In particular, discuss the special case of a thin cylinder ($ka \ll 1$) and determine the differential and total scattering cross-sections $d\sigma_s$ and σ_s for the cylinder.

419. Let $d\sigma_{\parallel}$ and $d\sigma_{\perp}$ be the differential scattering cross-sections of an infinitely long cylinder for plane waves with the electric vector $\mathbf{E}$ parallel and perpendicular to the axis of the cylinder respectively. Find the differential scattering cross-section $d\sigma'_s$ in the case of a wave whose electric vector $\mathbf{E}$ is at an angle φ to the axis of the cylinder, and also the differential scattering cross-section $d\sigma''_s$ for an unpolarised wave.

Hint: Use the principle of superposition of waves.

420. An unpolarised plane wave is scattered by a perfectly conducting thin cylinder ($ka \ll 1$). Determine the degree of depolarisation ρ of the scattered waves as a function of the angle of scattering.

421*. Solve Problem 418 in the case of diffraction of a plane wave by an infinitely long cylinder without assuming that the cylinder is perfectly conducting, but regarding its surface

impedance ζ as small. Use the approximate boundary condition given by Eqn. (VIII.10).

422. Determine the average loss of energy Q and the absorption cross-section σ_a per unit length of the cylinder considered in the preceding problem. Investigate the limiting case $ka \ll 1$ and explain your results.

423*. Investigate the diffraction of a plane monochromatic wave by a dielectric cylinder. The cylinder lies in a vacuum and has a radius a , permittivity ϵ , and magnetic permeability μ . The wave is incident at right angles to a generator of the cylinder and the vector $\mathbf{E}$ is perpendicular to its axis. Determine the resultant field.

424*. A linearly polarised, plane monochromatic wave is scattered by a sphere of radius a which is much smaller than the wavelength λ . Express the components of the scattered electromagnetic wave in the wave zone in terms of the electric and magnetic polarisabilities of the sphere. Determine the effective differential scattering cross-section.

Hint: In view of the fact that $a \ll \lambda$, the external field near the sphere may be looked upon as uniform, and it is sufficient to consider the emission of radiation by the induced electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$.

425. Calculate the differential and total scattering cross-sections $d\sigma_s$ and σ_s and also the degree of depolarisation ρ of the secondary radiation when an unpolarised wave is scattered by a sphere of radius a which is much smaller than the wavelength λ . Express the result in terms of the electric and magnetic polarisabilities β_e and β_m of the sphere.

426. Using the results of the preceding problem, determine the differential and total scattering cross-sections $d\sigma_s$ and σ_s for unpolarised light and a small dielectric sphere of permittivity ϵ ($\mu = 1$), and also the degree of depolarisation of the scattered radiation. Sketch out graphs showing these quantities as functions of the angle of scattering θ . Indicate the limits of applicability of the solutions. Solve the same problem for a perfectly conducting sphere with $\mu = 1$.

427. A plane monochromatic wave is incident at an angle $\pi/2 - \alpha$ on a perfectly conducting thin disc whose radius a is much smaller than the wavelength λ . Determine the differential and total scattering cross-sections $d\sigma_s$ and σ_s for different polarisations of the incident wave and also the scattering cross-section for an unpolarised wave.

428. A uniform dielectric of permittivity ϵ ($\mu = 1$) contains a cavity in the form of a thin disc of radius a and thickness $2h$. Unpolarised light of wavelength $\lambda \gg a$ is incident normally on one of the planes defining the cavity. Find the differential and total scattering cross-sections $d\sigma_S$ and σ_S .

429*. Find the differential and total cross-sections for the scattering of a plane wave of wavelength λ by a perfectly conducting cylinder of length $2h$ and radius $a \ll h$, $h \ll \lambda$. Investigate the various types of polarisation of the incident wave. The cylinder may be approximated by a prolate ellipsoid of revolution with semi-axes a and h .

Hint: Use the solutions of Problems 198, 200, and 378.

430. Solve the preceding problem for a dielectric cylinder whose length $2h$ is much less than the wavelength λ inside the cylinder.

431*. A plane wave is scattered by a dielectric body of arbitrary form and permeability $\mu = 1$. Obtain the integral equation for the resultant electric field. To do this, consider the secondary radiation emitted by a volume element of the body, and obtain the scattered wave in the form of an integral over the body.

432*. Using the integral equation derived in the preceding problem, investigate the scattering of an electromagnetic wave by a dielectric sphere on the first approximation in α where $\alpha = (\epsilon - 1)/4\pi \ll 1$. Determine the differential scattering cross-section $d\sigma_S$ and the degree of depolarisation ρ of the scattered radiation. Consider the special case of a very large sphere ($ka \gg 1$).

433. Determine the total scattering cross-section σ_S for the dielectric sphere discussed in the preceding problem in the limit $ka \gg 1$ and compare with $ka \ll 1$.

434*. A plane monochromatic wave is scattered by a set of charges, for example, a macroscopic body. The electric field at large distances from the scatterer is of the form

$$\mathbf{E} = E_0 \left[\mathbf{e} e^{ikz} + \mathbf{F}(\mathbf{n}) \frac{e^{ikr}}{r} \right],$$

where $\mathbf{n} = \mathbf{r}/r$, $\mathbf{e} = \mathbf{E}_0/E_0$, $k = \omega/c$, $\mathbf{E}_0$ is the amplitude of the incident wave, and $\mathbf{F}(\mathbf{n})$ is the scattering amplitude which characterises the properties of the scatterer and is a function of frequency.

Show that

$$\sigma_t = \frac{4\pi}{k} \operatorname{Im} [\mathbf{e} \cdot \mathbf{F}(\mathbf{n}_0)],$$

where $\sigma_t = \sigma_s + \sigma_a$ is the total cross-section for the interaction of the wave with the set of charges, σ_s is the scattering cross-section, σ_a is the absorption cross-section, and $\mathbf{F}(\mathbf{n}_0)$ is the forward-scattering amplitude.

435*. A plane monochromatic wave is incident on a macroscopic particle whose linear dimensions are much smaller than the wavelength λ . The electric and magnetic polarisabilities are $\beta_e = \beta'_e + i\beta''_e$ and $\beta_m = \beta'_m + i\beta''_m$. Since the polarisabilities are complex, the scattering is accompanied by absorption. Calculate the absorption cross-section.

Hint: The energy absorbed per unit time is equal to the flux of the Poynting vector through the surface of a large sphere surrounding the particle.

436. Calculate the cross-section for the absorption of electromagnetic waves by a conducting sphere with a small surface impedance $\zeta = \zeta' + i\zeta''$. The radius of the sphere, b , is small compared with the wavelength λ .

437. A plane monochromatic wave is incident on a macroscopic body of given absorption and differential scattering cross-sections σ_a and $d\sigma_s/d\Omega$. Express the time average of the force $\overline{\mathbf{F}}$ acting on the body in terms of these cross-sections.

438*. Determine the average force $\overline{\mathbf{F}}$ acting on a small sphere of radius a placed in the field of a plane monochromatic wave. Consider the examples of a perfectly conducting sphere and a dielectric sphere of permittivity ϵ and magnetic permeability $\mu = 1$.

439. A point source of light is placed on a line which passes through the centre of a circular opaque screen of radius a and is perpendicular to it. Assuming that the geometrical optics approximation may be used ($\lambda \ll a$), find the intensity of light I at a point P on the axis.

440. In the preceding problem, consider the diffraction at the complementary screen, *i.e.* a circular aperture in an infinite opaque screen.

441. A parallel beam of light is incident normally on a circular aperture in an opaque screen. Find the intensity distribution along the axis.

442. Find the angular distribution of the intensity of light in the case of Fraunhofer diffraction at an annular aperture (radii $a > b$) in an infinite opaque screen. Assume that the beam is incident normally on the plane of the aperture and consider the special case of diffraction by a circular aperture.

443. Find the angular distribution of the intensity of light for oblique incidence of a parallel beam on a circular aperture (Fraunhofer diffraction).

444. A plane linearly polarised wave is incident normally on the rectangular aperture $-a \leq x \leq a$, $-b \leq y \leq b$ in an infinite thin screen. The amplitudes of the electric and magnetic fields are $E_y = E_0$, $H_x = -E_0$, $H_y = E_x = 0$. Determine the field radiated from the aperture and also the angular distribution of the radiation.

445. A plane linearly polarised wave $\mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ is incident normally on a circular aperture of radius a in an infinite thin screen. Determine the field of the radiation emitted from the aperture and the angular intensity distribution.

3. Plane waves in anisotropic and gyrotropic media

When the permittivity and the magnetic permeability of a medium are tensors, the medium is referred to as optically anisotropic. Optical anisotropy may be a consequence of the crystalline structure of the body, but it may also be produced by an external electric field (*cf.* Problems 313 and 314) or by an external mechanical effect. In the absence of an external magnetic field, the tensors $\varepsilon_{ik}(\omega)$ and $\mu_{ik}(\omega)$ are symmetric:

$$\varepsilon_{ik} = \varepsilon_{ki}, \quad \mu_{ik} = \mu_{ki}. \quad (\text{VIII.29})$$

We shall not consider the effects associated with the spatial non-uniformity of the field which leads to a dependence of the permittivity and the magnetic permeability on the wave vector $\mathbf{k}$ (*cf.* Problem 458).

In an anisotropic medium, two plane monochromatic waves of the same frequency and polarised linearly in perpendicular planes may be propagated in a given direction with different phase velocities. The directions along which both waves have the same velocity of propagation are called optic axes. The direction of propagation of a wave, which is defined by the normal to the wave

surface, is not in general the same as the ray direction, *i.e.* the direction of the Poynting vector.

Crystals in which the principal values of the permittivity tensor are equal [$\epsilon^{(1)} = \epsilon^{(2)} = \epsilon_{\perp}$, $\epsilon^{(3)} = \epsilon_{\parallel}$] are known as uniaxial. Their optic axis lies along the $x_3 = z$ axis. The wave vectors of the two waves propagating at an angle θ to the optic axis are then given by

$$k_1 = \frac{\omega}{c} \sqrt{\epsilon_{\perp} \mu}, \quad k_2 = \frac{\omega}{c} \sqrt{\frac{\epsilon_{\perp} \epsilon_{\parallel} \mu}{\epsilon_{\perp} \sin^2 \theta + \epsilon_{\parallel} \cos^2 \theta}}. \quad (\text{VIII.30})$$

The first of these waves is known as the ordinary wave. The induction vector $\mathbf{D}$ and the electric field vector $\mathbf{E}$ of this wave are parallel to each other and perpendicular to the wave vector $\mathbf{k}_1$ and the plane containing the wave vector and the optic axis (principal plane). The second wave is known as the extraordinary wave. In this wave the induction vector $\mathbf{D}$ lies in the principal plane and is perpendicular to the corresponding wave vector $\mathbf{k}_2$. The vector $\mathbf{E}$ lies in the principal plane but is not parallel to $\mathbf{D}$.

In the presence of an external constant magnetic field, the tensors ϵ_{ik} and μ_{ik} are no longer symmetric. In non-absorbing media, to which we shall confine our attention in this section, these tensors are Hermitian, *i.e.*

$$\epsilon_{ik} = \epsilon_{ki}^*, \quad \mu_{ik} = \mu_{ki}^*. \quad (\text{VIII.31})$$

The relation between the field strengths and the inductions is then of the form (*cf.* Problem 316)

$$\mathbf{D} = \hat{\epsilon}' \mathbf{E} + l(\mathbf{E} \times \mathbf{g}_e), \quad \mathbf{B} = \hat{\mu}' \mathbf{H} + l(\mathbf{H} \times \mathbf{g}_m), \quad (\text{VIII.32})$$

where $\mathbf{g}_e$ and $\mathbf{g}_m$ are the gyration vectors (electric and magnetic respectively), and $\hat{\epsilon}' \mathbf{E}$ is a vector with components equal to $\epsilon'_{ik} E_k$. Media for which the field vectors are given by Eqn. (VIII.32) are known as gyrotropic media.

Two plane waves of the same frequency can propagate in a gyrotropic medium in a given direction with different phase velocities. These waves are elliptically polarised in opposite directions, and the polarisation ellipses have the same ratio of axes but are rotated relative to each other by $\pi/2$.

The boundary conditions on the surface of an anisotropic or gyrotropic body are the same as on the separation boundary between two isotropic media [*cf.* Eqns. (III.9) and (V.6)].

446. The direction of propagation of the extraordinary wave in a uniaxial crystal is at an angle θ to the optic axis. Determine the angle α between the wave vector $\mathbf{k}$ and the electric field $\mathbf{E}$, and also the angle ϑ between the ray direction (Poynting vector) and the optic axis of the crystal.

447. A plane wave is incident on the plane surface of a uniaxial crystal placed in a vacuum. The optic axis of the crystal is at right angles to its surface. Find the directions of propagation of the ordinary and extraordinary rays in the crystal if the angle of incidence is θ_0 .

448. Solve the preceding problem in the case where the optic axis of the crystal is parallel to its surface and makes an angle α with the plane of incidence.

449. A plane monochromatic wave propagates in an infinite ferrite medium which is magnetised to saturation at angle θ to a constant magnetic field. The magnetic permeability of the ferrite is given by the tensor

$$\mu_{ik} = \begin{pmatrix} \mu_{\perp} & -i\mu_a & 0 \\ i\mu_a & \mu_{\perp} & 0 \\ 0 & 0 & \mu_{\parallel} \end{pmatrix},$$

where the z -axis is parallel to the constant magnetic field. Assuming that the permittivity of the ferrite ϵ may be looked upon as a scalar[†], find the phase velocities of propagation $v_{1,2}$.

450. A plane monochromatic wave propagates in a dielectric placed in a constant uniform magnetic field. Assuming that the permeability of the dielectric is $\mu = 1$, and that the permittivity tensor (*cf.* Problem 318) is of the form

$$\epsilon_{ik} = \begin{pmatrix} \epsilon_{\perp} & -i\epsilon_a & 0 \\ i\epsilon_a & \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{pmatrix},$$

find the phase velocities of propagation.

451. Investigate the polarisation of waves which can propagate in an infinite ferrite medium magnetised to saturation. Consider the special cases where the propagation occurs along the constant magnetic field and where the propagation takes place at right angles to this field.

[†] This is so because the constant magnetic field has a much greater effect on the magnetic properties of the ferrite than on the electrical properties.

452. A dielectric is placed in an external magnetic field. A plane monochromatic wave propagates in the direction of the magnetic field (the z -axis) and is linearly polarised at the point $z = 0$. Determine the polarisation of the wave at points $z \neq 0$.

Hint: Use the permittivity tensor obtained in the solution of Problem 318.

453. A plane, circularly polarised wave is incident normally on the plane boundary of a ferrite placed in a vacuum. The ferrite is magnetised in the direction of incidence of the wave. Determine the polarisation and the amplitude of the reflected and transmitted waves.

Hint: Use the boundary conditions for the field vectors $\mathbf{E}$ and $\mathbf{H}$.

454. Solve the preceding problem in the case where the incident wave is linearly polarised.

455*. An artificial dielectric consists of thin perfectly conducting circular discs which are similarly oriented and placed in a vacuum. A constant magnetic field $\mathbf{H}_0$ is applied at right angles to the planes of the discs and a plane electromagnetic wave propagates in the direction parallel to the magnetic field. Determine the phase velocities of propagation, assuming that the dielectric may be looked upon as a continuous medium.

Hint: Take into account the Hall effect which appears as a result of the presence of the external magnetic field.

456*. Consider the propagation of longitudinal vibrations in a plasma for which the electric field vector $\mathbf{E}$ is parallel to the direction of propagation of the waves. Find the frequency of such oscillations.

Hint: Use the expression for the permittivity of a plasma obtained in Problem 312.

457. An ionised gas is placed in a constant magnetic field. A transverse plane wave propagates in the direction parallel to the field. Find the phase velocities of propagation. Consider the special case of low frequencies ($\omega \rightarrow 0$) and the nature of the electromagnetic waves, taking into account the motion of positive ions.

Hint: Use the expression for the permittivity tensor of an ionised gas in a constant magnetic field which was obtained in Problem 321.

458. Determine the magnetic permeability tensor $\mu_{ik}(\omega, \mathbf{k})$ of a ferroelectric without neglecting the term $q\nabla^2 \mathbf{M}$ in Eqn.(VI.16)

for the effective magnetic field. To do this, consider the motion of the magnetisation vector under the action of a plane monochromatic wave. The ferroelectric is magnetised to saturation by a constant magnetic field $\mathbf{H}_0$.

Hint: Confine the analysis to the case of small amplitudes and linearise the equation of motion of the magnetisation vector.

459. Find the dispersion relation for electromagnetic waves propagating in an isotropic ferroelectric magnetised to saturation without neglecting the term $q\nabla^2\mathbf{M}$ in the expression for the effective field [Eqn. (VI.16)]. Show that three types of waves with different dispersion relations can propagate in such a medium. Determine the explicit form of the dispersion relation for these waves, subject to the condition $\omega^2\epsilon/(ck)^2 \ll 1$. Estimate the relative magnitude of the electric and magnetic fields for this branch of the oscillations.

460. Determine the surface impedance ζ of a ferromagnetic conductor placed in a constant magnetic field which is parallel to its surface. Use the magnetic permeability tensor given in Problem 449 and assume that the components of the electrical conductivity tensor are $\sigma_{11} = \sigma_{22} = \sigma_1$, $\sigma_{33} = \sigma_3$, $\sigma_{12} = -\sigma_{21} = -i\sigma_2$, $\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0$.

Hint: The surface impedance is a tensor of rank 2 and should be determined from

$$E_{\tau i} = \zeta_{ik} (\mathbf{H}_{\tau} \times \mathbf{n})_k,$$

where $i, k = 1, 2$, $\mathbf{E}_{\tau i}$ and $\mathbf{H}_{\tau}$ are the tangential components of the field vectors near the surface of the conductor, and $\mathbf{n}$ is a unit normal.

461. Solve the preceding problem in the case where the constant magnetic field is normal to the surface of the ferromagnetic conductor.

Chapter

9

ELECTROMAGNETIC OSCILLATIONS IN BOUNDED BODIES

A part of space bounded on all sides by metal walls is called a cavity resonator. A system of standing waves with definite frequencies ω (natural frequencies of the resonator) may exist within the resonator. When the resonator is not filled with a dielectric and its walls are perfectly conducting, the standing wave system may be obtained by solving the equations

$$\Delta \mathbf{E} + \frac{\omega^2}{c^2} \mathbf{E} = 0, \quad \text{div } \mathbf{E} = 0. \quad (\text{IX.1})$$

subject to the boundary condition

$$\mathbf{E}_\tau = 0. \quad (\text{IX.2})$$

The actual natural frequencies of a resonator may differ from those computed from Eqns. (IX.1) and (IX.2) by an amount $\Delta\omega$ because of losses in the walls. For small attenuation, which may be described in terms of the surface impedance $\zeta^\dagger$ of the resonator walls, this difference is given by

$$\Delta\omega = -\frac{ic\zeta}{2} \cdot \frac{\oint |H|^2 dS}{\int |H|^2 dV}. \quad (\text{IX.3})$$

The integral in the numerator is evaluated over the inner surface and the integral in the denominator is taken over the volume of the resonator. The quantity $\Delta\omega = \Delta\omega' + i\Delta\omega''$ has a real and an imaginary part, and hence wall losses lead to a shift in the natural frequencies and to a damping of the natural oscillations.

In distinction to a resonator, a waveguide takes the form of a cavity (duct) of indefinite length. Travelling waves propagate along the axis of the waveguide (z -axis), while standing waves are

$\dagger$ Surface impedance is defined in Section 1, Chapter VIII.

set up in the transverse direction. In general, the waves in a waveguide will not be transverse. Waves in which $E_z \neq 0$, $H_z = 0$ are called electric-type or E -waves, while waves with $H_z \neq 0$, $E_z = 0$ are called magnetic-type or H -waves. Purely transverse waves are only possible in waveguides whose cross-sections are multiply connected.

The type of waves which may propagate in a given waveguide is determined by solving Maxwell's equations subject to the appropriate boundary conditions. A wave travelling along the axis of a waveguide may be described by

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{E}(x, y) e^{i(kz - \omega t)}, \quad \mathbf{H}(\mathbf{r}, t) = \mathcal{H}(x, y) e^{i(kz - \omega t)}, \quad (\text{IX.4})$$

where ω is the frequency and k is the component of the wave vector in the direction of the axis of the waveguide. The quantity k is also called the propagation constant.

In the case of electric-type waves (E -waves) $\mathcal{H}_z = 0$ and $\mathcal{E}_z$ is a solution of the equation

$$\Delta \mathcal{E}_z + \kappa^2 \mathcal{E}_z = 0, \quad (\text{IX.5})$$

where $\kappa^2 = \omega^2 \epsilon \mu / c^2 - k^2$, κ is the transverse component of the wave vector, ϵ and μ are the permittivity and magnetic permeability of the dielectric filling the waveguide, and the boundary condition on the waveguide wall is

$$\mathcal{E}_z = 0. \quad (\text{IX.6})$$

In the case of magnetic-type waves (H -waves) $\mathcal{E}_z = 0$ and $\mathcal{H}_z$ is the solution of

$$\Delta \mathcal{H}_z + \kappa^2 \mathcal{H}_z = 0, \quad (\text{IX.7})$$

which satisfies the boundary condition on the waveguide wall

$$\mathcal{E}_\tau = 0 \quad \text{or} \quad \frac{\partial \mathcal{H}_z}{\partial n} = 0. \quad (\text{IX.8})$$

The symbol Δ in Eqns. (IX.5) and (IX.7) is the two-dimensional Laplace operator. Strictly speaking, the boundary conditions (IX.6) and (IX.8) hold only for waveguides whose walls are perfectly conducting.

The transverse components of the vectors $\mathcal{E}$ and $\mathcal{H}$ may be expressed in terms of the longitudinal components of these vectors with the aid of Maxwell's equations.

An E -wave or an H -wave of a given type (*i.e.* with a given κ) can only propagate in a waveguide with a simply connected cross-section when its frequency is greater than a certain limiting frequency ω_0 . The corresponding "vacuum wavelength" $\lambda_0 = 2\pi c/\omega_0$ is of the order of the transverse linear dimensions of the waveguide. When $\omega < \omega_0$, the propagation constant k is purely imaginary, and wave propagation is impossible. However, even when $\omega > \omega_0$ the propagation constant k is in general complex. This is due to the fact that the walls of the waveguide have a finite conductivity and energy dissipation takes place within them, with the result that the electromagnetic wave is attenuated in accordance with the $\exp(-\alpha z)$ law. The attenuation coefficient α , which is the imaginary part of k , is equal to the energy dissipated per unit time within the walls of the waveguide per unit length divided by twice the energy flux along the waveguide. When the surface impedance $\zeta = \zeta' + i\zeta''$ is small, the attenuation coefficient for E -waves is approximately

$$\alpha = \frac{\omega \zeta''}{2\kappa k c} \cdot \frac{\oint |\nabla \mathcal{E}_z|^2 dl}{\int |\mathcal{E}_z|^2 dS}, \quad (\text{IX.9})$$

and the corresponding result for H -waves is

$$\alpha = \frac{c\kappa^2 \zeta''}{2k\omega} \cdot \frac{\oint [|\mathcal{H}_z|^2 + (k^2/\kappa^4) |\nabla \mathcal{H}_z|^2] dl}{\int |\mathcal{H}_z|^2 dS}. \quad (\text{IX.10})$$

where $\mathcal{E}_z$ and $\mathcal{H}_z$ are the field components for $\zeta = 0$, *i.e.* for perfectly conducting walls, dl is an element of the perimeter of the transverse cross-section of the waveguide, and dS is an area element of the cross-section.

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462. Determine the types of waves which can propagate in a rectangular waveguide with perfectly conducting walls and sides a and b . Find the corresponding dispersion relation and the field configurations, *i.e.* the dependence of the field components on the coordinates.

463. Determine the attenuation coefficients α of the various types of wave in a rectangular waveguide, given that the surface impedance of the waveguide walls is ζ .

464. An infinite layer of a dielectric material of permittivity ϵ and magnetic permeability μ fills the region $-a \leq x \leq a$. a . Show that the layer will act as a waveguide, *i. e.* the field of a travelling electromagnetic wave will be largely concentrated within the layer. Determine the types of wave which may propagate in such a waveguide. Confine the analysis to the special case where the field vectors are independent of the y coordinate.

465. A dielectric layer with permittivity ϵ and magnetic permeability μ fills the region $0 \leq x \leq a$ and is in contact with a perfectly conducting solid on one side. The region $x > a$ is evacuated. Determine the types of electromagnetic wave, with amplitudes decreasing with distance from the layer, which may propagate along the layer. Compare the possible types of wave with the wave system obtained in the preceding problem.

466. Find the possible types of wave in a circular waveguide of radius a assuming that the walls are perfectly conducting. Determine the limiting frequency ω_0 for this waveguide.

467. Using the result of the preceding problem, find the attenuation coefficients α of the various types of wave in a circular waveguide, assuming that the surface impedance is ζ .

468. Determine the phase and group velocities v_ϕ and v_g of waves in rectangular and circular waveguides with perfectly conducting walls and also the dependence of the wave velocities on $\lambda = 2\pi c/\omega$.

469. Determine the phase and group velocities v_ϕ and v_g of waves in a waveguide by a geometrical method. To do this, consider the simplest H_{10} wave in a rectangular waveguide, resolve it into plane waves, and investigate the reflection of these waves from the waveguide walls.

470. Investigate the structure of the transverse electromagnetic wave in a perfectly conducting coaxial line of inner and outer radii a and b respectively. Determine the average energy flux $\bar{\gamma}$ along the line. Investigate the limiting case of a single, perfectly conducting cylindrical conductor.

471. Determine the possible types of transverse electromagnetic waves in a coaxial line with perfectly conducting walls of radii a and $b > a$.

472. Determine the attenuation coefficient α of the transverse electromagnetic wave in a coaxial line of radii a and $b > a$ and surface impedance $\zeta = \zeta' + i\zeta''$.

Hint: Use the relation between the attenuation coefficient and the energy losses given at the beginning of this chapter.

473*. Discuss the propagation of an axially symmetric E -wave along an infinitely long cylindrical conductor having a finite conductivity and placed in a vacuum. Determine the phase velocity and show that in the case of a perfectly conducting cylinder the wave will become identical with the transverse electromagnetic wave considered in Problem 470. Use the approximate boundary condition given by (VIII.10)

474. An axially symmetric E -wave propagates in a circular waveguide of radius b which is partly filled with a dielectric. The dielectric has a permittivity ϵ and occupies the region $a \leq r \leq b$. If $a \ll b$, determine the phase velocity as a function of frequency and find the limiting frequency. Under what conditions will the phase velocity be less than c ? Consider the limiting case of a waveguide filled completely by the dielectric.

475. A rectangular waveguide with sides a and b and perfectly conducting walls is filled with a ferrodielectric. A constant magnetic field is applied at right angles to the wider wall of the waveguide (along the y -axis). The permittivity and magnetic permeability tensors of the ferrodielectric are given by

$$\epsilon_{ik} = \begin{pmatrix} \epsilon_{\perp} & 0 & -l\epsilon_a \\ 0 & \epsilon_{\parallel} & 0 \\ l\epsilon_a & 0 & \epsilon_{\perp} \end{pmatrix}, \quad \mu_{ik} = \begin{pmatrix} \mu_{\perp} & 0 & -i\mu_a \\ 0 & \mu_{\parallel} & 0 \\ i\mu_a & 0 & \mu_{\perp} \end{pmatrix}$$

(cf. the solution of Problem 329). Determine the components of the electromagnetic field, the propagation constant, and the limiting frequency when the field is independent of y .

476. The electric and magnetic fields in a waveguide with perfectly conducting walls are described by

$$\mathbf{E}_0 = \mathfrak{E}_0(x, y) e^{i(k_0 z - \omega t)}, \quad \mathbf{H}_0 = \mathfrak{H}_0(x, y) e^{i(k_0 z - \omega t)}.$$

When a dielectric core in the form of a cylinder of arbitrary cross-section is inserted into the waveguide with its longitudinal axis parallel to the axis of the waveguide, the fields are given by

$$\mathbf{E} = \mathfrak{E}(x, y) e^{i(kz - \omega t)}, \quad \mathbf{H} = \mathfrak{H}(x, y) e^{i(kz - \omega t)}.$$

The dielectric may in general be characterised by the tensorial parameters ϵ_{ik} , μ_{ik} . Show, with the aid of Maxwell's equations, that the change in propagation constant is given by

$$\Delta k = k - k_0 = \frac{\omega \int (\Delta \epsilon_{ik} \cdot \mathfrak{E}_k \mathfrak{E}_{0i}^* + \Delta \mu_{ik} \cdot \mathfrak{H}_k \mathfrak{H}_{0i}^*) dS}{c \int_S [(\mathfrak{E}_0^* \times \mathfrak{H}) + (\mathfrak{E} \times \mathfrak{H}_0^*)] \cdot \mathbf{e}_z dS},$$

where $\Delta\epsilon_{ik} = \epsilon_{ik} - 1$, $\Delta\mu_{ik} = \mu_{ik} - 1$. The integral in the numerator is evaluated over the cross-section of the dielectric rod (ΔS) and the integral in the denominator is taken over the cross-section of the waveguide (S).

477. A ferroelectric plate of thickness $d \ll a$ and magnetised along the axis of the waveguide is inserted into a rectangular waveguide with perfectly conducting walls (Fig. 26). Use the formula obtained in the preceding problem to determine the change Δk in the propagation constant for H_{10} waves to within terms of the order of d . Assume that the permittivity of the wall is a scalar, and the magnetic permeability tensor is as given in Problem 449.

478. A thin ferrite plate ($d \ll a, b$) is inserted into the coaxial waveguide shown in Fig. 27 and is magnetised in the

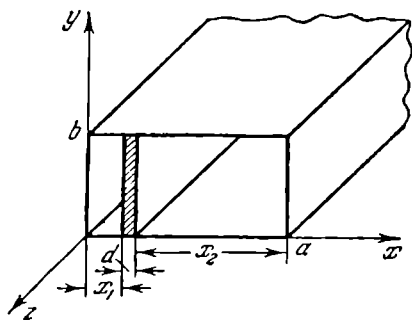


Fig. 26.

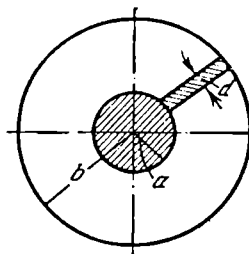


Fig. 27.

direction of the axis of the waveguide. Determine the change Δk in the propagation constant of the transverse electromagnetic wave.

Hint: Use the method employed in the preceding problem to find the amplitude of the disturbed fields.

479. Solve the preceding problem for a constant magnetising field H_0 applied at right angles to the axis of the waveguide. Consider two directions of this field, namely, H_0 perpendicular to the longer side of the plate and H_0 perpendicular to the shorter side of the plate.

480. Determine the types of proper oscillations in a cavity resonator with perfectly conducting walls. The resonator is in the form of a rectangular parallelepiped of sides a, b, h .

481. Determine the number of proper oscillations $\Delta N(\omega)$ per frequency interval $\Delta\omega$ in the hollow resonator considered in the preceding problem. Assume that $\Delta\omega \ll \omega$ and $\Delta N \gg 1$, and that the total volume of the resonator is V .

482. The walls of a rectangular resonator have a small surface impedance ζ . Determine the shift $\Delta\omega'$ in the proper frequency and the attenuation decrement $\Delta\omega''$ for oscillations with $n_1 = 0$, $n_2 = n_3 = 1$.

483. A given resonator is in the form of a right-circular cylinder of height h and radius a . Assuming that the walls are perfectly conducting, find the frequency of the proper oscillations. Consider the E and H oscillations.

Hint: Use the solution for a circular waveguide of Problem 466.

Chapter

10

SPECIAL THEORY OF RELATIVITY

1. Lorentz transformation

The space and time coordinates in two inertial reference frames S and S' are related by the Lorentz transformation formulae

$$x = \gamma(x' + Vt'), \quad y = y', \quad z = z', \quad t = \gamma\left(t' + \frac{Vx'}{c^2}\right), \quad (\text{X.1})$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ and $\beta = V/c$. It is assumed that S' moves with a constant velocity V along the Ox -axis relative to S in such a way that the coordinate axes remain parallel to each other, the x - and x' -axes coincide, and the origins of S and S' coincide at times $t = t' = 0$. The reverse Lorentz transformation is obtained by changing the sign of V :

$$x' = \gamma(x - Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{Vx}{c^2}\right). \quad (\text{X.2})$$

The quantities (x_1, x_2, x_3) and $x_4 = ict$ are the coordinates of the world point

$$\left. \begin{aligned} x_i &= (r, ict), \\ r &= (x_1, x_2, x_3). \end{aligned} \right\} \quad (\text{X.3})$$

The four-dimensional vector (4-vector) $A_i = (\mathbf{A}, A_4) = (A_1, A_2, A_3, A_4)$ is defined as the set of quantities A_1, A_2, A_3, A_4 which transform as space and time coordinates in accordance with (X.1) so that (cf. Section 2 of this chapter)

$$A_1 = \gamma(A'_1 - i\beta A'_4), \quad A_2 = A'_2, \quad A_3 = A'_3, \quad A_4 = \gamma(A'_4 + i\beta A'_1). \quad (\text{X.4})$$

The vector $\mathbf{A} = (A_1, A_2, A_3)$ is called the space component of the 4-vector A_i and the quantity A_4 is called the time component of this vector.

The squares of 4-vectors, A_i^2 , and their scalar products are invariant with respect to Lorentz transformations:

$$A_i^2 = \text{inv.}, \quad A_i B_i = \text{inv.} \quad (\text{X.5})$$

A 4-vector A_i is called spacelike if $A_i^2 > 0$ and timelike if $A_i^2 < 0$.

The invariant quantity

$$s_{12} = [(\mathbf{r}_1 - \mathbf{r}_2)^2 - c^2(t_1 - t_2)^2]^{\frac{1}{2}} \quad (\text{X.6})$$

is defined as the interval between two events with coordinates $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$.

The time indicated by a clock moving together with a given object is called the proper time of the object. If the object moves relative to the frame S with a velocity V , then the proper time interval $d\tau$ may be expressed in terms of the time interval dt in the frame S , and is given by

$$d\tau = dt \sqrt{1 - V^2/c^2}. \quad (\text{X.7})$$

The quantity $dt\sqrt{1 - \beta^2}$ is invariant under the Lorentz transformation.

If a given rod has a length l_0 at rest, then when it moves with a velocity v in the direction of its longitudinal axis, its length as measured by an observer at rest is given by

$$l = l_0 \sqrt{1 - v^2/c^2}. \quad (\text{X.8})$$

The four-dimensional velocity is defined as the 4-vector whose components are given by

$$u_i = \frac{dx_i}{d\tau} = \left(\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \frac{ic}{\sqrt{1 - v^2/c^2}} \right), \quad (\text{X.9})$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the ordinary particle velocity. It is clear from Eqn. (X.9) that

$$u_i^2 = -c^2. \quad (\text{X.10})$$

The 4-velocity transforms in accordance with (X.4) just like any other 4-vector.

The components of ordinary velocity are not the space components of any particular 4-vector, and transform in accordance with the following expressions ($\mathbf{V} \parallel x$):

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2}, \quad v_y = \frac{v'_y \sqrt{1 - V^2/c^2}}{1 + v'_x V/c^2}, \quad v_z = \frac{v'_z \sqrt{1 - V^2/c^2}}{1 + v'_x V/c^2}. \quad (\text{X.11})$$

If the particle velocity is at an angle ϑ and ϑ' to the x -axis in S and S' respectively, then

$$\tan \vartheta = \frac{v' \sqrt{1 - V^2/c^2} \sin \vartheta'}{v' \cos \vartheta' + V}, \quad v' = \sqrt{v_x'^2 + v_y'^2 + v_z'^2}. \quad (\text{X.12})$$

The four-dimensional acceleration of a particle is the 4-vector with components

$$w_l = \frac{du_l}{d\tau} = \frac{d^2 x_l}{d\tau^2}. \quad (\text{X.13})$$

The wave vector $\mathbf{k}$ and the frequency ω of a plane electromagnetic wave are the components of the wave 4-vector

$$k_l = \left(\mathbf{k}, \frac{i\omega}{c} \right). \quad (\text{X.14})$$

It follows that the phase of a plane wave $\varphi = k_i x_i$ is an invariant quantity.

It follows from (X.4) that the transformation formulae for the angle ϑ between the light ray and the x -axis are

$$\tan \vartheta = \frac{\sin \vartheta'}{\gamma (\cos \vartheta' + \beta)} \quad \text{or} \quad \cos \vartheta = \frac{\cos \vartheta' + \beta}{1 + \beta \cos \vartheta'}. \quad (\text{X.15})$$

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484. Suppose that a system S' moves relative to a system S with a velocity V along the x -axis. When clocks at $S' (x'_0, y'_0, z'_0)$ and $S (x_0, y_0, z_0)$ pass each other they indicate times t'_0 and t_0 respectively. Write down the Lorentz transformation formulae for this case.

485. A system S' moves relative to a system S with a velocity V . Show that when clocks in S and S' are compared, the clock is one of the systems, whose readings are successively compared with the readings of two clocks in the second frame (synchronised within that frame), will always lag behind. Express one of the two time intervals in terms of the other. (The readings of the moving clocks are compared when they pass each other.)

486. The length of a rod moving in the direction of its longitudinal axis in a given reference frame may be found by determining the time interval taken by the rod to travel past a point fixed in the frame and multiplying it by the velocity of the rod.

Show that this method of measuring the length of a rod will yield the usual Lorentz contraction [cf. Eqn. (X.8)].

487. A system S' moves relative to a system S with a velocity V . Clocks at rest at the origins of the two frames are found to be synchronised ($t = t' = 0$) when they pass each other. Find the coordinates in each system of the world point which has the property that clocks at rest in S and S' indicate the same time $t = t'$ at this point. Determine the law of motion of the world point.

488. A train $A'B'$ (cf. Fig. 73 in the solutions) has a length $l_0 = 8.64 \times 10^8$ km in the system in which it is at rest. The train travels past a platform AB having an equal length in its own rest frame with the velocity $V = 240\,000$ km sec⁻¹. Identical synchronised clocks are placed at A , B and A' , B' . When the head of the train (B') becomes level with the beginning of the platform (A), the corresponding clocks both indicate 12 h 00 min. Answer the following questions: (a) Is there a reference frame in which all four clocks will show 12 h 00 min? (b) What will be the time shown by the four clocks when A' becomes level with A ? (c) What will be the time shown by the clocks when the head of the train B' becomes level with B ?

489. Determine the time (as measured by a clock at rest on the earth's surface) taken by a rocket to reach a distant star and return to earth with a constant velocity $v = \sqrt{0.9999}c$, if the distance to the star is four light years†. Use the result to determine the time interval which may be used as a basis for estimating the amount of food and other supplies for the journey. Calculate the kinetic energy of the rocket if its mass is 10 tons.

490. Two rulers, each having a length l_0 at rest, move in opposite directions with uniform velocities along the x -axis. An observer at rest relative to one of the rulers notes that the time interval between instants at which the left and right ends of the rulers pass each other is Δt . What is the relative velocity of the two rulers? In what order will the ends of the rulers pass each other for observers at rest relative to either of the rulers, and also for an observer with respect to whom both rulers move with equal velocities in opposite directions?

† A light year is defined as the distance travelled by a light beam in vacuum in one year.

491. Derive the Lorentz transformation formulae without assuming that the velocity of S' relative to S is parallel to the x -axis. Give the result in a vector form.

Hint: Resolve the position vector $\mathbf{r}$ into the longitudinal and transverse components relative to the direction of the relative velocity $\mathbf{V}$ and use the Lorentz transformation (X.1).

492. Write down the Lorentz transformation formulae for an arbitrary 4-vector $A_i = (A, A_4)$ without assuming that the velocity $\mathbf{V}$ of the system S' relative to S is parallel to the x -axis.

493. Derive the velocity addition formulae when the velocity $\mathbf{V}$ of a system S' relative to S has an arbitrary direction. Give the result in a vector form.

494. Three reference frames, S , S' , and S'' are such that S'' moves relative to S' with a velocity V' in the direction parallel to the x' -axis, and S' moves relative to S with a velocity V in the direction parallel to the x -axis. The corresponding axes of all the three systems remain parallel. Find the Lorentz transformation connecting S'' and S and use it to obtain a formula for the addition of parallel velocities.

495. Prove the formula

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{V \sqrt{1 - v'^2/c^2} \cdot \sqrt{1 - V^2/c^2}}{1 + \mathbf{v}' \cdot \mathbf{V}/c^2},$$

where $\mathbf{v}$ and $\mathbf{v}'$ are the velocities of a particle relative to frames S and S' , and $\mathbf{V}$ is the velocity of S' relative to S .

496. Prove the relation

$$v = \frac{V(\mathbf{v}' + \mathbf{V})^2 - (\mathbf{v}' \times \mathbf{V})^2/c^2}{1 + \mathbf{v}' \cdot \mathbf{V}/c^2},$$

where $\mathbf{v}$ and $\mathbf{v}'$ are the velocities of a particle relative to frames S and S' , and $\mathbf{V}$ is the velocity of S' relative to S .

497. Two rulers, each of which has a length l_0 in its own rest frame, move towards each other with equal velocities v relative to a given reference system. Find the length l of each of the rulers in the reference frame in which the other ruler is at rest.

498. Solve the preceding problem on the assumption that the rulers move with equal velocities v at 90° to each other in a stationary reference frame.

499. Two electron beams travel along the same straight line but in opposite directions with velocities $v = 0.9c$ relative

to the laboratory system. Find the relative velocity V of the electrons as measured by an observer at rest in the laboratory and also by an observer moving together with one of the electron beams.

500. The effects which occur during a collision between two elementary particles are independent of the uniform motion of the two particles as a whole. Such effects depend only on their relative velocity. A given relative velocity may be communicated to the colliding particles in two ways (we shall assume for simplicity that they have the same mass): (1) an accelerator is used to accelerate one of the particles to an energy $\mathcal{E}$ while the other is at rest, or (2) both particles are accelerated towards each other by identical machines until they reach an energy $\mathcal{E}_0 < \mathcal{E}$. Compare the two energies $\mathcal{E}$ and $\mathcal{E}_0$ and consider in particular the ultra-relativistic case.

501. Find the transformation formulae for the acceleration $\dot{\mathbf{v}}$ in the case when the system S' moves relative to the system S with an arbitrary velocity $\mathbf{V}$. Express your result in vector form.

502. Express the components of the four-dimensional acceleration w_i in terms of the ordinary acceleration $\dot{\mathbf{v}}$ and velocity $\mathbf{v}$. Find w_i^2 . Is the four-dimensional acceleration space-like or timelike?

503. Express the acceleration $\dot{\mathbf{v}}'$ of a particle in the inertial frame in which it is instantaneously at rest in terms of its acceleration $\dot{\mathbf{v}}$ in the laboratory system. Consider the cases when $\mathbf{v}$ varies only in magnitude or only in direction.

504. The rocket considered in Problem 489 is accelerated from rest to a velocity $v = \sqrt{0.9999}c$. The acceleration of the rocket in the system in which it is instantaneously at rest is $|\dot{\mathbf{v}}| = 20 \text{ m sec}^{-2}$. Determine the time taken by the rocket to reach this velocity as measured by clocks in the frame in which it was originally at rest, and clocks at rest in the rocket.

Hint: Neglect the effect of inertial forces on the clocks in the rocket, *i.e.* determine the sum of the proper time intervals $d\tau = dt\sqrt{1 - v^2/c^2}$ in the inertial frames in which the rocket is successively instantaneously at rest.

505. A system S' moves with a velocity $\mathbf{V}$ relative to a system S . The velocities of two bodies relative to S are respectively $\mathbf{v}_1$ and $\mathbf{v}_2$. Find the angle α between the velocities of the two bodies as measured in S and in S' .

Hint: Use the results of Problems 493 and 495.

506. What happens to the angle between the two velocities in the preceding question when the velocity of S' relative to S tends to the velocity of light c ?

507. At a particular instant of time the direction of the rays of light from a star are at an angle ϑ to the orbital velocity $\mathbf{v}$ of the earth in the frame in which the sun is at rest. Find the change in the direction of the earth-star line after a period of six months without assuming that v/c is small. This phenomenon is known as the aberration of light.

508. Find the form of the apparent curve, described by a star on the celestial sphere, which is due to the annual aberration. Assume that the polar coordinates of the star in the frame in which the sun is at rest are ϑ, α , where the polar axis is drawn at right angles to the plane of the earth's orbit. The orbital velocity of the earth may be assumed to be very much smaller than the velocity of light.

509. A beam of light subtends a solid angle $d\Omega$ in a given reference system. Find the change in this angle when it is measured in another inertial reference system.

510. Assuming that the stars in the part of the galaxy which is nearest to us are distributed uniformly, determine their distribution $dN/d\Omega'$ as measured by an observer in a rocket moving with a velocity approaching the velocity of light.

511. Find the transformation formulae for the frequency ω and the wave vector $\mathbf{k}$ of a plane monochromatic light wave between two inertial systems. Assume that the direction of the relative velocity $\mathbf{V}$ is arbitrary.

512. Find the frequency ω of a light wave as observed in the case of the transverse Doppler effect (the direction of propagation is at right angles to the direction of motion of the source in the system in which the detector of the radiation is at rest). Find the direction of propagation of the wave in the system in which the source is at rest.

513. The wavelength of light emitted by a source in the system in which it is at rest is λ_0 . Find the wavelength λ measured by an observer approaching the source with a velocity V , and an observer moving with the same velocity away from the source.

514*. A mirror moves in a direction at right angles to its plane with a velocity $\mathbf{V}$. Find the law of reflection of a plane monochromatic wave from the mirror, and also the change in the frequency on reflection. Consider in particular the case $V \rightarrow c$.

515. Solve the preceding problem in the case when the mirror travels without rotation in the direction parallel to its own plane.

516. Set up the wave 4-vector describing the propagation of a plane monochromatic wave in a medium having a refractive index n and moving with a velocity V (the phase velocity of the wave in a stationary medium may be taken to be $v' = c/n$). Find the transformation formulae for the frequency, the angle of propagation, and the phase velocity.

517. A plane wave propagates in a medium moving with a velocity V in the direction of displacement of the medium. The wavelength in a vacuum is λ . Find the velocity v of the wave in the laboratory system (Fizeau experiment). The refractive index n is determined in the system S' in which the medium is at rest and depends on the wavelength λ' in this system. Neglect terms of second order in V/c .

2. Four-dimensional vectors and tensors

In going over from one inertial system S' to another inertial system S , the coordinates of a world point $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$ transform in accordance with the formulae

$$x_l = \alpha_{lk} x'_k. \quad (\text{X.16})$$

The transformation matrix is orthogonal, so that

$$\alpha_{il} \alpha_{kl} = \delta_{lk}, \quad \alpha_{il} \alpha_{lk} = \delta_{lk}, \quad (\text{X.17})$$

and hence the reverse transformation may be written

$$x'_l = \alpha_{kl} x_k. \quad (\text{X.18})$$

The quantity $s = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}$ is invariant under the Lorentz transformation.

The transformation matrix corresponding to (X.4) is of the form

$$\hat{\alpha} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \quad (\text{X.19})$$

When the Lorentz transformation is applied repeatedly, the corresponding transformation matrices form a product in accordance

with the usual rule for the multiplication of matrices (*cf.* Section 1 of Chapter 1).

The components A_i of a four-dimensional vector, *i.e.* a 4-vector, transform as the coordinates of a world point:

$$A_i = \alpha_{ik} A'_k, \quad A'_i = \alpha_{ki} A_k. \quad (\text{X.20})$$

A four-dimensional tensor (4-tensor) of rank N is defined as the set of 4^N quantities $T_{ik\dots l}$, which transform as the product of the corresponding coordinates $(x_i, x_k, \dots, x_l)$:

$$T_{ik\dots l} = \alpha_{ip} \alpha_{kr} \dots \alpha_{ls} T'_{pr\dots s}. \quad (\text{X.21})$$

The determinant $|\alpha_{ik}|$ of the matrix $\hat{\alpha}$ may be equal either to ± 1 or to -1 . When the determinant is equal to $+1$, the transformation may be looked upon as a rotation in the four-dimensional space. When the determinant is equal to -1 , the transformation includes reflection of one or three coordinates.

A pseudotensor of rank N is defined as the set of 4^N quantities $P_{ik\dots l}$ which transform in accordance with the formulae

$$P_{ik\dots l} = \alpha_{ip} \alpha_{kr} \dots \alpha_{ls} |\alpha_{mn}| P_{pr\dots s}. \quad (\text{X.22})$$

The skew-symmetric unit pseudotensor of rank 4 (*cf.* Problem 521 below) is an example of a pseudotensor. Its components e_{iklm} are defined by the following conditions: (a) e_{iklm} changes sign when any pair of subscripts is reversed, and (b) $e_{1234} = 1$. It follows that the components of e_{iklm} will vanish when all the subscripts are identical, or will be equal to ± 1 when the subscripts are all different.

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518. Show that the components A_1, A_2, A_3 of the four-dimensional tensor $A_i = (A_1, A_2, A_3, A_4)$ transform on space rotation as the components of the three-dimensional vector $\mathbf{A} = (A_1, A_2, A_3)$ and also that the component A_4 is a three-dimensional scalar.

519. Find the three-dimensional tensors into which a 4-tensor of rank 2 is found to divide as a result of space rotations.

520. Show that the components of a skew-symmetric 4-tensor of rank 2 transform on space rotation as the components of two independent three-dimensional vectors.

521. Show that the quantity e_{iklm} which was defined in the introduction to this section does in fact transform as a pseudo-tensor.

522. Show that

$$\begin{aligned} e_{iklm}e_{lmrs} &= 2(\delta_{ir}\delta'_{ks} - \delta_{is}\delta_{kr}); \\ e_{iklm}e_{klmn} &= -6\delta_{ln}. \end{aligned}$$

523. Show that

$$e_{iklm}e_{lmrs}A_iB_kC_rD_s = 2(A_iC_i)(B_kD_k) - 2(A_iD_i)(B_kC_k).$$

524. Find the transformation laws for the following quantities: (a) $\partial\varphi/\partial x_i$, (b) $\partial A_i/\partial x_k$, and (c) $T_{ik}A_k$, where φ is a scalar, A_i a vector, and T_{ik} a tensor.

525. Two 4-vectors A_i and B_i are called parallel if

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3} = \frac{A_4}{B_4}.$$

Show that the ratio of the corresponding components of parallel 4-vectors is invariant under a Lorentz transformation.

526. Find the number of different components of a 4-tensor of rank 3 which is skew-symmetric with respect to the transposition of any pair of subscripts. Show that the components transform on rotation as the components of a four-dimensional pseudovector.

527. Three reference systems S , S' , and S'' are such that S'' moves relative to S' with a velocity V' in the direction parallel to the x' -axis and S' moves relative to S with a velocity V in the direction of the x -axis. Corresponding axes of the three systems remain parallel. By multiplying together the corresponding matrices, find the matrix for the transformation from S'' to S . Hence deduce the formula for the addition of parallel velocities [cf. (X.11)].

528. Write down the Lorentz transformation (X.1) in terms of the variables $x_1, x_2, x_3, x_0 = ct$ by expressing the magnitude of the relative velocity V in terms of the angle α given by $\tanh \alpha = V/c$.

529. A system S' moves relative to a system S with a velocity $\mathbf{V}$ ($\tanh \alpha = V/c$) in the direction defined by the polar angles ϑ, φ . The corresponding axes of S and S' remain parallel. Find the matrix $\hat{g}$ for the transformation from S' to S by multiplying together the simple transformation matrices.

3. Relativistic electrodynamics

In this section we shall summarise the fundamental formulae of relativistic electrodynamics in a vacuum. The three-dimensional current density $\mathbf{j} = \rho \mathbf{v}$ and the charge density ρ form the current density 4-vector

$$j_i = (\mathbf{j}, i c \rho). \quad (\text{X.23})$$

The electric and magnetic fields are the components of the skew-symmetric electromagnetic-field 4-tensor

$$F_{ik} = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}. \quad (\text{X.24})$$

The field components in two systems S' and S , whose x and x' axes are parallel to the relative velocity of the two systems, transform in accordance with the following formulae

$$\begin{aligned} E_x &= E'_x, \quad E_y = \gamma(E'_y + \beta H'_z), \quad E_z = \gamma(E'_z - \beta H'_y); \\ H_x &= H'_x, \quad H_y = \gamma(H'_y - \beta E'_z), \quad H_z = \gamma(H'_z + \beta E'_y). \end{aligned} \quad (\text{X.25})$$

The quantities

$$H^2 - E^2 = \text{inv}, \quad \mathbf{E} \cdot \mathbf{H} = \text{inv} \quad (\text{X.26})$$

are invariant under the Lorentz transformation. The vector and scalar potentials $\mathbf{A}$ and φ form the potential 4-vector

$$A_i = (\mathbf{A}, i\varphi). \quad (\text{X.27})$$

The components of the energy-momentum tensor in a vacuum are defined by

$$T_{ik} = \frac{1}{4\pi} \left(F_{il} F_{kl} - \frac{1}{4} \delta_{ik} F_{lm}^2 \right). \quad (\text{X.28})$$

The nine space components of the tensor T_{ik} form the three-dimensional Maxwell stress tensor

$$T_{\alpha\beta} = \frac{1}{4\pi} (-E_\alpha E_\beta - H_\alpha H_\beta) + \frac{1}{8\pi} (E^2 + H^2) \delta_{\alpha\beta}. \quad (\text{X.29})$$

The space-time components of T_{ik} are proportional to the components of the energy flux density $\mathbf{S}$ and the field momentum

density $\mathbf{g}$:

$$\left. \begin{aligned} T_{4a} &= \frac{i}{c} S_a, \quad \mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}), \\ T_{a4} &= ic g_a, \quad \mathbf{g} = \frac{1}{4\pi c} (\mathbf{E} \times \mathbf{H}) = \frac{1}{c^2} \mathbf{S}. \end{aligned} \right\} \quad (\text{X.30})$$

The time component of T_{ik} is related to the field energy density W by the formula

$$T_{44} = -w = -\frac{1}{8\pi} (E^2 + H^2). \quad (\text{X.31})$$

The divergence of the tensor T_{ik} determines the volume density of forces $f_i = \left[\mathbf{f}, \frac{i}{c} (\mathbf{v} \cdot \mathbf{f}) \right]$ acting on the charges:

$$\frac{\partial T_{ik}}{\partial x_k} = -f_i = -\frac{1}{c} F_{ik} J_k. \quad (\text{X.32})$$

Let us consider now the formulae of electrodynamics in the presence of media. The field vectors $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$ form two skew-symmetric four-dimensional tensors of rank 2, *i.e.*, the field tensor

$$F_{ik} = \begin{pmatrix} 0 & B_z & -B_y & iE_x \\ -B_z & 0 & B_x & iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \quad (\text{X.33})$$

and the induction tensor

$$H_{ik} = \begin{pmatrix} 0 & H_z & -H_y & iD_x \\ -H_z & 0 & H_x & iD_y \\ H_y & -H_x & 0 & -iD_z \\ iD_x & iD_y & iD_z & 0 \end{pmatrix}. \quad (\text{X.34})$$

The polarisation and magnetisation vectors $\mathbf{P}$ and $\mathbf{M}$ form the following 4-tensor

$$M_{ik} = \begin{pmatrix} 0 & M_z & -M_y & iP_x \\ -M_z & 0 & M_x & iP_y \\ M_y & -M_x & 0 & iP_z \\ -iP_x & -iP_y & -iP_z & 0 \end{pmatrix}. \quad (\text{X.35})$$

The formulae $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ and $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ may be joined into the single relation

$$H_{ik} = F_{ik} - 4\pi M_{ik}. \quad (\text{X.36})$$

The 4-force f_i per unit volume of the medium is given by

$$f_i = \left\{ \mathbf{f}, \frac{1}{c} [Q + (\mathbf{f} \cdot \mathbf{v})] \right\}, \quad (\text{X.37})$$

where $\mathbf{f}$ is the pondermotive force per unit volume and Q is the Joule heat liberated per unit volume per unit time.

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530. Write down the transformation formulae for the field vectors $(\mathbf{E}, \mathbf{B})$, $(\mathbf{D}, \mathbf{H})$ and the polarisations $(\mathbf{P}, \mathbf{M})$ in the case where the S' system moves relative to the S system with an arbitrarily oriented velocity $\mathbf{V}$. Express the transformation formulae in a vector form.

Hint: Use the expression for the transformation coefficients in Problem 529 and the asymmetry of the tensors F_{ik} , H_{ik} , and M_{ik} .

531. A uniform electromagnetic field $\mathbf{E}, \mathbf{H}$ exists in the reference system S . Find the velocity of the system S' relative to S in which $\mathbf{E}' \parallel \mathbf{H}'$. Does this problem always have a solution, and if so, is it always unique? Find the absolute values of $\mathbf{E}'$ and $\mathbf{H}'$.

532. The electric and magnetic fields in a reference system S are mutually perpendicular ($\mathbf{E} \perp \mathbf{H}$). Find the velocity of the system S' relative to S in which only the electric or only the magnetic field is present. Does this problem always have a solution, and if so, is it unique?

533. An infinitely long circular cylinder is uniformly charged with a density κ per unit length. A uniformly distributed current $\mathcal{I}$ flows in the direction of the axis of the cylinder. The permittivity and the magnetic permeability are equal to unity in all space. Find the reference system in which there is only an electric or only a magnetic field, and determine the magnitude of these fields.

534. Find the magnitude of the e.m.f. induced in a conductor moving in a magnetic field $\mathbf{B}$, using either the transformation formulae for the field strength or the transformation formulae for the potentials.

535. Find the potentials $\varphi, \mathbf{A}$ and fields $\mathbf{E}, \mathbf{H}$ due to a point charge e moving with a uniform velocity $\mathbf{V}$ by carrying out the Lorentz transformation from the reference system in which the charges are at rest.

536. Show that the electric field due to a uniformly moving point charge is "compressed" in the direction of motion, and that the field E on the line of motion of the charge is reduced as compared with the Coulomb field. Is this reduction in the field consistent with the transformation formula $E_{||} = E'_{||}$?

537. An electric dipole having a moment $\mathbf{p}_0$ in the reference system in which it is at rest, moves with a uniform velocity $\mathbf{V}$. Find the potentials ϕ , $\mathbf{A}$ and the fields $\mathbf{E}$, $\mathbf{H}$ due to the dipole.

538. Derive the transformation formulae for the electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$ of a polarised and magnetised body between the inertial system in which the body is at rest and another inertial system.

Hint: Use the transformation formulae for the polarisation and magnetisation vectors $\mathbf{P}$ and $\mathbf{M}$.

539. An uncharged wire loop in the form of a rectangle of sides a and b carries a current $\mathcal{J}'$ and moves with a uniform velocity $\mathbf{V}$ which is parallel to the side of length a . The wire has a finite cross-section. Find the distribution of electric charges in the loop and its electric and magnetic moments in the laboratory system.

540. Deduce the relativistic transformation law for the Joule heat Q from the definition of the 4-vector of force.

541. Find the transformation formulae for the energy momentum tensor T_{ik} under the Lorentz transformation (X.1).

542. Find the sum of the diagonal elements (*i.e.* the spur) of the energy-momentum tensor (X.28).

543*. An electromagnetic field exists only within a finite volume of space V in which there are no charges. Show that the total energy and momentum of the field transform as a 4-vector.

544*. The total angular momentum of a system consisting of an electromagnetic field in vacuum and a set of point charges may be defined by

$$K_{ik} = -\frac{i}{c} \int (x_i T_{kl} - x_k T_{il}) dS_l + \sum (x_i p_k - x_k p_i), \quad \dagger$$

† It can be easily verified using the definition of the tensor T_{ik} that the space part $K_{\alpha\beta}$ of the tensor K_{ik} is a skew-symmetric tensor equivalent to the vector $\mathbf{K} = \int (\mathbf{r} \times \mathbf{g}) dV + \sum \mathbf{r} \times \mathbf{p}$, where $\mathbf{g} = \frac{1}{4\pi c} (\mathbf{E} \times \mathbf{H})$ is the field momentum density.

where the integral is taken over the entire hypersurface $x_4 = ict = \text{const.}$ The summation is carried out over all the particles, and the values of x_i, p_k are taken at the points of intersection of the world lines of the corresponding charges and the hypersurface $x_4 = \text{const.}$ Prove the conservation of the total angular momentum K_{ik} of the system, taking into account the fact that $\partial T_{ik}/\partial x_k = -F_{ik}j_k/c$.

545*. A particular system, which consists of particles and an electromagnetic field, occupies a finite evacuated region of space. By considering the balance of the total angular momentum $K_{\alpha\beta}$ of the system, find an expression for the flux density $\mathfrak{H}$ of the angular momentum of the field. Use the expression for K_{ik} given in the preceding problem.

Chapter

11

RELATIVISTIC MECHANICS

1. Energy and momentum

The momentum $\mathbf{p}$ of a relativistic particle is given by

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (\text{XI.1})$$

where $\mathbf{v}$ is the velocity of the particle and m is its rest mass. The total energy $\mathcal{E}$ of a freely moving particle can be expressed in terms of the velocity:

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (\text{XI.2})$$

or in terms of the momentum:

$$\mathcal{E} = c \sqrt{p^2 + m^2 c^2}. \quad (\text{XI.3})$$

The kinetic energy T of a particle is given by

$$T = \mathcal{E} - mc^2, \quad (\text{XI.4})$$

where $\mathcal{E}_0 = mc^2$ is the rest energy. The energy, the momentum, and the velocity of a particle are related by

$$\mathcal{E}\mathbf{v} = c^2\mathbf{p}. \quad (\text{XI.5})$$

The energy and the momentum of a particle are the time and space components of the energy-momentum 4-vector (4-momentum)

$$p_i = \left(\mathbf{p}, \ i \frac{\mathcal{E}}{c} \right). \quad (\text{XI.6})$$

In the transition from one inertial system into another, the momentum transforms in accordance with (X.4). The square of

the 4-momentum is a relativistic invariant:

$$p_i^2 = p^2 - \frac{\mathcal{E}^2}{c^2} = -m^2c^2. \quad (\text{XI.7})$$

A particle is referred to as non-relativistic if its kinetic energy is small, and ultra-relativistic if its kinetic energy is very large, as compared with the rest energy. The velocity of an ultra-relativistic particle approaches the velocity of light, and its energy is given by

$$\mathcal{E} = cp. \quad (\text{XI.8})$$

Particles with zero rest mass and zero rest energy (photons and neutrinos) are always ultra-relativistic and their velocity is exactly equal to the velocity of light c .

The energy and momentum of a photon in a vacuum are given by

$$\mathcal{E} = \hbar\omega, \quad p = \frac{\hbar\omega}{c} = \hbar k, \quad (\text{XI.9})$$

where ω is the frequency and $\hbar = 1.05 \times 10^{27}$ erg sec is Planck's constant divided by 2π .

The total energy and momentum of an isolated system of particles is conserved. It follows that if there is no interaction between the particles before and after a particular reaction (disintegration or collision) then the initial total 4-momentum $p_i^{(0)}$ is equal to the final total 4-momentum p_i :

$$\sum_a p_{ai}^{(0)} = \sum_b p_{bi}, \quad (\text{XI.10})$$

where the summation is carried out over all the particles existing before and after the reaction.

In collision theory it is convenient to use either the laboratory system S , or the centre of mass system S' (CM system) in which the total momentum $\mathbf{p}$ is zero. The useful device consisting of making use of the invariance of the squares of 4-momenta should be noted (*cf.* solutions to Problems 561, 574, and 580).

There are two types of collisions, namely, elastic, in which there is no change in the internal state and therefore no change in the masses of particles, and inelastic, in which there is a change in the internal energy (mass) of the colliding particles, and creation or annihilation of particles may take place. In an inelastic

collision between two particles, the total mass $m_1 + m_2$ of the colliding particles differs from the total mass M_k of the particles emerging from the reaction by the amount given by

$$\Delta M = m_1 + m_2 - M_k. \quad (\text{XI.11})$$

The quantity $Q = c^2 \Delta M$ is known as the Q value of the reaction.

Table (XI.1) gives the rest energy $\mathcal{E}_0$ of a number of elementary particles. The rest energy of the electron $m_e c^2$ is a convenient unit of energy. Another convenient unit is the electron-volt which is defined by $1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg}$,

TABLE XI.1

Particle	$\mathcal{E}_0$	
	in units of $m_e c^2$	in MeV
Electron	1	0.511
μ -meson	207	106
π^+ -meson	273	139
π^0 -meson	264	135
K -meson	966	494
Proton	1836	938
Λ -hyperon	2181	1115

TABLE XI.2

Element	B , MeV
H_1^2	2.18
He_2^4	28.11
Li_3^7	38.96

$1 \text{ MeV} = 10^6 \text{ eV}$, $1 \text{ GeV} = 10^9 \text{ eV}$. Table (XI.2) gives the binding energies of three light nuclei. The binding energy is defined by

$$B = \Delta M c^2 = \sum \mathcal{E}_{0n} - \mathcal{E}_{0N},$$

where $\mathcal{E}_{0n}$ is the rest energy of a nucleon and $\mathcal{E}_{0N}$ is the rest energy of the nucleus.

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546. Express the momentum p of a relativistic particle in terms of its kinetic energy T .

547. Express the velocity $\mathbf{v}$ of a particle in terms of its momentum $\mathbf{p}$.

548. A particle of rest mass m has an energy $\mathcal{E}$. Find the velocity v of the particle. Discuss in particular the non-relativistic and the ultra-relativistic limits.

549. Find approximate expressions for the kinetic energy T of a particle of rest mass m in terms of (a) its velocity v , and (b) its momentum p , to within terms of the order of v^4/c^4 and $p^4/m^4 c^4$ respectively, where $v \ll c$.

550. Find the velocity v of a particle of rest mass m and charge e after it has traversed a potential difference V (assume that the initial velocity is zero). Consider the non-relativistic and the ultra-relativistic situations.

551. Find the particle velocity v for: (a) electrons in an electronic valve ($\mathcal{E} = 300$ eV), (b) electrons in a 300 MeV synchrotron, (c) protons in a 680 MeV synchrocyclotron, and (d) protons in a 10 GeV proton synchrotron.

552. In a linear accelerator, the particles are accelerated in the gaps between hollow cylindrical electrodes (the so-called drift tubes), and travel along the common axis of these electrodes. The acceleration is produced by a high-frequency electric field having a frequency $\nu = \text{const.}$ Find the lengths of the drift tubes which will ensure that a particle of charge e and rest mass m will traverse all the accelerating gaps at instants of time at which the maximum voltage V_0 acts across the gaps. Estimate also the total length of the accelerator with N drift tubes.

553. A beam of monochromatic μ -mesons produced in the upper atmosphere is incident vertically on the earth's surface. Find the ratio of the intensity of the beam at a height h above sea level to the intensity at sea level, by assuming that the reduction in the beam intensity in the particular layer of air of thickness h occurs only as a result of the spontaneous disintegration of the μ -mesons (energy of the μ -mesons $\mathcal{E} = 4.2 \times 10^8$ eV, $h = 3$ km, half-life of μ -meson at rest $\tau_0 = 2 \times 10^{-6}$ sec).

554. A reference system S' moves with a velocity $\mathbf{V}$ relative to a system S . A particle of rest mass m having an energy $\mathcal{E}'$ and velocity $\mathbf{v}'$ in S' moves at an angle ϑ' to the direction of $\mathbf{V}$. Find the angle ϑ between the momentum $\mathbf{p}$ of the particle and the direction of $\mathbf{V}$ in the system S . Express the energy and the momentum of the particle in S in terms of ϑ' , $\mathcal{E}'$ or ϑ' , v' . Consider in particular the ultra-relativistic case $\mathcal{E}' \gg mc^2$, $V \simeq c$. Show that in the ultra-relativistic case there will be a range of angles in which it is possible to use the approximate formula $\vartheta \simeq \gamma^{-1} \tan(\vartheta'/2)$.

555*. A system S' moves with a velocity $\mathbf{V}$ relative to a system S . The angular distribution of particles of equal energy $\mathcal{E}'$ in S' is described by the function $dW/d\Omega' = F'(\vartheta', \alpha')$ where dW is the fraction of particles which move within the solid angle $d\Omega'$ in the system S' . This function is usually normalised so that

$$\int dW = \int F'(\vartheta', \alpha') d\Omega' = 1,$$

where the angle ϑ' is measured from the direction of $\mathbf{V}$. Find the angular distribution of such particles in the system S . Consider in particular the ultra-relativistic case.

556. A π^0 -meson moving with a velocity v decays in flight into two γ -rays. Find the angular distribution of the γ -rays, $dW/d\Omega$, in the laboratory system, assuming that the distribution in the rest system of the π^0 -meson is spherically symmetric.

557. Express the energy of the π^0 -meson considered in the preceding problem in terms of the ratio f of the number of γ -rays emitted into the forward and backward hemispheres.

558*. Find the dependence of the energy of the γ -rays appearing as a result of the disintegration of the π^0 -meson (cf. Problem 556) on the angle ϑ between the direction of motion of the γ -ray and the direction of motion of the π^0 -meson. Determine the energy spectrum of the γ -rays in the laboratory system.

Hint: It follows from the laws of conservation of energy and momentum that in the rest system of the π^0 -meson, the energy of the γ -rays is $\mathcal{E}' = \frac{1}{2}mc^2$, where m is the rest mass of the π^0 -meson.

559. Show that whatever the form of the energy spectrum of π^0 -mesons, the energy spectrum of the γ -rays which are produced as a result of the disintegration of these mesons will, in the laboratory system, have a maximum at $\mathcal{E} = \mathcal{E}'$, $\mathcal{E}' = \frac{1}{2}mc^2$, where m is the rest mass of the π^0 -meson. Express the rest mass m of the π^0 -meson in terms of $\mathcal{E}_1$ and $\mathcal{E}_2$, which are two arbitrary values of the γ -ray energy on either side of the above maximum and correspond to equal values of the distribution function.

Hint: Use the γ -ray energy spectrum deduced in Problem 558.

560. A particle of energy $\mathcal{E}$ and rest mass m_1 moves towards a stationary particle of rest mass m_2 . Find the velocity v of the centre of mass relative to the laboratory system.

561*. A particle of rest mass m_1 and energy $\mathcal{E}_0$ undergoes an elastic collision with a stationary particle (rest mass m_2). Express the angles of scattering ϑ_1, ϑ_2 of the particles in the laboratory system in terms of their energies $\mathcal{E}_1, \mathcal{E}_2$ after the collision.

562. Use the solution of the preceding problem to express the energy of the particles undergoing the elastic collision in terms of the scattering angles in the laboratory system.

563. A particle of rest mass m is elastically scattered by a stationary particle of equal rest mass. Express the kinetic energy T_1 of the scattered particle in terms of the kinetic energy T_0 of the incident particle and the scattering angle θ_1 .

564. Use the result of Problem 562 to find (on the non-relativistic approximation) the dependence of the final kinetic energies T_1 and T_2 of the particles undergoing an elastic collision on the initial kinetic energy T_0 of the incident particle, and the scattering angles θ_1, θ_2 in the laboratory system. Assume that the target particle was stationary prior to the collision.

565. Particles of rest mass m_1 and m_2 undergo an elastic collision. Their velocities in the centre of mass system are v'_1 and v'_2 , the angle of scattering is θ' , and the velocity of the centre of mass system relative to the laboratory system is V . Determine the angle χ between the directions of motion of the particles after the collision in the laboratory system. Consider in particular the case $m_1 = m_2$.

566. A photon of frequency ω_0 is scattered by a uniformly moving free electron. The momentum of the electron $\mathbf{p}_0$ is at an angle θ_0 to the direction of motion of the photon. Find the frequency ω of the scattered photon as a function of its direction of motion. Consider in particular the case where the electron was originally at rest (Compton effect).

567*. Particles of type 1 having a velocity $\mathbf{v}_1$ in a system S are scattered by stationary particles of type 2. Find the transformation law for the scattering cross-section $d\sigma_{12}$ to a reference system S' in which particles of type 2 have a velocity $\mathbf{v}'_2$ and particles of type 1 a velocity $\mathbf{v}'_1$. Consider in particular the case where $\mathbf{v}'_1$ and $\mathbf{v}'_2$ are parallel.

Hint: The scattering cross-section $d\sigma_{12}$ is defined as the ratio of the number of particles scattered per unit time by a single scattering centre into a solid angle $d\Omega$, to the intensity of incident particles $I_{12} = n_1 v_0$, where n_1 is the number of incident particles per unit volume and v_0 is the initial relative velocity of particles of type 1 and 2 (cf. Problem 499).

568. Determine the rest mass m of a particle given that it disintegrates into two particles of rest masses m_1 and m_2 . It is known from experiment that the momenta of the two particles are p_1 and p_2 and the angle between their directions of motion is θ .

569. Determine the rest mass m_1 of a particle given that it is one of two particles produced as a result of the disintegration

of a particle having a rest mass m and momentum p . Assume that the momentum p_2 , the rest mass m_2 , and the direction of motion $\mathbf{v}_2$ of the second particle produced in the disintegration are also known.

570. A particle of rest mass m_1 and velocity v collides with a stationary particle of rest mass m_2 and is absorbed by it. Find the rest mass m and the velocity V of the resultant particle.

571. A stationary body of rest mass m_0 disintegrates into two particles with rest masses m_1 and m_2 . How is the disintegration energy $\Delta\mathcal{E} = (m_0 - m_1 - m_2)c^2$ shared between the two particles?

572. Find the distribution of the disintegration energy $\Delta\mathcal{E}$ in the rest system of the disintegrating particles between (a) the alpha-particle and the daughter nucleus in the alpha-decay of U^{238} , (b) the μ -meson and the neutrino in the disintegration of the π -meson ($\pi \rightarrow \mu + \nu$), and (c) the γ -ray and the recoil nucleus in the emission of γ -radiation. Consider in particular the case $A = 20$, $\Delta\mathcal{E} = 2 \text{ MeV}$.

573. A stationary excited nucleus of rest mass m (excitation energy $\Delta\mathcal{E}$) emits a γ -ray. Find the frequency ω of the γ -ray.

574*. A particle of rest mass m collides with a stationary particle of rest mass m_1 . The collision results in the production of a number of particles with a total rest mass M . If $m + m_1 < M$, then at low kinetic energies of the incident particle the reaction will not proceed, because it would violate the law of conservation of energy. Determine the minimum kinetic energy of the incident particle at which the reaction becomes energetically possible, *i.e.* the threshold kinetic energy T_0 for the reaction.

575. Find the energy thresholds T_0 for the following reactions: (a) π -meson production in nucleon-nucleon scattering ($N + N \rightarrow N + N + \pi$), (b) photoproduction of π -mesons on nucleons ($N + \gamma \rightarrow N + \pi$), (c) production of a K -meson and a Λ -hyperon ($\pi + N \rightarrow \Lambda + K$), and (d) proton-antiproton pair production in the scattering of a proton of rest mass m_p by a nucleus of rest mass m . Consider in particular the case of proton-proton scattering. Estimate the threshold for the production of an antiproton in the collision between a proton and a nucleus of mass number A , assuming that $m \approx m_p A$.

576. Find an approximate expression for the energy threshold T_0 of reactions in which the change in the rest mass ΔM of the colliding particles is a small fraction of their total rest

mass M (this situation occurs in the case of a collision between two non-relativistic particles). Use this formula to determine the energy threshold T_0 of the following reactions: (a) photo-disintegration of the deuteron ($\gamma + \text{H}_1^2 \rightarrow p + n$) and (b) the reaction $\text{He}_2^4 + \text{He}_2^4 \rightarrow \text{Li}_3^7 + p$. Compare the approximate expressions with the exact expressions (*cf.* Problem 574).

577. Show that the electron-positron pair production by γ -rays is only possible if a particle of finite rest mass participates in the reaction. (This particle remains unchanged in the reaction, its sole purpose being to take up a fraction of the energy and momentum, so that the corresponding conservation laws may be satisfied.) Find the threshold T_0 for the pair-production reaction.

578. Show that the annihilation of an electron-positron pair with the emission of a single γ -ray is forbidden by the conservation of energy-momentum, but the annihilation of a pair with the emission of two photons is not forbidden.

579. Use the law of conservation of energy-momentum to show that the emission and absorption of light by a free electron in a vacuum is impossible.

580*. Show that the uniform motion of a free charged particle of rest mass m , charge e , and velocity $\mathbf{v}$ in a medium with a refractive index $n(\omega)$ may be accompanied by the emission of electromagnetic waves (Cherenkov effect†). Express the angle θ between the direction of propagation of the wave and the velocity $\mathbf{v}$ in terms of v , ω , and $n(\omega)$ (*cf.* Problem 727).

Hint: The energy and momentum of a photon in a medium with a refractive index $n(\omega)$ is $\mathcal{E} = \hbar\omega$ and $p = n(\omega)\hbar\omega/c$.

581. Show that a free electron moving with a velocity v in a medium, can absorb electromagnetic waves of frequency ω provided $v > c/n(\omega)$, where $n(\omega)$ is the refractive index.

582*. A particle which, in general, may have a complicated structure and may contain a number of electric charges (for example an atom), moves with a uniform velocity $\mathbf{v}$ through a medium with a refractive index $n(\omega)$. The particle is at first in an excited state, and then emits a photon of frequency ω_0 (in the rest system) when it undergoes a transition to the ground state.

† An analogous effect may occur for a neutral particle which has a finite electric or magnetic moment.

The photon is observed in the laboratory system at an angle θ to the direction of motion of the particle. Find the frequency ω of the photon in the laboratory system (this is the Doppler effect in a refracting medium). Consider in particular the case $\omega_0 \rightarrow 0$.

Hint: Neglect terms which are of the second order in $\hbar$ and assume that $\hbar\omega_0 \ll mc^2$, where m is the rest mass of the particle.

583*. The particle considered in Problem 582 moves with a uniform velocity through a medium. The particle is in the ground state and the remaining conditions are the same as those of Problem 582. Show that the emission of radiation may still occur, accompanied by the excitation of the particle. Elucidate the conditions necessary for this behaviour, and find the frequency ω of the emitted radiation.

584. The beam current in an accelerator is $\mathcal{I}$ and the kinetic energy at the output of the accelerator is T . Find the force F exerted by the beam on a target placed in its path, and the power W dissipated in the target.

585. A body moves with a relativistic velocity v through a gas containing N slowly moving particles of rest mass m per unit volume. Find the pressure p exerted by the gas on a surface element of the body which is perpendicular to its velocity.

2. The motion of charged particles in an electromagnetic field

The force acting on a particle of charge e , which is moving with a velocity $\mathbf{v}$ in an electromagnetic field $\mathbf{E}$, $\mathbf{H}$ is given by the Lorentz formula

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{H}. \quad (\text{XI.12})$$

The change in the kinetic energy of the particle per unit time is given by

$$\mathbf{F} \cdot \mathbf{v} = e\mathbf{E} \cdot \mathbf{v} = \dot{\mathcal{E}} = \frac{d\mathcal{E}}{dt}, \quad (\text{XI.13})$$

where $\mathcal{E}$ is the energy of the particle (*cf.* Section 1).

The magnetic field does no work on the particle, since the force associated with it is always perpendicular to the particle velocity. The quantities $\mathbf{F}$ and $d\mathcal{E}/dt$ may be used to set up the

following 4-vector (the Minkowski force)

$$F_l = \left(\frac{\mathbf{F}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t \frac{\mathbf{F} \cdot \mathbf{v}}{c \sqrt{1 - \frac{v^2}{c^2}}} \right). \quad (\text{XI.14})$$

The 4-force may be expressed in terms of the electromagnetic field tensor: $F_l = (e/c) F_{ik} u_k$, where u_k is the 4-velocity of the particle and F_{ik} is the field tensor.

The four-dimensional form of the differential equation of motion of a particle is

$$\frac{dp_l}{d\tau} = e F_l \quad \text{or} \quad m \frac{du_l}{d\tau} = \frac{e}{c} F_{ik} u_k. \quad (\text{XI.15})$$

On resolving these equations along the space and the time axes, we obtain the equations of motion in the three-dimensional form, and the law of conservation of energy:

$$\dot{\mathbf{p}} = e\mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{H}, \quad \dot{T} = e\mathbf{v} \cdot \mathbf{E}. \quad (\text{XI.16})$$

where $T = \mathcal{E} - mc^2$ is the kinetic energy of the particle, $\mathbf{p}$ is its momentum. m is the rest mass, and the dot over the symbols indicates differentiations with respect to time t . The equations given by (XI.16) hold for any velocity of the particle.

The relativistic Lagrange function for a charged particle in an electromagnetic field having a potential φ , $\mathbf{A}$ is of the form

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - U. \quad (\text{XI.17})$$

In the non-relativistic case

$$L = \frac{mv^2}{2} - U. \quad (\text{XI.18})$$

In general

$$U = -\frac{e}{c} \mathbf{A} \cdot \mathbf{v} + e\varphi. \quad (\text{XI.19})$$

The quantity U plays the role of a potential energy of interaction between the particle and the external field. The equations of motion of the particle may then be written in the Lagrangian form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} = 0. \quad (\text{XI.20})$$

where $q_i, \dot{q}_i$ are the generalised coordinates and velocities.

The current associated with the orbital motion of a point charged particle round a given centre may be characterised by the magnetic moment†

$$\mathbf{m} = \kappa \mathbf{l}, \quad (\text{XI.21})$$

where $\kappa = e/2mc$ is the gyromagnetic ratio, m is the mass of the particle, and $\mathbf{l} = \mathbf{r} \times m\mathbf{v}$ is the angular momentum. The particle will experience a couple $\mathbf{N} = \mathbf{m} \times \mathbf{H}$ in an external magnetic field $\mathbf{H}$, and this will produce a change in the angular momentum $\mathbf{l}$ which is given by $d\mathbf{l}/dt = \mathbf{N}$. It follows that the rate of change of the magnetic moment is given by

$$\frac{d\mathbf{m}}{dt} = \kappa \mathbf{m} \times \mathbf{H}. \quad (\text{XI.22})$$

In addition to the angular momentum and the magnetic moment associated with orbital motion, micro-particles also exhibit an intrinsic (spin) angular momentum $\mathbf{s}$ and a magnetic moment $\mathbf{m}_0$ which are either parallel or anti-parallel and are related by

$$\mathbf{m}_0 = \kappa_0 \mathbf{s}. \quad (\text{XI.23})$$

For an electron $\kappa_0 = e/mc < 0$, where e is the charge on the electron and m its mass. The rate of change of $\mathbf{m}_0$ is given by an equation similar in form to Eqn. (XI.22), except that κ is replaced by κ_0 and $\mathbf{m}$ by $\mathbf{m}_0$.

The neutron is not electrically charged but does, nevertheless, possess a spin magnetic moment $\mathbf{m}_0$. According to quantum theory, the direction of this moment in an external magnetic field $\mathbf{H}(\mathbf{r})$ can only be parallel or anti-parallel to the field, and the initial orientation of the moment is conserved under certain conditions‡. The motion of neutrons with magnetic moments parallel (or anti-parallel) to the field can then be looked upon as

† The classical theory described below can only be applied to micro-particles with certain reservations. A rigorous theory of motion of elementary magnetic moments can only be developed on the basis of the quantum theory.

‡ This will be true if the angle of rotation of the field per unit time in the system in which the neutron is at rest is small compared with the precession frequency $\omega_L = 2\pi \mathbf{m}_0 \mathbf{H}/\hbar$ of the magnetic moment $\mathbf{m}_0$ in the field $\mathbf{H}$ (adiabatic approximation).

the motion of classical particles in a force field with a potential energy

$$U = \mp m_0 H, \quad (\text{XI.24})$$

where $H = |\mathbf{H}(\mathbf{r})|$. The energy U is usually very small and hence a magnetic field will only have an appreciable effect on the motion of very slow (cold) neutrons.

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586. Write down the relativistic equation of motion for a particle acted upon by a force $\mathbf{F}$ by expressing the momentum explicitly in terms of the velocity $\mathbf{v}$. Consider in particular the following cases: (1) the direction of the velocity remains constant, (2) the magnitude of the velocity remains constant, and (3) $v \ll c$.

587. Derive the expression relating the force $\mathbf{F}$ on a particle in the laboratory system and the force $\mathbf{F}'$ in the rest system. Assume that the velocity of the particle $\mathbf{v}$ is given.

588. A body of rest mass m is stationary relative to a rocket which moves with a relativistic velocity v in a circular orbit of radius R . Find the force F on the body as measured by an observer in the frame in which the body is instantaneously at rest.

589. Two charges e and e' move parallel to the x -axis with equal constant velocities $\mathbf{v}$. Using the results of Problem 535, show that the electromagnetic force between the charges may be derived from the so-called convection potential $\psi = (1 - \beta^2)e/R$ where

$$R = \sqrt{(x_1 - x_2)^2 + (1 - \beta^2)[(y_1 - y_2)^2 + (z_1 - z_2)^2]}.$$

$\mathbf{r}_1$, $\mathbf{r}_2$ are the position vectors of the two charges, and the force $\mathbf{F}$ is given by $\mathbf{F} = -e' \text{grad} \psi$. What will happen when $v \rightarrow c$?

590. Find the convection potential ψ for an infinitely long, uniformly charged, straight conductor†. The linear density of charge in the frame in which the conductor is at rest is κ . The

† The convection potential of a system of charges moving as a whole is defined as that function of the coordinates which, when differentiated, yields the components of the Lorentz force in the laboratory system per unit test charge moving together with the system of charges.

conductor moves without rotation at an angle α to its axis with a velocity v (in the laboratory frame). Consider in particular $\alpha = 0$ and $\alpha = \pi/2$.

591. An infinitely long, uniformly charged, straight line carrying a charge λ per unit length in the frame in which it is at rest moves with a velocity v in the direction parallel to itself. A point charge at a distance r from the line moves with an equal velocity in the direction parallel to the line. Find the electro-magnetic force F on the point charge for an arbitrary velocity v .

592. The distribution of electrons in a parallel beam is axially symmetric and is characterised by a volume charge density ρ in the reference frame in which the electrons are at rest. The electrons are accelerated by a potential difference V and the total beam current is $\mathcal{I}$. Find the electromagnetic force F on each of the electrons in the laboratory frame.

Hint: Use the result of Problem 591.

593. Find the spread Δa over a path length L of the electron beam considered in the preceding problem, assuming that it is due to the mutual repulsion between the electrons. Assume that the broadening is small ($\Delta a \ll L$) and that the cross-section of the beam is circular with a radius a .

594*. A particle of charge e and rest mass m moves with an arbitrary velocity in a uniform constant electric field $\mathbf{E}$. At the initial instant of time, $t = 0$, the particle was at the origin and had a momentum $\mathbf{p}_0$. Determine the three-dimensional coordinates and the time t of the particle in the laboratory system as functions of its proper time τ . By eliminating τ express the three-dimensional coordinates as functions of t (the problem can also be solved directly by integrating the equations of motion in the three-dimensional form). Consider in particular the non-relativistic and the ultra-relativistic limits.

595. Find the trajectory of a charged particle of charge e and rest mass m in a uniform constant electric field $\mathbf{E}$, using the results of Problem 594. Consider in particular the non-relativistic limit.

596. Find the range l of a relativistic charged particle of charge e , rest mass m , and potential energy $\mathcal{E}$ in a retarding uniform electric field E which is parallel to the initial velocity of the particle.

597*. A relativistic particle of charge e and rest mass m moves in a constant uniform magnetic field $\mathbf{H}$. At the initial

instant of time, $t = 0$, the particle was at the point with position vector $\mathbf{r}_0$ and had a momentum $\mathbf{p}_0$. Determine the law of motion of the particle.

598*. A non-relativistic particle of charge e and rest mass m moves in constant crossed electric and magnetic fields $\mathbf{E} = (0, E_y, E_z)$ and $\mathbf{H} = (0, 0, H)$ respectively. At the initial instant of time, $t = 0$, the particle was at the origin and had a velocity $\mathbf{v} = (v_{0x}, 0, v_{0z})$. Determine the functions $x(t)$, $y(t)$, and $z(t)$ and sketch out the possible trajectories of the particle.

Hint: The substitution $u = x + iy$ will simplify the integration.

599. A relativistic particle moves in parallel uniform constant electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ ($\mathbf{E} \parallel \mathbf{H} \parallel z$). At $t = 0$ the particle was at the origin and had a momentum $\mathbf{p}_0 = (p_{0x}, 0, p_{0z})$. Determine the quantities x , y , z , t as functions of the proper time τ of the particle.

600. Determine the law of motion of a particle in mutually perpendicular uniform constant electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$. Use two methods, namely, (a) the Lorentz formula and assume that the motion of a particle in a purely electric or purely magnetic field is known (*cf.* Problems 594 and 597), and (b) integrate Eqn. (XI.15).

601. Find the kinetic energy T of a particle as a function of the proper time τ for the types of motion considered in Problems 594, 599, and 600.

602. A particle having a small initial velocity v_0 ($v_0 \ll c$) moves in crossed constant uniform electric and magnetic fields $\mathbf{E} = (0, E_y, E_z)$, $\mathbf{H} = (0, 0, H)$, $E \ll H$. Determine the law of motion of the particle by using the Lorentz transformation and assuming that its motion in parallel electric and magnetic fields is known (*cf.* Problem 599). Use the results of Problem 530 and compare with the result of Problem 598.

603. Determine the law of motion of a particle of charge e and rest mass m in the field of a plane electromagnetic wave $\mathbf{E}(t')$, $\mathbf{H}(t')$ where $t' = t - (\mathbf{n} \cdot \mathbf{r})/c$ and $\mathbf{n}$ is a unit vector in the direction of propagation. Assume that at $t = 0$, the particle was at rest at the origin.

Hint: Note that the proper time τ of the particle is the same as the argument t' of the plane wave.

604. A non-relativistic charged particle of charge e and mass m passes through a two-dimensional electrostatic field

whose potential is $\varphi = k(x^2 - y^2)$ where $k = \text{const} > 0$ (*i.e.* a strong focusing lens). Assume that at $t = 0$ the particle was at the point (x_0, y_0, z_0) and the initial velocity v_0 was parallel to the z -axis. Determine the motion of the particle.

605. Find the differential equation of motion for a relativistic particle in an electromagnetic field, starting from the Lagrange function in terms of cylindrical coordinates.

Hint: In evaluating the time derivative in the Lagrange equations, note that the derivative is taken along the trajectories of the particles, so that r, α, z should be looked upon as functions of time.

606*. A potential difference V is maintained between the plates of a cylindrical capacitor whose radii are a and b ($a < b$). An axially symmetric magnetic field, which is parallel to the axis of the capacitor, is set up between the plates. The inner plate (cathode) emits electrons with zero initial velocity. Find the critical magnetic flux Φ_{cr} at which the electrons will no longer reach the anode.

607. A long straight cylindrical cathode of radius a carries a uniformly distributed current $\mathcal{J}$ and emits electrons with zero initial velocity. The cathode is surrounded by a long coaxial anode of radius b which is maintained at a potential V relative to the cathode. Find the minimum potential difference between the cathode and the anode which will ensure that the electrons will reach the anode in spite of the effect of the magnetic field due to the current $\mathcal{J}$.

608. An infinitely long straight cylindrical conductor of radius a carries a current $\mathcal{J}$. The surface of the conductor emits electrons whose initial velocity v_0 is parallel to the axis of the conductor. Find the maximum distance b from the axis of the conductor which the electrons can reach.

609. Solve Problem 607 by using the Lorentz transformation from the frame in which there is only one field ($\mathbf{E}$ or $\mathbf{H}$).

Hint: Use the results of Problems 533 and 608.

610*. A relativistic particle of charge $-e$ and rest mass m moves in the field of a stationary point charge Ze . Find the equation of the trajectory of the particle. Investigate the possible trajectories when the angular momentum $K > Ze^2/c$.

Hint: Use the law of conservation of energy and the equations obtained in Problem 605.

611. Investigate the possible trajectories of the particle considered in the preceding problem if $K \leq Ze^2/c$.

612*. A relativistic particle of charge e and rest mass m moves in the field of a heavy point charge Ze . Find the trajectory of the particle and discuss the solution.

613. Show that when a particle moves in an attractive Coulomb field (*cf.* Problem 610) its velocity will tend to c when $r \rightarrow 0$ ($Ze^2 \geq Kc$).

614. Determine the trajectory of the relative motion of non-relativistic particles of charges e, e' , masses m_1, m_2 , and energy $\mathcal{E}$. Investigate the solution.

615*. Find the differential scattering cross-section $\sigma(\theta)$ for non-relativistic particles of charge e in the field of a stationary point charge e' . Assume that the velocity of the particles at a large distance from the scattering centre is v_0 .

616. Determine the angle θ through which a relativistic particle of charge e , energy $\mathcal{E} > mc^2$, and angular momentum $K > |ee'|/c$ is deflected in the Coulomb field of a heavy stationary charge e' (*cf.* Problems 610 and 612).

617. A relativistic particle of charge e , rest mass m , and velocity v_0 at large distances is scattered through a small angle by the Coulomb field of a fixed charge e' . Determine the differential scattering cross-section $\sigma(\theta)$.

618. An electron of charge e and rest mass m travels through the evacuated space above a plane uncharged surface of a dielectric of permittivity ϵ . Initially the electron moves in a direction parallel to the surface of the dielectric with a velocity v and at a distance a from it. Find the projected distance x from the initial position to the point at which the electron will enter the dielectric.

619*. While an electron is being accelerated in the betatron, the magnetic field is continuously increasing, giving rise to the induced e.m.f. which accelerates the electron in a stationary orbit. Show that in order to accelerate an electron in an orbit of constant radius, it is necessary that the total magnetic flux cutting the orbit should be twice the flux associated with a magnetic field which is uniform within the orbit and zero along the orbit (the 2 : 1 betatron rule).

620*. Show that, to within terms of the order of v^2/c^2 , the energy of the retarded interaction between two charged particles is of the form

$$U(t) = \frac{e_1 e_2}{R} \left\{ 1 - \frac{1}{2c^2} [\mathbf{v}_1 \cdot \mathbf{v}_2 + (\mathbf{v}_1 \cdot \mathbf{n})(\mathbf{v}_2 \cdot \mathbf{n})] \right\},$$

where $\mathbf{R}$ is the relative position vector of the two particles, $\mathbf{n} = \mathbf{R}/R$, and $\mathbf{v}_1, \mathbf{v}_2$ are the velocities of the particles (this is the so-called Breit formula; an analogous expression is used in an approximate quantum description of the retarded interaction). All the quantities on the right-hand side of the equation are taken at time t .

Hint: Use the expansions for the Lienard-Wiechert potentials (*cf.* Problem 658) and retain only those terms which are independent of the accelerations and their derivatives. Carry out a gradient transformation of the potentials in such a way that the scalar potential becomes identical with the Coulomb potential.

621. Find an approximate expression for the Lagrange function for two interacting particles, having charges e_1, e_2 and rest masses m_1, m_2 , by taking into account the retarding effect to within terms of the order of v^2/c^2 .

622. A particle having a magnetic moment $\mathbf{m}$ and gyromagnetic ratio κ is placed in a uniform magnetic field $\mathbf{H}$. Determine the motion of the magnetic moment of the particle.

623*. A particle of charge e and rest mass m executes a periodic motion in a centrally symmetric electrostatic potential $\varphi(r)$. The intrinsic magnetic moment in the rest system of the particle is $\mathbf{m} = \frac{e}{mc} \mathbf{s}$, where $\mathbf{s}$ is the spin of the particle. In the laboratory system the moving particle has in addition an electric moment $\mathbf{p} = \frac{1}{c}(\mathbf{v} \times \mathbf{m})$, so that it experiences both a force and couple due to the electric field. Determine the couple $\mathbf{N}$ and the corresponding potential energy U of the magnetic moment in the electric field averaged over one period (retain only those terms which are linear in v/c). The effect of magnetic forces on the translational motion of the particle may be neglected.

624*. A particle of charge e , rest mass m , and intrinsic angular momentum $\mathbf{s}$ executes a periodic motion in a centrally symmetric electrostatic potential $\varphi(r)$ (*cf.* Problem 623). Find the differential equation for the average (over one period) change in the angular momentum $\mathbf{s}$ per unit time. What is the nature of this motion?

625. A neutron (magnetic moment m_0 , kinetic energy $\mathcal{E}_0$) enters a magnetic field defined by $H = \text{const}$, $x \geq 0$ and $H = 0$, $x < 0$. Under what conditions will the neutron be reflected from the field?

626. Investigate the possible trajectories of a neutron (rest mass m , dipole moment $\mathfrak{m}_0$) in the field of an infinitely long straight conductor carrying a current $\mathcal{J}$.

627. A beam of cold neutrons (velocity v_0 , magnetic moment $\mathfrak{m}_0$, rest mass m) is scattered by the magnetic field of an infinitely long straight conductor carrying a current $\mathcal{J}$. Determine the differential scattering length $l(\alpha) = \left| \frac{ds}{d\alpha} \right|$ where $s(\alpha)$ is the impact parameter for which the neutron is scattered through an angle α .

Hint: Use the method employed in the solution of Problem 615.

Chapter 12

EMISSION OF ELECTROMAGNETIC WAVES

1. The Hertz vector and the multipole expansion

The problem of the determination of a variable electromagnetic field in a vacuum for a given distribution of charges $\rho(\mathbf{r}', t)$ and currents $\mathbf{j}(\mathbf{r}', t)$ may be solved by evaluating the retarded potentials

$$\varphi(\mathbf{r}, t) = \int \frac{\rho\left(\mathbf{r}', t - \frac{R}{c}\right)}{R} dV', \quad (\text{XII.1})$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{\mathbf{j}\left(\mathbf{r}', t - \frac{R}{c}\right)}{R} dV', \quad (\text{XII.2})$$

where $R = |\mathbf{r} - \mathbf{r}'|$, $\mathbf{r}$ is the position vector of the point at which the field is to be determined, $\mathbf{r}'$ is the position vector of the field source, and dV' is a volume element of the field source. These potentials satisfy the d'Alembert equations:

$$\Delta\varphi - \frac{1}{c^2} \cdot \frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho, \quad (\text{XII.3})$$

$$\Delta\mathbf{A} - \frac{1}{c^2} \cdot \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}, \quad (\text{XII.4})$$

and are related by the Lorentz condition

$$\text{div } \mathbf{A} + \frac{1}{c} \cdot \frac{\partial \varphi}{\partial t} = 0. \quad (\text{XII.5})$$

The number of unknown functions may be reduced if the potentials $\mathbf{A}(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$, which are related by Eqn. (XII.5), are replaced by the single vector function $\mathbf{Z}(\mathbf{r}, t)$ which is known as the Hertz vector, or the polarisation potential, and is related to $\mathbf{A}$ and φ by the expressions

$$\varphi = -\text{div } \mathbf{Z}, \quad (\text{XII.6})$$

$$\mathbf{A} = \frac{1}{c} \cdot \frac{\partial \mathbf{Z}}{\partial t}. \quad (\text{XII.7})$$

The distribution of charges and currents can then be conveniently described with the aid of a single vector function $\mathbf{P}(\mathbf{r}', t)$ which is related to ρ and $\mathbf{j}$ by the expressions

$$\rho = -\text{div } \mathbf{P}, \quad (\text{XII.8})$$

$$\mathbf{j} = \frac{\partial \mathbf{P}}{\partial t}. \quad (\text{XII.9})$$

This definition of $\mathbf{P}$ ensures that the continuity condition $\text{div } \mathbf{j} + \partial \rho / \partial t = 0$ is satisfied. The quantity $\mathbf{P}$ is often referred to as the polarisation (but must not be confused with the polarisation of a dielectric).

The Hertz vector $\mathbf{Z}$ satisfies the d'Alembert equation

$$\Delta \mathbf{Z} - \frac{1}{c^2} \cdot \frac{\partial^2 \mathbf{Z}}{\partial t^2} = -4\pi \mathbf{P}. \quad (\text{XII.10})$$

The vectors $\mathbf{E}$ and $\mathbf{H}$ are given by

$$\left. \begin{aligned} \mathbf{E} &= \text{curl curl } \mathbf{Z} - 4\pi \mathbf{P} \\ \mathbf{H} &= \frac{1}{c} \cdot \frac{\partial \text{curl } \mathbf{Z}}{\partial t} \end{aligned} \right\} \quad (\text{XII.11})$$

In order to determine the electromagnetic field for given ρ and $\mathbf{j}$ with the aid of the Hertz vector, the polarisation vector $\mathbf{P}$ must first be determined from (XII.8) and (XII.9). Owing to the analogy between Eqns. (XII.3)–(XII.4) and (XII.10), the Hertz vector can then be expressed in terms of $\mathbf{P}$ in a way similar to that whereby φ and $\mathbf{A}$ are expressed in terms of ρ and $\mathbf{j}$:

$$\mathbf{Z}(\mathbf{r}, t) = \int \frac{\mathbf{P}\left(\mathbf{r}', t - \frac{R}{c}\right)}{R} dV'. \quad (\text{XII.12})$$

If a system of charges and currents is confined to a finite region, its linear dimensions are of the order of a , and if the wavelengths λ which contribute significantly to the spectral expansions of the potentials are such that

$$\frac{a}{\lambda} \ll 1 \quad \text{and} \quad \frac{a}{r} \ll 1, \quad (\text{XII.13})$$

then the integrands can be expanded in powers of a/λ and a/r .

If one retains only the first term of this expansion then

$$\mathbf{Z}(\mathbf{r}, t) = \frac{\mathbf{p}(t')}{r}, \quad (\text{XII.14})$$

where $t' = t - (r/c)$ is the retarded time of the centre of the system. The quantity

$$\mathbf{p}(t') = \int \mathbf{r}' \rho(\mathbf{r}', t') dV' \quad (\text{XII.15})$$

is the electric dipole moment of the charge distribution (*cf.* Problems 643 and 644). The corresponding expression for $\mathbf{A}$ and φ can then be obtained with the aid of Eqns. (XII.6) and (XII.7).

The field at large distances r from the system of charges is of particular interest. In this case we have the following inequality in addition to (XII.13):

$$r' \ll \lambda \ll r. \quad (\text{XII.16})$$

This defines the so-called wave zone. The field can then be determined by expanding the vector potential in powers of a/λ . To within terms of the order of v^2/c^2 , this expansion is of the form

$$\mathbf{A}(\mathbf{r}, t) = \frac{\dot{\mathbf{p}}(t')}{cr} + \frac{\ddot{\mathbf{Q}}(t')}{2c^2 r} + \frac{\dot{\mathbf{m}}(t') \times \mathbf{n}}{cr}, \quad (\text{XII.17})$$

where $\mathbf{n} = \mathbf{r}/r$ is a unit vector in the direction of propagation of the electromagnetic waves and where the dots represent differentiation with respect to t' . The magnetic dipole moment is given by

$$\mathbf{m} = \frac{1}{2c} \int \mathbf{r}' \times \mathbf{j} dV' \quad (\text{XII.18})$$

and the components of the quadrupole moment are

$$Q_\alpha = \sum_\beta Q_{\alpha\beta} n_\beta, \quad Q_{\alpha\beta} = \int \rho(\mathbf{r}') x'_\alpha x'_\beta dV'. \quad (\text{XII.19})$$

The form of the dependence of the vector potential in the wave zone on the distance r from the system is a characteristic feature. It ensures (see below) that the energy flux at infinity remains finite. This means that the vector potential describes the emission of electromagnetic energy.

The second and third terms of the vector potential, which represent the electric quadrupole and the magnetic dipole respectively, are smaller by a factor of a/λ as compared with the first term (electric dipole) and may be neglected unless there are some special reasons responsible for a large reduction in the first term.

In the wave zone, the field in a sufficiently small region will be of the form of a plane wave travelling away from the source. The corresponding field strengths may be evaluated from the formulae

$$\mathbf{H} = \frac{1}{c} \dot{\mathbf{A}} \times \mathbf{n}, \quad \mathbf{E} = \mathbf{H} \times \mathbf{n}. \quad (\text{XII.20})$$

The angular distribution of the emitted radiation may be characterised by the amount of energy flowing into a unit solid angle per unit time and this is given by

$$\frac{dI}{d\Omega} = \frac{c}{4\pi} H^2 r^2. \quad (\text{XII.21})$$

The total intensity I is obtained by integrating (XII.21) over all directions.

Using the expansion given by (XII.17), the final expression for the total intensity is given by

$$I = \frac{2}{3c^3} (\ddot{\mathbf{p}})^2 + \frac{1}{60c^5} \left[3 \sum_{\alpha\beta} \ddot{Q}_{\alpha\beta}^2 - \left(\sum_{\beta} \ddot{Q}_{\beta\beta} \right)^2 \right] + \frac{2}{3c^3} (\ddot{\mathbf{m}})^2. \quad (\text{XII.22})$$

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628. Use a direct substitution to verify that the retarded potentials satisfy the d'Alembert equation and the Lorentz condition.

629. Use the results of Problem 32 to obtain the formula given by (XII.22).

630. Write down the equations which are satisfied by the electromagnetic potentials φ and $\mathbf{A}$ if the Lorentz condition (XII.5) is replaced by the condition $\text{div } \mathbf{A} = 0$ (this is the so-called Coulomb calibration).

631. Show that when the Lorentz condition is satisfied then, in the wave zone, the scalar potential of a finite radiating system may be expressed in terms of the vector potential by the formula $\varphi = \mathbf{n} \cdot \mathbf{A}$.

632. Using the solution of Problem 545 [Eqns. (2) and (3)], find an expression for the loss of angular momentum per unit time by a system radiating as an electric dipole.

633. Find the equations for the lines of force of the electric and magnetic fields of a point electric dipole oscillator having a dipole moment $\mathbf{p} = \mathbf{p}_0 \cos \omega t$. Investigate the change in the field configuration between the wave zone and the region in the immediate neighbourhood of the oscillator.

Hint: If the polar axis lies along $\mathbf{p}_0$ then the electromagnetic field of the oscillator is of the form

$$\begin{aligned} E_r &= \frac{2p_0 \cos \vartheta}{r^2} \left[\frac{\cos(kr - \omega t)}{r} + k \sin(kr - \omega t) \right], \\ E_\vartheta &= \frac{p_0 \sin \vartheta}{r} \left[\left(\frac{1}{r^2} - k^2 \right) \cos(kr - \omega t) + \frac{k}{r} \sin(kr - \omega t) \right], \\ E_\alpha &= H_r = H_\vartheta = 0, \\ H_\alpha &= -\frac{p_0 k^2}{r} \cos(kr - \omega t) + \frac{p_0 k}{r^2} \sin(kr - \omega t). \end{aligned}$$

634. Find the electromagnetic field $\mathbf{H}$, $\mathbf{E}$ due to a charge e moving with a uniform velocity on a circle of radius a . Assume that the motion is non-relativistic, the angular velocity is ω , and the distance to the point of observation r is much greater than the radius a . Find the time average of the angular distribution and of the total intensity of the radiation, and investigate its polarisation.

635. Investigate the effect of interference on the emission of electromagnetic waves by a system of charges in the following situation. Two identical electric charges e move with a uniform velocity at an angular frequency ω on a circular orbit of radius a in such a way that they remain at the opposite ends of a diameter. Find the polarisation, the angular distribution, and the intensity of the radiation. What will be the effect of the removal of one of these charges? (cf. the solution of Problem 634).

636. How should the position of the particles in the preceding problem be adjusted on the circle in order that the intensities of the electric dipole and quadrupole radiations should be equal?

637. Two electric dipole oscillators vibrate with the same frequency ω but their phases differ by $\pi/2$. The amplitudes of the dipole moments are both equal to p_0 and are at an angle φ to each other. Assuming that the distance between the oscillators is small compared with the wavelength, find the field $\mathbf{H}$ in the wave zone, the average angular distribution, and the average total intensity of the emitted radiation.

638. Investigate the polarisation of the radiation field due to the oscillator system considered in the preceding problem, using the method employed in the solution of Problem 387.

639*. Find the time average of the energy flux at large distances from the charge considered in Problem 634, retaining terms of the order of r^{-3} . Find the couple $\mathbf{N}$ on a totally absorbing spherical screen of large radius with the charge at its centre.

640. A uniformly magnetised sphere of radius a and magnetisation $\mathbf{M}$ rotates with a constant angular velocity ω about an axis passing through the centre of the sphere and making an angle φ with the direction of $\mathbf{M}$. Find the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ and investigate the polarisation of the field. Determine the average angular distribution and total intensity of the radiation.

641. A uniformly charged drop pulsates with its density remaining constant. The surface of the drop is described by

$$R(\vartheta) = R_0 [1 + aP_2(\cos \vartheta) \cos \omega t],$$

where $a \ll 1$. The total charge on the drop is q . Find the average angular distribution and total intensity of the radiation emitted.

642. An electric charge q is arranged in a continuous spherically symmetric distribution in a bounded region and executes radial pulsations. Find the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ outside the charge distribution.

643. Find expressions for the electric dipole, electric quadrupole, and magnetic dipole terms in the expansion for the Hertz vector, which hold for an arbitrary dependence of the charges and currents on time at distances $r \gg a$ and $\lambda \gg a$ (the condition $r \gg \lambda$ need not be satisfied).

644. Find, in vector form, expressions for the electromagnetic field strengths due to electric and magnetic dipole oscillators at distances which are large compared with their dimensions.

Hint: When differentiating with respect to $\mathbf{r}$, note that the moments should be taken at the retarded time $t' = t - (r/c)$ and are, therefore, functions of $\mathbf{r}$.

645. Two identical electric dipole moments lie along a given straight line and oscillate in antiphase with a frequency ω and amplitude p_0 . The distance between the centres a is such that $\lambda \gg a$. Find the electromagnetic field at distances $r \gg a$. Find also the average angular distribution of the radiation and its total intensity.

646*. A standing current wave of amplitude $\mathcal{J}_0$ and frequency ω is excited in a linear antenna of length l with nodes at the ends. The number of current half-waves in the antenna is m . Find the average angular distribution of the emitted radiation.

647. Find the total radiation intensity $\bar{I}$ and the radiation resistance $R = 2\bar{I}/\mathcal{J}_0^2$ of the antenna considered in the preceding problem.

Hint: The result can be expressed in terms of the integral cosine

$$\text{Ci}(x) = C + \ln x + \int_0^x \frac{\cos t - 1}{t} dt,$$

where $C = 0.577$ is the Euler constant.

648. A travelling current wave $\mathcal{J} = \mathcal{J}_0 \exp[i(k\xi - \omega t)]$ propagates in a linear antenna of length l , where $k = \omega/c$ and ξ is the coordinate of a point on the antenna. The loads at the ends of the antenna are chosen so that there are no reflected waves. Find the average angular distribution and the total intensity of the radiation.

649*. A standing current wave of the form $\mathcal{J} = \mathcal{J}_0 \sin n\alpha' \times \exp(-i\omega t)$ is excited in a circular wire loop of radius a . Find the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ in the wave zone.

650*. The centres of two electric dipole oscillators of frequency ω and equal amplitudes $\mathbf{p}_0 \parallel x$ lie on the z -axis at equal distances from the origin and distances $a = \lambda/4$ from each other. The phase difference between the oscillators is $\pi/2$. Find the average angular distribution of the radiation.

651. A system of charges B having a charge density $\rho(\mathbf{r}, t)$ and current density $\mathbf{j}(\mathbf{r}, t)$ is reflected in the $z = 0$ plane in such a way that (a) each point $\mathbf{r} = (x, y, z)$ transforms into $\mathbf{r}' = (x, y, -z)$, and (b) the charge density changes sign so that $\rho(\mathbf{r}, t) = -\rho'(\mathbf{r}', t)$ where ρ' is the charge density in the reflected system B' . Find the law of transformation on reflection of the current density $\mathbf{j}(\mathbf{r}, t)$ and the electric ($\mathbf{p}$, $\mathbf{Q}$) and magnetic ($\mathbf{m}$) moments of the system, and also the electromagnetic field $\mathbf{E}$, $\mathbf{H}$.

652. Show that the electromagnetic field due to an arbitrary system of charges B , placed near a perfectly conducting plane, may be obtained as a superposition of the fields due to B and a system B' reflected in this plane (cf. preceding problem). Consider in particular the radiation emitted by an electric dipole oscillator having a moment $\mathbf{p}(t) = \mathbf{p}_0 f(t)$, where $|\mathbf{p}_0| = 1$, $f(t)$ is

an arbitrary function, and the dipole is at a distance $b \ll \lambda$ from the plane, and is at an angle $\varphi_0 = \text{const}$ to it. Use the electric-dipole approximation.

653. An electric dipole oscillates with a frequency ω and an amplitude p_0 . It is placed at a distance of $a/2$ from a perfectly conducting plane where $a \ll \lambda$ and $\mathbf{p}_0$ is parallel to the plane. Find the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ at distances $r \gg \lambda$ and the average angular distribution of the emitted radiation.

654. Show that if a function $u(r, \vartheta, \alpha)$ satisfies the Helmholtz equation $\Delta u + k^2 u = 0$, then the Hertz potential for a monochromatic electric wave ($H_r = 0$) of frequency $\omega = kc$ in a source-free space is of the form $\mathbf{Z} = u\mathbf{r} + \text{grad} \chi$ where $\chi = k^{-2} \partial(ru)/\partial r$. Find also the components of the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ along the axes of a spherical system of coordinates in terms of $u(r, \vartheta, \alpha)$ (the function u is known as the Debye potential).

Hint: In showing that $\Delta \mathbf{Z} + k^2 \mathbf{Z} = 0$, note that $\Delta \chi + k^2 \chi + 2u = 0$.

655. Show that the field due to a bound electric dipole oscillator placed at the point $\mathbf{r}_0$ ($\mathbf{r}_0 \parallel \mathbf{p}_0$) and having a moment $\mathbf{p}_0 \exp(-i\omega t)$ may be described by a Debye potential (cf. Problem 654) which is of the form $u = (p_0/r_0 R) \exp(ikR)$, where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$.

Hint: The Hertz vector $\mathbf{Z} = u\mathbf{r} + \text{grad} \chi$ corresponding to the potential u differs from the expression $(p_0/R) \exp(ikR)$ [cf. Eqn. (XII.14)] but leads to the same expressions for the fields $\mathbf{E}$, $\mathbf{H}$.

656. An electric dipole oscillator placed at a distance b from the centre of a perfectly conducting sphere of radius a has a dipole moment $p_0 \exp(-i\omega t)$. The dipole lies along the line joining it to the centre of the sphere. Using the Debye potential u (cf. Problem 654), find the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ and the average angular distribution.

2. The electromagnetic field of a moving point charge

A point charge e moving with a velocity $\mathbf{v}(t')$, whose position vector at time t' is $\mathbf{r}_0(t')$, will give rise to an electromagnetic field whose potentials at a time t at a point having a position vector $\mathbf{r}$ are given by the Lienard-Wiechert formulae

$$\varphi(\mathbf{r}, t) = \frac{e}{R(1 - \mathbf{R} \cdot \mathbf{v}/c)} \Big|_{t'}, \quad \mathbf{A}(\mathbf{r}, t) = \frac{e\mathbf{v}}{c(R - \mathbf{R} \cdot \mathbf{v}/c)} \Big|_{t'}, \quad (\text{XII.23})$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$. The retarded time t' is defined by

$$c(t - t') = |\mathbf{R}|. \quad (\text{XII.24})$$

The field strengths derived from the Lienard-Wiechert potentials are

$$\mathbf{E}(\mathbf{r}, t) = e \frac{(1 - \beta^2)(\mathbf{n} - \mathbf{v}/c)}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^3 R^2} + \frac{e\mathbf{n} \times [(\mathbf{n} - \mathbf{v}/c) \times \dot{\mathbf{v}}]}{c^2(1 - \mathbf{n} \cdot \mathbf{v}/c)^3 R} \Big|_{t'}, \quad (\text{XII.25})$$

$$\mathbf{H} = \mathbf{n} \times \mathbf{E} \Big|_{t'},$$

where

$$\mathbf{n} = \frac{\mathbf{R}}{R}, \quad \beta = \frac{v}{c}.$$

The first term in the expression for $\mathbf{E}$ and the corresponding term of $\mathbf{H}$ describe a field which falls off as R^{-2} (quasi-stationary field) and moves together with the charge. The second term in $\mathbf{E}$ and the corresponding term in $\mathbf{H}$ describe a field which falls off as R^{-1} (radiation field). The energy flux associated with the latter field is independent of R . This means that the radiation field is not rigidly attached to the charge responsible for it. At large distances from the charge, *i. e.* in the wave zone, the quasi-stationary field is negligible compared with the radiation field. As can be seen from Eqn. (XII.25), the condition that the radiation field should appear is that the acceleration should be finite ($\dot{\mathbf{v}} \neq 0$).

The intensity of radiation in the direction $\mathbf{n} = \mathbf{R}/R$ in the wave zone is given by

$$\frac{dI_{\mathbf{n}}(t)}{d\Omega} = \frac{c}{4\pi} E^2(t) R^2 = \frac{e^2}{4\pi c^3} \left[\frac{2(\mathbf{n} \cdot \mathbf{v})(\mathbf{v} \cdot \dot{\mathbf{v}})}{c(1 - \mathbf{n} \cdot \mathbf{v}/c)^5} + \frac{\dot{\mathbf{v}}^2}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^4} - \frac{(1 - v^2/c^2)(\mathbf{n} \cdot \dot{\mathbf{v}})^2}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^6} \right]. \quad (\text{XII.26})$$

If the velocity, v , of the charge is small compared with the velocity of light, then the radiation field can be expanded into a multipole expansion and can be calculated with the aid of Eqns. (XII.17)–(XII.22).

As a result of the emission of radiation, the accelerated particle loses energy and momentum which are transferred to the electromagnetic field. The loss of the i -th component of the 4-vector of energy-momentum of the particle $p_i = (\mathbf{p}, i\mathcal{E}/c)$ per unit proper time τ may be expressed in terms of the 4-velocity u_i

and the 4-acceleration w_i of the particles is given by

$$-\frac{dp_i}{d\tau} = \frac{2e^2}{3c^3} w_h^2 u_i. \quad (\text{XII.27})$$

The energy lost by the particle per unit time in the laboratory system, *i. e.* the rate of loss of energy $d\mathcal{E}/dt'$, differs from (XII.27) by the factor $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$, since $dt' = \gamma d\tau$. The intensity of the radiation as determined with the aid of (XII.26) is not in its turn identical to the rate of loss of energy (*cf.* Problems 663 – 669).

The vector potential $\mathbf{A}(\mathbf{R}_0, t)$ due to a charge executing a periodic motion in a closed orbit $\mathbf{r} = \mathbf{r}_0(t')$ with a period $2\pi/\omega_0$ may be expanded into a Fourier series: $\mathbf{A}(\mathbf{R}_0, t) = \sum_{l=-\infty}^{\infty} \mathbf{A}_l \exp(-i\omega_0 l t)$.

At large distances from the orbit, the Fourier component $\mathbf{A}_l$ is given by

$$\mathbf{A}_l = e \frac{e^{ikR_0}}{cR_0 T} \oint e^{i[l\omega_0 t' - \mathbf{k} \cdot \mathbf{r}_0(t')]} \mathbf{v}(t') dt', \quad (\text{XII.28})$$

where

$$\omega_0 = \frac{2\pi}{T}, \quad \mathbf{k} = \frac{l\omega_0}{c} \mathbf{n}.$$

The integral in the above expression may be evaluated over the whole trajectory of the charge.

Charged particles which experience a collision move with a finite acceleration and, therefore, emit electromagnetic energy. The law of motion of the colliding particles, and hence the radiation emitted on collision, depends on the form of the interaction and the impact parameter, s (if the potential energy of the interaction between the two particles is a function of the distance between them only). The energy emitted in all directions when a beam of particles is scattered can be conveniently characterised by the total effective emission

$$\chi = 2\pi \int_0^{\infty} \Delta W(s) s ds, \quad (\text{XII.29})$$

where $\Delta W(s)$ is the energy emitted in a single collision between two particles and s is the corresponding impact parameter.

The angular distribution of the radiation may be characterised by the differential effective emission $d\kappa_{\mathbf{n}}$ which is defined by

$$\frac{d\kappa_{\mathbf{n}}}{d\Omega} = 2\pi \int_0^{\infty} \frac{d[\Delta W_{\mathbf{n}}(s)]}{d\Omega} s ds, \quad (\text{XII.30})$$

where $d[\Delta W_{\mathbf{n}}(s)]/d\Omega$ is the energy emitted per unit solid angle in the direction $\mathbf{n}$ for a single collision of two particles with an impact parameter s , which is averaged with respect to the azimuth in the plane perpendicular to the particle beam. There is an analogous formula for the effective differential emission per unit frequency interval $d\kappa_{\omega}/d\omega$. When dipole radiation predominates, then Eqn. (XII.30) becomes

$$\frac{d\kappa_{\mathbf{n}}}{d\Omega} = \frac{1}{4\pi c^3} [A + BP_2(\cos \vartheta)], \quad (\text{XII.31})$$

where $P_2(\cos \vartheta) = \frac{3}{2} \cos^2 \vartheta - \frac{1}{2}$ is the Legendre polynomial (*cf.* Appendix II), ϑ is the polar angle between the direction $\mathbf{n}$ of the radiation and the direction of the particle beam, and the quantities A and B are given by

$$\begin{aligned} A &= \frac{2}{3} \int_0^{\infty} 2\pi s ds \int_{-\infty}^{+\infty} \ddot{\mathbf{p}}^2 dt, \\ B &= \frac{1}{3} \int_0^{\infty} 2\pi s ds \int_{-\infty}^{+\infty} (\ddot{\mathbf{p}}^2 - 3\ddot{p}_z^2) dt. \end{aligned} \quad (\text{XII.32})$$

The low frequency spectrum of the radiation emitted during a collision occupying a time interval τ may be found from ($\omega\tau \ll 1$)

$$\frac{d\Delta W_{\omega}}{d\omega} = \frac{2}{3\pi c^3} \left[\sum e(v_2 - v_1)^2 \right], \quad (\text{XII.33})$$

where the sum is evaluated over all the colliding particles and $\mathbf{v}_1$, $\mathbf{v}_2$ are the velocities of the particles before and after the collision ($v_1, v_2 \ll c$).

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657*. Derive the Lienard-Wiechert potentials [*cf.* Eqn. (XII.23)] from the general formulae for the retarded potentials.

Hint: The charge distribution in a point particle may be characterised by a volume charge density $\rho(\mathbf{r}', t) = e\delta[\mathbf{r}' - \mathbf{r}_0(t)]$

where $\mathbf{r}_0(t)$ is the position vector of the particle at time t and e is its charge. In evaluating the volume integral with respect to $dV' = dx'dy'dz'$, it will be necessary to use the new variable $\mathbf{R}_1 = \mathbf{r}' - \mathbf{r}_0$.

658*. Find the expansion for the Lienard-Wiechert potentials in powers of R/c by expanding the general expressions for the retarded potentials (XII.1) and (XII.2) in powers of $1/c$.

659. Derive the potentials due to a uniformly moving point charge from the Lienard-Wiechert potentials by expressing the retarded time t' in the latter in terms of the time t of observation of the field (cf. Problems 535 and 711).

660. Find the field strengths due to a uniformly moving point charge using the general formulae (XII.25). Express the field in terms of the time, t , of observation by eliminating the retarded time t' (cf. Problem 535).

661. A charge e moves with a small velocity $\mathbf{v}$ and acceleration $\dot{\mathbf{v}}$ in a bounded region. Find approximate expressions for the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ due to the particle at a point whose distance from the particle is large compared with the linear dimensions of the region within which the particle is moving. Determine the position of the boundary between the quasi-stationary zone and the wave zone.

662. Determine angular distribution and the total intensity of the radiation emitted by the charge considered in the preceding problem.

663*. Express the energy lost by radiation per unit solid angle by a particle in terms of the intensity (flux) given by the Poynting vector. Solve the problem (a) analytically, by considering the relation between the retarded time, t' , and the time of observation, t , and (b) geometrically, by considering the form of the region of space within which the electromagnetic energy emitted by the particle in a time dt' is localised.

664. Show that if a particle executes a periodic motion, then the average rate of energy loss per period is equal to the average intensity of the radiation.

665. Prove the formula given by (XII.26).

666. Find the total rate of emission of energy in all directions by a charged particle, and express it in terms of (a) the velocity $\mathbf{v}(t')$ and acceleration $\dot{\mathbf{v}}(t')$, and (b) the velocity $\mathbf{v}(t')$ and the external field $\mathbf{E}$, $\mathbf{H}$ producing the acceleration of the particle. Assume that the mass, m , and the charge, e , of the particle are given.

667. Express the rate of loss of momentum by a radiating charged particle in terms of the total rate of loss of energy.

668. A radiating particle is observed from two reference systems which experience uniform relative motion. Compare the total rates of energy loss in the two systems.

669. The velocity $\mathbf{v}$ of a relativistic particle at a retarded time t' is parallel to its acceleration $\dot{\mathbf{v}}$. Find the instantaneous angular distribution of the emitted radiation, the total instantaneous intensity, and the total rate of emission of energy.

670. The velocity of a particle decreases from v_0 to zero in a time interval τ . Find the angular distribution of the radiation emitted during this time interval, assuming that the deceleration is constant. Determine the length of the pulse as recorded by stationary instruments.

671. A relativistic particle of charge e , rest mass m , and momentum p moves in a circular orbit in a constant uniform magnetic field $\mathbf{H}$. The radius of the orbit is $a = cp/eH$. Find the total rate of loss of energy by the particle.

672. Find the instantaneous angular distribution of the intensity of radiation emitted by a relativistic particle whose velocity at the retarded time t' is perpendicular to its acceleration. Sketch out the polar diagram for $v \ll c$ and $v \sim c$. Determine the directions in which the radiation intensity is zero.

673*. A particle of charge e and rest mass m moves with a velocity v on a circular orbit in a constant uniform magnetic field $\mathbf{H}$. Find the average angular distribution of the intensity of the emitted radiation. Investigate the ultra-relativistic case $v \sim c$.

Hint: Use the results of the preceding problem. Transform to polar coordinates with the origin at the centre of the circular trajectory and the polar axis parallel to $\mathbf{H}$.

674*. Find the Fourier components $\mathbf{A}_n$, $\mathbf{H}_n$ of the radiation field due to a charge e moving on a circular orbit of radius a with a relativistic velocity v . Investigate the polarisation of the Fourier components.

Hint: Use Eqns. (11) and (9) of Appendix III.

675. Explain the presence of higher harmonics in the spectrum of the field radiated by a charge moving with a constant velocity in a circular orbit (*cf.* preceding problem). What will be the change in the intensity of these harmonics when $\beta = v/c \rightarrow 0$? What will then be the form of the radiation field?

676*. A charge e moves on a circular orbit of radius a with a velocity $v = \beta c$. Find the spectral expansion of the intensity of the radiation emitted in a given direction.

677*. N electrons move simultaneously on a given circular orbit (*cf.* Problem 674). Consider the effect of the interference between the fields produced by these electrons on the intensity of the n -th Fourier component of the emitted radiation. Consider the following special cases: (a) the distribution of the electrons over the circular orbit is random, (b) the electrons are uniformly distributed and lie at angular distances $2\pi/N$ from each other, and (c) the electrons are arranged in a bunch whose dimensions are small compared with the radius of the orbit (the result will then depend on the ratio of the wavelength to the dimensions of the bunch).

678*. Two particles having charges e_1, e_2 and masses m_1, m_2 ($e_1/m_1 \neq e_2/m_2$) execute elliptical motion (*cf.* Problem 614). Find the time average of the total intensity of the radiation emitted by them.

679. Find the average rate of loss of angular momentum of a system of two particles executing an elliptical motion (*cf.* preceding problem).

Hint: A general formula for the change in the angular momentum was obtained in the solution of Problem 632.

680*. Find the effective differential emission $d\mathbf{x}_n/d\Omega$ for the scattering of a beam of particles with charges e_1 , masses m_1 , and velocities v_0 by a similarly charged particle of charge e_2 and mass m_2 .

Hint: In evaluating the integrals for A and B given by (XII.31) replace the integration with respect to t by integration with respect to r where $dt = dr/\dot{r}$, $\dot{r} = v_0[1 - (2a/r) - (s/r)^2]^{1/2}$. s is the impact parameter, and $2a$ is the minimum distance of approach of the two particles (this distance is reached when $s = 0$). Integrate first with respect to s and then with respect to r . In evaluating B , it will be necessary to use the equation for the trajectory of the relative motion, and this will be found in the solution of Problem 614.

681*. A particle of charge e_1 and mass m is scattered by another particle of charge e_2 and mass much greater than m . The impact parameter is s and the kinetic energy of the incident particle is large compared with the potential energy of interaction e_1e_2/r . The velocity v of the incident particle may, therefore,

be looked upon as constant during the entire event and need not be small compared with the velocity of light. Find the angular distribution of the total emission $d\Delta W_{\mathbf{n}}/d\Omega$ and consider in particular the case $\beta = v/c \ll 1$.

Hint: Use the general formula for the angular distribution of total radiation (XII.26). The acceleration of the particle $\dot{\mathbf{v}}$ should be expressed in terms of the Coulomb force acting upon it and also the velocity $\mathbf{v}$, using the formulae $\mathbf{v} = c^2 \mathbf{p}/\mathfrak{E}$ and $\dot{\mathbf{p}} = e_1 e_2 \mathbf{r}/r^3$.

682. Determine the total loss of energy ΔW and momentum $\Delta \mathbf{p}$ by the particle during the collision considered in the preceding problem. Obtain these quantities both by direct integration of the angular distribution found in the preceding problem, and by using the formulae obtained in the solution of Problems 666 and 667.

683*. A particle of charge e_1 and rest mass m collides with a heavy particle of charge e_2 . The impact parameter s is large, so that the kinetic energy of the particle during the motion is large compared with its potential energy, and its velocity is much less than the velocity of light. Find the spectrum of the radiation emitted by the particle.

Hint: Use Eqn. (15) of Appendix III.

684. A beam of particles each of charge e and velocity $v \ll c$ is scattered by a perfectly hard sphere of radius a . Find the effective emission $d\mathfrak{x}_{\omega}$ in the frequency range $d\omega$. Find also the total effective emission $\mathfrak{x}$.

685*. A beam of particles having charges e_1 and masses m_1 is scattered by a particle of charge e_2 and mass m_2 ($e_1/m_1 = e_2/m_2$). Express the effective differential emission $d\mathfrak{x}_{\mathbf{n}}/d\Omega$ in terms of the components $Q_{\alpha\beta}$ of the quadrupole moment of the system. Give the result in a form analogous to (XII.31) and (XII.32).

686*. Find the total effective emission $\mathfrak{x}$ for a beam of charged particles (charge e , rest mass m , velocity v_0) scattered by a similar particle.

3. Interaction of charged particles with radiation

A radiating system of particles, transmitting energy and momentum to the radiation field, experiences a reaction due to this field. If the radiation is of the electric dipole type, then each particle of charge e experiences a radiative deceleration

(radiative friction) and the corresponding force is given by

$$\mathbf{f} = \frac{2e}{3c^3} \ddot{\mathbf{p}}, \quad (\text{XII.34})$$

where $\mathbf{p}$ is the electric dipole moment of the system. For a single charge having a velocity $v \ll c$

$$\mathbf{f} = \frac{2e^2}{3c^3} \ddot{\mathbf{v}}. \quad (\text{XII.35})$$

In the ultra-relativistic case when $v \simeq c$ the radiative friction is of the form

$$f_x = -\frac{2e^4}{3(mc^2)^4} [(E_y - H_z)^2 + (E_z + H_y)^2] \mathcal{E}^2, \quad (\text{XII.36})$$

where the x -axis is taken in the direction of the velocity of the particle, $\mathbf{E}$, $\mathbf{H}$ are the components of the external field in which the particle is moving, and $\mathcal{E} = mc^2/(1 - v^2/c^2)^{1/2}$ is the energy of the particle.

The radiative friction as given by Eqns. (XII.34)–(XII.36) is only approximate. The concept of radiative friction can only be used if the corresponding force is small compared with other forces acting on the particle in its rest system. This condition is satisfied for a particle of charge e and mass m , moving in a given electromagnetic field $\mathbf{E}$, $\mathbf{H}$, provided†

$$\lambda \gg r_0, \quad (\text{XII.37})$$

$$H \ll \frac{m^2 c^4}{e^3} = \frac{e}{r_0^2}, \quad (\text{XII.38})$$

where λ is the wavelength of the radiation emitted by the particle and $r_0 = e^2/mc^2 = 2.8 \times 10^{-13}$ cm is the classical radius of the electron. These conditions indicate that classical electrodynamics gives rise to internal contradictions at small distances (large frequencies) and in strong fields. It should, however, be noted that owing to quantum effects, classical electrodynamics ceases to hold even before these internal contradictions become important. This occurs at distances of the order of $\lambda_0 = h/mc = 137r_0$ and in fields $H \sim e/\lambda_0 r_0 = m^2 c^4/137e^3$.

An electromagnetic wave incident on a system of charges will give rise to an acceleration of these charges. As a result, the system becomes a source of secondary waves and scatters the incident radiation. The scattering process may be characterised

by the differential and total scattering cross-sections which were defined in Section 2 of Chapter VIII.

The electromagnetic field due to a moving charged particle possesses energy, momentum, and therefore mass (electromagnetic mass of the particle). The problem of the electromagnetic mass of elementary particles cannot be resolved on the basis of classical electrodynamics. However, the classical theory gives a satisfactory descriptive account of the *idea* of the electromagnetic mass, and Problems 687 to 690 will illustrate the fundamentals of this theory and also the associated difficulties.

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687*. Find the momentum of the electromagnetic field due to a particle of charge e moving with a uniform velocity $\mathbf{v}$. The particle may be looked upon as a solid sphere of radius r_0 in its own rest system S' (the Lorentz contraction will occur in the system in which the velocity of the particle is v). Introduce the electromagnetic rest mass m_0 of the particle, which is related to its field energy at rest by the Einstein relation, and discuss the difficulties which arise in this connection.

688. Find the energy W_m of the magnetic field and the total electromagnetic energy W of the particle considered in the preceding problem.

689*. Find the force $\mathbf{F}$ which a charged, spherically symmetric particle exerts on itself when it undergoes an accelerated, translational motion at a low velocity $v \ll c$. Retarded effects and the Lorentz contraction may be neglected.

Hint: Calculate the resultant force on small elements of charge de of the particle, using the Lienard-Wiechert expression for the field strength (XII.25).

690*. Find the correct expression for the self force $\mathbf{F}$ on a charged, spherically symmetric particle (*cf.* preceding problem). In deriving this expression, allow for the finite velocity of propagation of the interaction by including terms which are linear in the time of propagation of the interaction between the elements of the particle ($t' - t$). Consider in particular the limiting case of a point particle. Estimate the contribution due to higher order terms in this limiting situation.

691. Calculate the lifetime T which characterises the Rutherford model of the hydrogen atom, assuming that the

electron in the atom moves and radiates as a classical particle. Assume that the electron moves towards the proton in a quasi-circular spiral, so that at each instant it radiates as if it were moving on a circular orbit, *i.e.* assume that the radius of the orbit is a slow function of time. Investigate the conditions under which this approximation is satisfactory. (Initial radius of the atom may be taken to be 5×10^{-9} cm.)

692. A relativistic particle of charge e and rest mass m moves on a circular orbit in a constant, uniform, magnetic field $\mathbf{H}$, and loses energy by the emission of radiation. Find the energy of the particle and the radius of its orbit as a function of time, assuming that at time $t = 0$ the energy is $\mathcal{E}_0$ (*cf.* Problem 691).

693. An electron is accelerated in a betatron and travels on an orbit of constant radius a . The induced electric field responsible for the acceleration is due to an alternating magnetic field of frequency ω . Find the critical energy of the electron $\mathcal{E}_{cr}$ at which the energy lost by the emission of radiation is equal to the energy communicated to the electron by the induced electric field.

694*. A particle of charge e and rest mass m is attracted towards a fixed point with a force $-m\omega_0^2\mathbf{r}$. Free oscillations commence in this harmonic oscillator at a time $t = 0$. Find the law of attenuation of the oscillations by assuming that the radiative reaction is small. Determine the form of the spectrum of the oscillator and the width of a spectral line (natural width). Find the relation between the uncertainty in the energy of the emitted photons and the lifetime of the oscillator.

695. A gas consists of atoms of mass m . Owing to the thermal motion of the atoms and the Doppler effect, an observer at rest relative to the container will record a frequency which is different from the frequency ω_0 of the radiation emitted by these atoms at rest. Neglect the natural line width and find the form of the spectrum emitted by the gas at a temperature T .

Hint: The velocity distribution of the atoms may be assumed to be Maxwellian, *i.e.* given by

$$\frac{dN}{N} = \left(\frac{m}{2\pi kT} \right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}} dv_x dv_y dv_z,$$

where dN/N is the fraction of molecules whose velocity v lies within the range $dv_x dv_y dv_z$ and k is the Boltzmann constant (1.38×10^{-16} erg deg $^{-1}$). Since the velocity of the molecules is

much smaller than the velocity of light, all terms of order higher than v/c may be neglected in the formula for the Doppler frequency change (cf. Problem 511).

696. An emitting atom, which may be described by the harmonic oscillator model, is in translational motion in a gas. As a result of collisions with other atoms, there are discontinuous changes in its oscillations. The probability that the atom will not undergo a collision between time τ and $\tau + d\tau$ is given by $dW(\tau) = \frac{1}{2}\Gamma \exp(-\frac{1}{2}\Gamma\tau)d\tau$, so that the average time between collisions is $\bar{\tau} = 2/\Gamma$. Neglecting the natural line width, find the form of the spectrum emitted by the oscillator.

697*. A group of waves of total intensity $S = \int_0^\infty S_\omega d\omega$ (S is the total amount of energy passing through a unit area) is incident on a three-dimensional isotropic oscillator. The spectral width of the group is large compared with the natural width of a spectral line of the oscillator, γ , and the velocity of the electron is much smaller than the velocity of light. Find the energy absorbed by the oscillator from the wave group, taking into account the braking radiation (bremsstrahlung). What will be the effect on your result of the polarisation and the direction of propagation of the waves making up the group?

698. Find the total amount of energy ΔW absorbed by a one-dimensional oscillator of natural frequency ω_0 from a group of waves of spectral density S_ω in the following three cases: (a) a linearly polarised plane group of waves in which the $\mathbf{E}$ vector is at an angle ϑ to the axis of the oscillator, (b) an unpolarised plane group of waves travelling at an angle θ to the oscillator axis, and (c) an isotropic radiation field (plane waves with any polarisation and direction of propagation are incident on the oscillator with equal probability).

699*. A linearly polarised wave is incident on an isotropic harmonic oscillator. Assuming that the velocity of the electron is much less than the velocity of light, find the differential and total scattering cross-sections, taking into account the radiative friction force. Consider in particular the cases of weakly and strongly bound electrons.

700. A plane, circularly polarised, electromagnetic wave is scattered by a free charge. Determine the scattered field $\mathbf{H}$ and investigate its polarisation. Find also the differential and total scattering cross-sections.

701. An unpolarised plane wave is scattered by a free charge. Find the degree of depolarisation ρ of the scattered wave as a function of the scattering angle ϑ .

702*. A linearly polarised wave is scattered by a free charge, and moves with a relativistic velocity $\mathbf{v}$ in the direction of propagation of the wave. Find the differential scattering cross-section. Consider also the scattering of an unpolarised wave.

Hint: Use Eqn. (XII.26) and express $\dot{\mathbf{v}}$ in terms of $\mathbf{E}$ and $\mathbf{H}$.

703*. An isotropic, harmonic oscillator having a natural frequency ω_0 , charge e , and mass m is placed in a weak, uniform, constant magnetic field $\mathbf{H}$. Determine the motion of the oscillator, and investigate the polarisation of the radiation emitted by it. Note that this harmonic oscillator is a model of an atom in an external, magnetic field, and hence the solution of the problem will be equivalent to the classical theory of the Zeeman effect.

4. Expansion of an electromagnetic field in terms of plane waves

An electromagnetic field is a function of the two independent variables $\mathbf{r}$ and t . In many instances it is convenient to use the Fourier expansion for the field. The following expansions are frequently employed.

(1) Expansion in terms of monochromatic waves:

$$f(\mathbf{r}, t) = \int_{-\infty}^{\infty} f_{\omega}(\mathbf{r}) e^{-i\omega t} d\omega, \quad (\text{XII.39})$$

$$f_{\omega}(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\mathbf{r}, t) e^{i\omega t} dt. \quad (\text{XII.39}')$$

(2) Expansion in terms of plane waves

$$f(\mathbf{r}, t) = \int f_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{r}} (d\mathbf{k}), \quad (\text{XII.40})$$

$$f_{\mathbf{k}}(t) = \frac{1}{(2\pi)^3} \int f(\mathbf{r}, t) e^{-i\mathbf{k} \cdot \mathbf{r}} (d\mathbf{r}), \quad (\text{XII.40}')$$

where $(d\mathbf{k}) = dk_x dk_y dk_z$.

(3) Expansion in terms of plane monochromatic waves

$$f(\mathbf{r}, t) = \int f_{\mathbf{k}\omega} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} (d\mathbf{k}) d\omega, \quad (\text{XII.41})$$

$$f_{\mathbf{k}\omega} = \frac{1}{(2\pi)^4} \int f(\mathbf{r}, t) e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} (d\mathbf{r}) dt. \quad (\text{XII.41}')$$

It follows from Maxwell's equations that the frequency ω is a function of the wave vector $\mathbf{k}$. The equation $\omega = \omega(\mathbf{k})$ is known as the dispersion relation. Since the field components $f(\mathbf{r}, t)$ are real, we have

$$f_{\omega} = f_{-\omega}^*, \quad f_{\mathbf{k}} = f_{-\mathbf{k}}^*, \quad f_{\mathbf{k}\omega} = f_{-\mathbf{k}, -\omega}^*. \quad (\text{XII.42})$$

Eqns. (XII.40) and (XII.41) describe the field in all space. It follows that the integrals in these formulae are to be evaluated over all values of the wave vectors, and over all space.

Eqns. (14) and (15) of Appendix I are very useful in connection with the Fourier expansions. In particular, Eqn. (15) of Appendix I and Eqn. (XII.42) may be used to show that

$$\left. \begin{aligned} \int_{-\infty}^{\infty} f^2(t) dt &= 4\pi \int_0^{\infty} |f_{\omega}|^2 d\omega, \\ \int_{-\infty}^{\infty} f^2(\mathbf{r}, t) (d\mathbf{r}) &= (2\pi)^3 \int \int \int |f_{\mathbf{k}}|^2 (d\mathbf{k}). \end{aligned} \right\} \quad (\text{XII.43})$$

The expansion into plane monochromatic waves is of major importance in quantum electrodynamics where each of these waves is identified with a photon, *i.e.* with a particle moving with the velocity of light. The energy $\mathcal{E}$ and the momentum $\mathbf{p}$ of photons are related to the frequency ω and the wave vector $\mathbf{k}$ by the formulae

$$\mathcal{E} = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}. \quad (\text{XII.44})$$

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704. Prove the expressions given by Eqn. (XII.43).

705. Find the relation between the Fourier components of the fields $\mathbf{E}$, $\mathbf{H}$ and the potentials $\mathbf{A}$, φ (consider all three forms of the Fourier expansion).

706. Write down Maxwell's equations for the Fourier components for the three forms of Fourier expansion considered above. Assume that all space is filled by a homogeneous isotropic dispersive medium with permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$.

707. Write down the d'Alembert equations and the Lorentz condition for the Fourier components of the potentials $\mathbf{A}(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$. Consider all three forms of Fourier expansion and assume that all space is filled with a homogeneous isotropic medium with parameters $\epsilon(\omega)$ and $\mu(\omega)$.

708*. Find the plane-wave expansion for the potential φ of the Coulomb field due to a fixed point charge.

709. Find the plane-wave expansion for the electric field due to a point charge.

710. A point charge moves with a velocity $\mathbf{v} = \text{const}$ in a vacuum. Expand the potentials φ , $\mathbf{A}$ and the fields $\mathbf{E}$, $\mathbf{H}$ into plane monochromatic waves.

711*. Find the potentials $\varphi(\mathbf{r}, t)$, $\mathbf{A}(\mathbf{r}, t)$ due to a point charge in uniform motion (cf. solution to Problem 535), using the plane-wave expansions for these potentials obtained in the preceding problem.

Hint: In order to evaluate the integral with respect to $\mathbf{k}$ carry out the substitution $k_x \rightarrow k_x [1 - (v^2/c^2)]^{-\frac{1}{2}}$, $k_y \rightarrow k_y$, $k_z \rightarrow k_z$. Assume that the x -axis is parallel to the direction of the velocity and use the plane-wave expansion for the field of a fixed point charge (cf. Problem 708).

712*. A neutral point system of charges moves in a vacuum with a uniform velocity $\mathbf{v}$. Find the electromagnetic field potentials $\varphi(\mathbf{r}, t)$, $\mathbf{A}(\mathbf{r}, t)$ by using the Fourier expansion in terms of plane monochromatic waves and assuming that the electric and magnetic dipole moments of the system in the laboratory frame are given.

Hint: The electric charge density and the current density of the system are given by

$$\begin{aligned} \mathbf{j} &= c \text{curl} [\mathbf{m} \delta(\mathbf{r} - \mathbf{v}t)] + \frac{\partial}{\partial t} [\mathbf{p} \delta(\mathbf{r} - \mathbf{v}t)], \\ \rho &= -\text{div} [\mathbf{p} \delta(\mathbf{r} - \mathbf{v}t)]. \end{aligned}$$

713. Find the potentials of the field due to a uniformly moving magnetic dipole having a moment $\mathbf{m}_0$ in its own rest system for $\mathbf{m}_0 \parallel \mathbf{v}$ and $\mathbf{m}_0 \perp \mathbf{v}$.

Hint: Use the transformation formulae of Problem 538.

714. Find the field due to a uniformly moving electric dipole of moment $\mathbf{p}_0$ in its own rest system, using the results of Problem 712 (cf. solution to Problem 537).

715. Show that the plane-wave components of the Fourier expansion of an irrotational vector are parallel to $\mathbf{k}$, while the

Fourier components of a solenoidal vector are perpendicular to $\mathbf{k}$.

716*. Find the equations which are satisfied in a vacuum by the irrotational and solenoidal parts of the electromagnetic field vectors $\mathbf{E}$ and $\mathbf{H}$. Show that the irrotational part of the electric field $\mathbf{E}_{||}(\mathbf{r}, t)$ represents the instantaneous (unretarded) Coulomb field determined by the charge distribution at the same instant of time for which $\mathbf{E}_{||}$ is defined.

717*. Find the plane-wave expansion for the vector potential $\mathbf{A}(\mathbf{r}, t)$ in a vacuum when $\rho = 0$, $\mathbf{j} = 0$ ($\varphi = 0$). The field occupies all space. Write down the Fourier amplitudes of

these waves in the form $\mathbf{A}_{\mathbf{k}\lambda}(t) = \frac{c}{\pi\sqrt{2}} q_{\mathbf{k}\lambda}(t) \mathbf{e}_{\mathbf{k}\lambda}$, where $\mathbf{e}_{\mathbf{k}\lambda}$ is a unit vector which characterises the direction of polarisation of the given transverse wave, so that $\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}\lambda} = 0$ (*cf.* beginning of Section 1, Chapter VIII). On this representation there will be two independent polarisation unit vectors ($\lambda = 1, 2$) for each $\mathbf{k}$. The unit vectors $\mathbf{e}_{\mathbf{k}1}$ and $\mathbf{e}_{\mathbf{k}2}$ are orthogonal, *i. e.* $\mathbf{e}_{\mathbf{k}1} \cdot \mathbf{e}_{\mathbf{k}2}^* = \mathbf{e}_{\mathbf{k}1}^* \cdot \mathbf{e}_{\mathbf{k}2} = 0$. Find the equations which are satisfied by the complex "coordinates" $q_{\mathbf{k}\lambda}(t)$. Express the fields $\mathbf{E}$, $\mathbf{H}$, the energy W , and the momentum $\mathbf{G}$ of the field in terms of $q_{\mathbf{k}\lambda}$ and $\dot{q}_{\mathbf{k}\lambda}$.

718*. Using the results of the preceding problem, express the field vectors $\mathbf{A}$, $\mathbf{E}$, $\mathbf{H}$ in terms of the real oscillator coordinates

$$Q_{\mathbf{k}\lambda} = a_{\mathbf{k}\lambda} e^{-i\omega t} + a_{\mathbf{k}\lambda}^* e^{i\omega t}.$$

Find also the energy W and the momentum $\mathbf{G}$ of the field in terms of these coordinates.

719*. The electromagnetic radiation field may be described in terms of the oscillator coordinates $q_{\mathbf{k}\lambda}$ (*cf.* Problem 717). Write down the differential equations for the interaction between the field and a charged, non-relativistic particle.

720. Find the rate of change in the energy of a radiation field due to the interaction of a particle with the field. Express the rate of change in terms of the oscillator coordinates $q_{\mathbf{k}\lambda}$ and the forces $F_{\mathbf{k}\lambda}(t)$ (*cf.* solution of the preceding problem).

721*. A particle of charge e executes a simple harmonic oscillation $\mathbf{r} = \mathbf{r}_0 \sin \omega_0 t$, where $\mathbf{r}_0 = \text{const.}$ Use the field

oscillator method (*cf.* Problem 719) to find the angular distribution and the total intensity of the emitted radiation†.

722. A particle of charge e moves with a constant angular velocity ω_0 on a circle of radius a_0 . Using the field oscillator method (*cf.* Problem 719), investigate the polarisation of the radiation emitted by the particle, and hence find the angular distribution and the total intensity of the radiation (*cf.* Problem 634).

723*. A linearly polarised wave of frequency ω is incident on a harmonic oscillator whose natural frequency is ω_0 . Using the field oscillator method (*cf.* Problem 719), find the differential and total scattering cross-sections (neglect radiative friction). Investigate the polarisation of the scattered radiation.

724. Find by the field oscillator method the differential and total scattering cross-sections for linearly polarised, circularly polarised, and unpolarised monochromatic waves scattered by a free charge (*cf.* Problem 719, and also 699 and 700).

725. A free charge scatters (a) an unpolarised wave of frequency ω , and (b) a circularly polarised wave. Investigate the polarisation of the radiated field, using the field oscillator method (*cf.* Problem 719, and also 699 and 700).

† The problem can, of course, be solved in a much simpler way (*cf.* Section 1 of this chapter). The method suggested in the problem is closely related to that employed in the solution of the analogous problem in quantum electrodynamics.

Chapter

13

THE RADIATION EMITTED DURING THE INTERACTION OF CHARGED PARTICLES WITH MATTER

In this chapter we shall use the methods of classical macroscopic electrodynamics to discuss energy losses by fast particles in matter.

The macroscopic theory does not take into account the spatial dispersion of the permittivity and magnetic permeability, and may be used whenever the medium may be looked upon as continuous, *i.e.* when the incident particle interacts simultaneously with a large number of atoms. This means that the macroscopic equations may be used to determine the energy communicated to those electrons in the medium which are at sufficiently large distances r from the trajectory, so that $r \gg a$, where a is of the order of the interatomic distance. In condensed media, the quantity a is of the order of the linear dimensions of the atom ($\sim 10^{-8}$ cm).

The velocity of the particle v should satisfy the condition $v \gg v_{at}$, where v_{at} is the average velocity of the atomic electrons. At lower velocities, the particle will transmit its energy mainly to electrons near its trajectory, where the macroscopic equations do not hold.

Energy losses due to ionisation and excitation of atoms in the medium are referred to as ionisation losses. For a particle moving through a plasma, a considerable proportion of the energy lost is used in the excitation of oscillations of the electron gas as a whole (longitudinal plasma waves; *cf.* Problems 456 and 739).

The medium is also found to have an important effect on the emission of transverse electromagnetic waves by moving particles. Thus, if the velocity of a charged particle in a non-absorbing dielectric exceeds the ambient phase velocity of light, then the Vavilov-Cherenkov radiation will be produced.

The electromagnetic field produced in a medium by a moving particle may be determined from Maxwell's equations. The charge and current densities for these equations may conveniently

be written in the form $\rho = e\delta[\mathbf{r} - \mathbf{r}_0(t)]$ and $\mathbf{j} = e\dot{\mathbf{r}}_0\delta[\mathbf{r} - \mathbf{r}_0(t)]$, where e is the charge of the particle and $\mathbf{r}_0(t)$ is its position vector. The integration of Maxwell's equations for a dispersive medium is carried out by expanding the required quantities (field vectors) in terms of Fourier integrals with respect to the coordinates and time. A system of algebraic equations is then obtained for the Fourier components (*cf.*, for example, Problem 726).

In order to find the energy of the Vavilov-Cherenkov radiation emitted per unit length of the trajectory of a particle, it is necessary to determine the electromagnetic field due to this particle in the medium, and to compute the energy flux through a cylindrical surface of unit length and infinite radius surrounding the trajectory of the particle. The time integral of this energy flux will give the total energy emitted by the particle per unit path length in the form of electromagnetic waves. If the radius a of the cylindrical surface is finite, then the time integral of the energy flux will include not only the Vavilov-Cherenkov radiation, but also the energy communicated to electrons at distances $r > a$ from the trajectory of the particle.

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726*. A particle of charge e moves with a constant velocity through a homogeneous isotropic medium of permittivity $\epsilon(\omega)$ and magnetic permeability $\mu = 1$. Determine the components of the electromagnetic field due to this particle.

727*. A particle moves with a constant velocity $v = \beta c$ in a non-absorbing dielectric. Using the results of the preceding problem, investigate the field due to this particle at a large distance from its trajectory. Show that a sufficiently fast particle will emit transverse electromagnetic waves (Vavilov-Cherenkov effect). Find the conditions under which this radiation will be emitted, and the total Cherenkov loss per unit path length.

728. A particle of charge e moves with a constant velocity through a medium whose permittivity is given by the approximate formula

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}.$$

Determine the Vavilov-Cherenkov energy emitted per unit path length when the particle velocity satisfies the condition $v^2\epsilon_0 > c^2$,

where ϵ_0 is the static permittivity. Find the range of angles within which the radiation is concentrated. Obtain a numerical estimate, assuming that

$$\omega_0 = 6 \cdot 10^{15} \text{ sec}^{-1}, \quad \epsilon_0 = 2, \quad v = c.$$

729. Derive the condition $\cos \theta = (\beta n)^{-1}$, giving the direction of propagation of the Vavilov-Cherenkov radiation, by considering the interference between waves emitted by the particle at various points along its trajectory.

730. The Cherenkov radiation emitted by a particle may be looked upon as the result of a resonance between natural oscillations in the medium and an impressed force associated with the moving particle. Find the condition for the appearance of the Vavilov-Cherenkov effect by comparing the frequencies of the natural oscillations of the medium and those of the exciting force.

731. Show that the minimum velocity $v_{\min}$ of a particle at which the Vavilov-Cherenkov effect becomes possible in a particular direction, satisfies the condition

$$v_{\min} \cos \theta = v_g(\omega_m),$$

where v_g is the group velocity of electromagnetic waves in the dielectric, ω_m is the frequency at which the refractive index has a maximum, and θ is the angle between the direction of emission and the velocity of the particle. The dielectric may be regarded as non-absorbing.

732*. A particle moves with a constant velocity $v = \beta c$ in a non-dispersive medium which has a permittivity ϵ and magnetic permeability μ . Determine the electromagnetic potentials φ and $\mathbf{A}$. Consider the two situations $v < v_\varphi$ and $v > v_\varphi$, where v_φ is the phase velocity of electromagnetic waves in the medium under investigation.

733. A straight-line conductor lying along the x -axis is displaced along the y -axis with a constant velocity v in a non-absorbing medium of permittivity $\epsilon(\omega)$ and magnetic permeability $\mu(\omega)$. The conductor is electrically neutral in the laboratory frame, and carries a current $\mathcal{J}$ in the direction of the x -axis. Find the condition for the emission of the Vavilov-Cherenkov radiation. Determine the total energy of this radiation per unit path length, and find the retarding force $\mathbf{f}$ acting per unit length of the conductor due to the field produced by it.

Hint: The vector potential has the single component $A_x(y, z, t)$.

734. Two point charges e_1 and e_2 move with equal constant velocities v along the same straight line, but at a distance l (measured in the laboratory frame) from each other. Assuming that the motion takes place in a medium of permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu = 1$, find the energy of the Vavilov-Cherenkov radiation emitted per unit path length. Consider the cases $e_1 = e_2 = e$, and $e_1 = -e_2 = e$. By passing to the limit, find the Cherenkov energy losses for a point electric dipole lying along the direction of motion.

735*. Two point charges, $+e$ and $-e$, separated by a constant distance l , move with a constant velocity v in a medium of permittivity $\varepsilon(\omega)$ and permeability $\mu = 1$. The line joining the charges is at an angle α to the direction of motion (l and α are measured in the laboratory system). Use the method employed in the preceding problem to find the Vavilov-Cherenkov energy emitted per unit path length when l is very small.

736*. A magnetic dipole moves with a constant velocity $v = \beta c$ in a non-absorbing medium which has a permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$. The magnetic moment measured in the laboratory system is $\mathbf{m}$ and is parallel to the direction of the velocity. Determine the Vavilov-Cherenkov energy losses per unit path length.

Hint: Use the Fourier transformation to integrate the equations for the potentials. A moving magnetic dipole gives rise to a current $\mathbf{j}(\mathbf{r}, t) = c \operatorname{curl} \mathbf{m} \delta(\mathbf{r} - \mathbf{v}t)$.

737*. A fast particle of charge e moves through a non-absorbing dielectric of permittivity

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2},$$

where $\omega_p^2 = 4\pi e^2 N/m$. Determine the energy lost per unit path length at distances from the particle trajectory which are greater than the interatomic distance a . The parameter a should be chosen so that the macroscopic equations will hold in the region $r > a$. Explain the physical significance of the various terms in the expression for the energy loss.

738*. A charged particle of velocity $v = \beta c$ moves through a plasma of permittivity

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2},$$

where $\omega_p^2 = 4\pi e^2 N/m$. Find the energy losses per unit path length due to distant collisions, *i.e.* collisions for which $r > a$, where a is the distance at which the macroscopic equations begin to hold.

739. Show that the longitudinal plasma waves considered in Problem 456 take the form of an oscillation of the electron cloud relative to the set of ions. To do this, consider an electrically neutral (on the average) plasma filling a rectangular parallelepiped of volume $V = a_1 a_2 a_3$. Find the equations of motion for the Fourier components $\rho_{\mathbf{k}}$ of the electronic charge density (the positive ions may be looked upon as fixed). Show that for small $\mathbf{k}$ the density $\rho_{\mathbf{k}}$ will execute harmonic oscillations with the "plasma frequency" $\omega_p = (4\pi e^2 N/m)^{1/2}$, where N is the average number of electrons per unit volume, and e, m are the electronic charge and mass. Estimate the range of values of $\mathbf{k}$ for which the above description will hold.

740*. A point charge e moves in a vacuum in a direction perpendicular to the boundary of a perfect conductor. Determine the spectral and the angular distribution of the radiation emitted when the charge enters the conductor. The acceleration of the charge due to its interaction with the image induced in the conductor may be neglected.

Hint: The field in the vacuum is due to the charge and its image moving towards each other with equal constant velocities. When the particle reaches the boundary of the conductor, its charge is instantaneously screened by the free electrons in the conductor, and this is equivalent to a sudden deceleration of the charge and its image at the same point, *i.e.* on the boundary of the conductor.

741*. A point charge e has a velocity $v = \beta c$ and moves in a vacuum in a direction perpendicular to the boundary of a non-absorbing dielectric of permittivity $\epsilon(\omega)$ ($\mu = 1$). Neglecting the acceleration of the charge due to image forces, determine the spectral and angular distribution of the radiation emitted into the vacuum ($z > 0$ in Fig. 100) when the particle enters the dielectric.

Hint: The charge and current densities produced by the moving particle should be replaced by the equivalent set of harmonic oscillators. In order to determine the field in the wave zone, use the reciprocity theorem $\mathbf{p}_B \cdot \mathbf{E}_A(B) = \mathbf{p}_A \cdot \mathbf{E}_B(A)$, where $\mathbf{E}_B(A)$ is the field produced at the point A by the harmonic dipole oscillator $\mathbf{p}_B$ at B , and $\mathbf{E}_A(B)$ is the field produced at B by the oscillator $\mathbf{p}_A$ at A . Fresnel's formulae may be used to calculate

$\mathbf{E}_A(B)$, since the point of observation A is at a large distance from the point at which the charge enters the dielectric (wave-zone approximation).

Answers and Solutions

Chapter 1

VECTOR AND TENSOR CALCULUS

1. Vector and tensor algebra. Transformations of vectors and tensors

$$1. \quad \cos \theta = \mathbf{n} \cdot \mathbf{n}' = \cos \alpha \cos \alpha' + \sin \alpha \sin \alpha' \cos (\beta - \beta').$$

3. Since b_i ($i = 1, 2, 3$) are the components of a vector, it follows that when the system of coordinates is rotated, $b'_i = \alpha_{ik} b_k$. Substituting b'_i into $a'_i b'_i = \text{inv}$ and comparing with $a_k b_k = \text{inv}$, we find that $a_k = \alpha_{ik} a'_i$, *i. e.* a_k transform as the components of a vector when the system of coordinates is rotated. Since the invariant (a scalar) does not change sign on reflection, the components a_i and b_i should either both change sign (polar vectors) or should both remain unaltered (pseudovectors).

$$10. \quad (\mathbf{a} \times \mathbf{b})_0 = i(a_{-1}b_{+1} - a_{+1}b_{-1}), \quad (\mathbf{a} \times \mathbf{b})_{\pm 1} = \pm i(a_0b_{\pm 1} - a_{\pm 1}b_0),$$

$$(\mathbf{a} \cdot \mathbf{b}) = \sum_{\mu=-1}^{\mu=1} (-1)^\mu a_{-\mu} b_\mu, \quad r_\mu = \sqrt{\frac{3}{4\pi}} r Y_{1\mu}.$$

11. The tensor reciprocal to ϵ_{ik} satisfies the conditions

$$\epsilon_{ik} \epsilon_{kl}^{-1} = \delta_{il}. \quad (1)$$

The above is a set of algebraic equations involving the components of the reciprocal tensor ϵ_{ik}^{-1} . The solutions are of the form

$$\epsilon_{ik}^{-1} = \frac{\Delta_{kl}}{|\epsilon|}, \quad (2)$$

where Δ_{ik} is the cofactor of the element ϵ_{ik} in the determinant $|\epsilon|$. It follows from Eqn. (2) that the necessary condition for the existence of the reciprocal tensor is $|\epsilon| \neq 0$. Bearing in mind the well-known property of determinants $\Delta_{ki} \epsilon_{kl} = \delta_{il} |\epsilon|$, we see that the reciprocal tensor must, in addition to Eqn. (1), also satisfy the conditions

$$\epsilon_{ik}^{-1} \epsilon_{kl} = \delta_{il}. \quad (3)$$

If ϵ_{ik} is a symmetric tensor defined along the principal axes:

$\varepsilon_{ik} = \varepsilon^{(i)} \delta_{ik}$ (no summation over i), then

$$\varepsilon_{ik}^{-1} = \frac{1}{\varepsilon^{(i)}} \delta_{ik}.$$

14. T_{ik} form a tensor of rank 2.

15. In the transformation $\mathbf{e}_i \rightarrow \mathbf{e}'_i$, the coefficients $\alpha_{ik} = \mathbf{e}'_i \cdot \mathbf{e}_k$ in $\mathbf{e}'_i = \alpha_{ik} \mathbf{e}_k$ represent the components of the new unit vectors along the old unit vectors. On evaluating the components

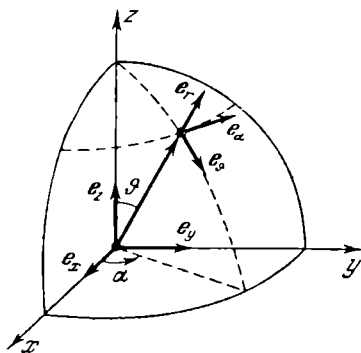


Fig. 28.

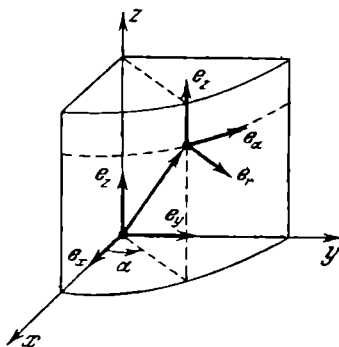


Fig. 29.

(Figs. 28 and 29) we obtain the following transformation matrices:

(a) transformation from cartesian to spherical coordinates:

$$\alpha = \begin{pmatrix} \sin \vartheta \cos \alpha & \sin \vartheta \sin \alpha & \cos \vartheta \\ \cos \vartheta \cos \alpha & \cos \vartheta \sin \alpha & -\sin \vartheta \\ -\sin \alpha & \cos \alpha & 0 \end{pmatrix},$$

$$\hat{\alpha}^{-1} = \begin{pmatrix} \sin \vartheta \cos \alpha & \cos \vartheta \cos \alpha & -\sin \alpha \\ \sin \vartheta \sin \alpha & \cos \vartheta \sin \alpha & \cos \alpha \\ \cos \vartheta & -\sin \vartheta & 0 \end{pmatrix};$$

(b) transformation from cartesian to cylindrical coordinates:

$$\hat{\alpha} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{\alpha}^{-1} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

16. Let $\hat{g}$ represent the matrix relating the components of the vector in coordinate systems S' and S ($A'_i = g_{ik} A_k$). We then have:

(a) for the case of reflection

$$\hat{g}_- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

(b) for the case of rotation

$$\hat{g}(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where α is positive in the clockwise direction (on looking in the direction of the z -axis).

17. Using the results of the preceding problem we have

$$\begin{aligned} \hat{g}(\alpha_1 \theta \alpha_2) &= \hat{g}(\alpha_2) \hat{g}(\theta) \hat{g}(\alpha_1) = \\ &= \begin{pmatrix} \cos \alpha_1 \cos \alpha_2 - \cos \theta \sin \alpha_1 \sin \alpha_2; & \sin \alpha_1 \cos \alpha_2 + \cos \theta \cos \alpha_1 \sin \alpha_2; & \sin \theta \sin \alpha_2 \\ -\cos \alpha_1 \sin \alpha_2 - \cos \theta \sin \alpha_1 \cos \alpha_2; & -\sin \alpha_1 \sin \alpha_2 + \cos \theta \cos \alpha_1 \cos \alpha_2; & \sin \theta \cos \alpha_2 \\ \sin \alpha_1 \sin \theta & -\sin \theta \cos \alpha_1 & \cos \theta \end{pmatrix}. \end{aligned}$$

18.

$$\hat{D}(\alpha_1 \theta \alpha_2) = \begin{pmatrix} \frac{1}{2}(1 + \cos \theta) e^{i(\alpha_1 + \alpha_2)}; & -\frac{i \sin \theta}{\sqrt{2}} e^{i\alpha_2}; & -\frac{1}{2}(1 - \cos \theta) e^{i(\alpha_2 - \alpha_1)} \\ -\frac{i \sin \theta}{\sqrt{2}} e^{i\alpha_1}; & \cos \theta; & -\frac{i}{\sqrt{2}} \sin \theta e^{-i\alpha_1} \\ -\frac{1}{2}(1 - \cos \theta) e^{i(\alpha_1 - \alpha_2)}; & -\frac{i}{\sqrt{2}} \sin \theta e^{-i\alpha_2}; & \frac{1}{2}(1 + \cos \theta) e^{-i(\alpha_1 + \alpha_2)} \end{pmatrix}.$$

19. Since the matrix corresponding to rotation through zero angle (identity transformation) is the unit matrix $\hat{1}$, it follows that for a rotation through a small angle $|\varepsilon_{ik}| \ll 1$. In order to prove the relation $\varepsilon_{ik} = -\varepsilon_{ki}$, we can use the invariance of $r^2 = \delta_{ik} x_i x_k$ with respect to rotation. Since $x'_i = \alpha_{ik} x_k = x_i + \varepsilon_{ik} x_k$, it follows that to within terms of the first order we have $r'^2 = r^2 + 2\varepsilon_{ik} x_i x_k$. It follows from the invariance of r^2 that $\varepsilon_{ik} x_i x_k = 0$ for arbitrary x_i , and this is only possible when $\varepsilon_{ik} = -\varepsilon_{ki}$. Consider now the vector $\delta\varphi$ with components $\delta\varphi_i = \frac{1}{2} \varepsilon_{ikl} \varepsilon_{kl}$. We then have $\mathbf{r}' = \mathbf{r} + \delta\varphi \times \mathbf{r}$, and it is clear that $\delta\varphi$ is a vector representing an infinitesimal rotation whose direction is parallel to the axis of rotation and whose magnitude is equal to the angle of rotation.

22. The proof is the same for any number of transformations. Let the transformation matrix be $\hat{\alpha}$ and its determinant $|\hat{\alpha}|$. Since the matrix $\hat{\alpha}$ is orthogonal, there are n^2 equations of the form $\alpha_{ik} \alpha_{lk} = \delta_{il}$. Since the left-hand sides of these equations contain the elements of the determinant which is equal to the product of two determinants $|\hat{\alpha}|$, we have $|\hat{\alpha}| \cdot |\hat{\alpha}| = |\hat{1}| = 1$, or $|\hat{\alpha}|^2 = 1$. Hence, $|\alpha| = \pm 1$.

It will now be shown that for rotations $|\hat{\alpha}| = +1$. For a rotation through a zero angle (identity) $|\hat{\alpha}| = |\hat{1}| = 1$. Since the

elements of the matrix $\hat{\alpha}$ are continuous functions of the parameters defining the rotation (e.g. the Euler angles; cf. Problem 17), it follows that the result $|\hat{\alpha}| = 1$ will also hold for finite rotations.

The form of $|\hat{\alpha}|$ which corresponds to reflections is

$$|\hat{\alpha}| = \begin{vmatrix} \pm 1 & 0 & 0 & \dots \\ 0 & \pm 1 & 0 & \dots \\ 0 & 0 & \pm 1 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

The negative signs of the diagonal elements correspond to reflected axes. It is clear that $|\hat{\alpha}| = +1$ for an even number of such axes, and $|\hat{\alpha}| = -1$ for an odd number.

24. Of all the 27 quantities e_{ikl} , only six have non-zero values. The remainder have at least two identical subscripts, and in view of the skew-symmetry, they are all equal to zero ($e_{iik} = -e_{iik} = 0$). The finite components are

$$e_{123} = e_{312} = e_{231} = -e_{321} = -e_{213} = -e_{132} = 1.$$

Consider now the expression $\alpha_{1i}\alpha_{2k}\alpha_{3l}e_{ikl}$. Recalling the definition of a third-order determinant and the definition of e_{ikl} , we may rewrite this expression in the form $\alpha_{1i}\alpha_{2k}\alpha_{3l}e_{ikl} = |\hat{\alpha}| = +1 = e'_{123}$. On interchanging two subscripts on the left, e.g. 1 and 2, we have

$$\alpha_{2i}\alpha_{1k}\alpha_{3l}e_{ikl} = -\alpha_{1k}\alpha_{2i}\alpha_{3l}e_{kil} = -e'_{123} = e'_{213} \dots$$

It is clear from these equations that e_{ikl} transform on rotation as a tensor of rank 3. The quantities e_{ikl} remain unaltered on reflection and hence they form an axial tensor of rank 3. This tensor has the interesting property that its components are the same in all coordinate systems.

25. The tensor A_{ik} may be written in the form of the table

$$A_{ik} = \begin{pmatrix} 0 & A_{12} & -A_{31} \\ -A_{21} & 0 & A_{23} \\ A_{31} & -A_{23} & 0 \end{pmatrix}.$$

Let $A_{23} = A_1$, $A_{31} = A_2$, and $A_{12} = A_3$. These three equations may also be written in the form $A_i = \frac{1}{2}e_{ikl}A_{kl}$, where e_{ikl} is a skew-symmetric unit tensor of rank 3 (cf. preceding problem). Since A_{kl} is a tensor of rank 2, it follows that the quantities

A_i ($i = 1, 2, 3$) form a vector. The vector A_i is called dual with respect to the tensor A_{ik} .

26. $(\mathbf{A} \times \mathbf{B})_i = e_{ikl} A_k B_l$, $\text{curl}_i \mathbf{A} = e_{ikl} \partial A_l / \partial x_k$. $\mathbf{A} \times \mathbf{B}$ and $\text{curl} \mathbf{A}$ may be looked upon as skew-symmetric tensors of rank 2, or as their dual vectors whose components do not change sign on reflection (pseudovectors).

28. a) $a^2(\mathbf{b} \cdot \mathbf{c}) + (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})$; b) $[(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \cdot [(\mathbf{a}' \times \mathbf{b}') \times \mathbf{c}']$.

30. $(\mathbf{a} \cdot \mathbf{a}')(\mathbf{b} \cdot \mathbf{b}')(\mathbf{c} \cdot \mathbf{c}') + (\mathbf{a} \cdot \mathbf{b}')(\mathbf{b} \cdot \mathbf{c}')(\mathbf{c} \cdot \mathbf{a}') +$
 $+ (\mathbf{b} \cdot \mathbf{a}')(\mathbf{c} \cdot \mathbf{b}')(\mathbf{a} \cdot \mathbf{c}') - (\mathbf{a} \cdot \mathbf{c}')(\mathbf{c} \cdot \mathbf{a}')(\mathbf{b} \cdot \mathbf{b}') - (\mathbf{a} \cdot \mathbf{b}')(\mathbf{b} \cdot \mathbf{a}')(\mathbf{c} \cdot \mathbf{c}') -$
 $- (\mathbf{b} \cdot \mathbf{c}')(\mathbf{c} \cdot \mathbf{b}')(\mathbf{a} \cdot \mathbf{a}')$.

31. Consider the proof for a vector and a tensor of rank 2.

(a) Since the components of the vector should, by definition, be the same in all systems of coordinates, it follows that for any rotation $A'_i = A_i$, *i.e.*

$$A'_x = A_x, \quad A'_y = A_y, \quad A'_z = A_z. \quad (1)$$

Let us rotate the system of coordinates about the z -axis through an angle π . Since, in general, the transformation formulae for rotations are $A'_i = \alpha_{ik} A_k$, we find that

$$A'_x = -A_x, \quad A'_y = -A_y, \quad A'_z = A_z. \quad (2)$$

Eqns. (1) and (2) are consistent only if $A_x = A_y = 0$. Next, consider the rotation of the coordinate system about the x -axis through an angle π . As before, we can show that $A_z = 0$, *i.e.* the vector $\mathbf{A} = 0$ if its components are independent of the choice of the coordinate system.

(b) An arbitrary tensor of rank 2 may be written down in the form of a sum of symmetric and skew-symmetric tensors: $T_{ik} = S_{ik} + A_{ik}$. The skew-symmetric tensor is equivalent to a pseudovector (*cf.* Problem 25), and in view of the above property of a vector, its components are independent of the choice of the coordinate system only if they are equal to zero. We shall therefore consider the symmetric tensor S_{ik} .

Let us select a coordinate system in which S_{ik} has the diagonal form $\lambda^{(i)} \delta_{ik}$. If the $\lambda^{(i)}$ are not all equal, then the components of the tensor will depend on the choice of the axes, *i.e.* on which subscript (1, 2, or 3) is used to represent a given axis. The components of the tensor will be independent of the choice of the axes only if $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = \lambda$. The tensor will then be of the form $\lambda \delta_{ik}$.

32. The required mean values are given by the following integrals

$$\overline{n_i} = \frac{1}{4\pi} \int n_i d\Omega, \quad \overline{n_i n_k} = \frac{1}{4\pi} \int n_i n_k d\Omega \dots \quad (1)$$

However, instead of the direct evaluation of these integrals it is more convenient to use a method based upon the transformation properties of the quantities under consideration. It is clear that the quantities $\overline{n_i}$, $\overline{n_i n_k}$, and so on, are tensors of rank 1, 2, 3, 4,, respectively. On the other hand, it follows from their definition [Eqn. (1)] that these quantities should be independent of the choice of the coordinate system. They will therefore be expressed in terms of tensors whose components are independent of the choice of the coordinate system.

Consider $\overline{n_i}$. Since the only vector whose components are independent of the choice of the coordinate system is the zero vector (*cf.* Problem 31), it follows that $\overline{n_i} = 0$.

The tensor $\overline{n_i n_k}$ should be capable of being expressed in terms of a symmetric tensor of rank 2 whose components are independent of the choice of the coordinate system. The only tensor satisfying this condition is δ_{ik} .

We therefore have

$$\overline{n_i n_k} = \lambda \delta_{ik}, \quad (2)$$

where λ can be determined by contracting the tensor:

$$\overline{n_i n_i} = \overline{n^2} = 1 = 3\lambda, \quad \lambda = \frac{1}{3}.$$

Similarly,

$$\begin{aligned} \overline{n_i n_k n_l} &= 0, \\ \overline{n_i n_k n_l n_m} &= \frac{1}{15} (\delta_{ik} \delta_{lm} + \delta_{il} \delta_{km} + \delta_{im} \delta_{kl}). \end{aligned} \quad (3)$$

$$\begin{aligned} 33. \quad & \frac{1}{3} a^2, \quad \frac{1}{3} \mathbf{a} \cdot \mathbf{b}, \quad \frac{1}{3} \mathbf{a}, \quad \frac{2}{3} a^2, \quad \frac{2}{3} \mathbf{a} \cdot \mathbf{b}, \\ & \frac{1}{15} [(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) + (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) + (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})]. \end{aligned}$$

$$34. \quad \mathbf{n} \cdot \mathbf{n}', \quad (\mathbf{n} \times \mathbf{n}') \cdot \mathbf{l}.$$

$$35. \quad \mathbf{n} \cdot \mathbf{l}, \quad \mathbf{n}' \cdot \mathbf{l}, \quad \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3).$$

2. Vector analysis

$$36. \quad \nabla_{\pm 1} = \mp \frac{1}{\sqrt{2}} e^{\pm i\alpha} \left(\sin \vartheta \frac{\partial}{\partial r} + \frac{\cos \vartheta}{r} \frac{\partial}{\partial \vartheta} \pm \frac{i}{r \sin \vartheta} \frac{\partial}{\partial \alpha} \right),$$

$$\nabla_0 = \cos \vartheta \frac{\partial}{\partial r} - \frac{\sin \vartheta}{r} \frac{\partial}{\partial \vartheta}.$$

$$37. \quad \operatorname{div} \mathbf{r} = 3, \quad \operatorname{curl} \mathbf{r} = 0, \quad \operatorname{grad} (\mathbf{l} \cdot \mathbf{r}) = \mathbf{l}, \quad (\mathbf{l} \cdot \nabla) \mathbf{r} = \mathbf{l}.$$

$$38. \quad \operatorname{curl} (\boldsymbol{\omega} \times \mathbf{r}) = 2\boldsymbol{\omega}.$$

$$41. \quad \operatorname{grad} \varphi(r) = \frac{\mathbf{r}}{r} \varphi'; \quad \operatorname{div} \varphi(r) \mathbf{r} = 3\varphi + r\varphi';$$

$$\operatorname{curl} \varphi(r) \mathbf{r} = 0; \quad (\mathbf{l} \cdot \nabla) \varphi(r) \mathbf{r} = \mathbf{l}\varphi + \frac{\mathbf{r}(\mathbf{l} \cdot \mathbf{r})}{r} \varphi'.$$

Here, and also below, differentiation with respect to r is indicated by an apostrophe.

$$42. \quad \varphi(r) = \frac{\text{const}}{r^3}.$$

$$43. \quad \operatorname{div} (\mathbf{r} \cdot \mathbf{a}) \mathbf{b} = \mathbf{a} \cdot \mathbf{b}, \quad \operatorname{curl} (\mathbf{r} \cdot \mathbf{a}) \mathbf{b} = \mathbf{a} \times \mathbf{b}, \quad \operatorname{div} (\mathbf{a} \cdot \mathbf{r}) \mathbf{r} =$$

$$= 4(\mathbf{a} \cdot \mathbf{r}), \quad \operatorname{curl} (\mathbf{a} \cdot \mathbf{r}) \mathbf{r} = \mathbf{a} \times \mathbf{r}, \quad \operatorname{div} (\mathbf{a} \times \mathbf{r}) = 0, \quad \operatorname{curl} (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a},$$

$$\operatorname{div} \varphi(r) (\mathbf{a} \times \mathbf{r}) = 0, \quad \operatorname{curl} \varphi(r) (\mathbf{a} \times \mathbf{r}) = (2\varphi + r\varphi') \mathbf{a} - \frac{\mathbf{r}(\mathbf{a} \cdot \mathbf{r})}{r} \varphi',$$

$$\operatorname{div} \mathbf{r} \times (\mathbf{a} \times \mathbf{r}) = -2(\mathbf{a} \cdot \mathbf{r}), \quad \operatorname{curl} \mathbf{r} \times (\mathbf{a} \times \mathbf{r}) = 3(\mathbf{r} \times \mathbf{a}).$$

$$44. \quad \operatorname{grad} \mathbf{A}(r) \mathbf{r} = \mathbf{A} + \frac{\mathbf{r}}{r} (\mathbf{r} \cdot \mathbf{A}'), \quad \operatorname{grad} \mathbf{A}(r) \cdot \mathbf{B}(r) =$$

$$= \frac{\mathbf{r}}{r} (\mathbf{A}' \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{B}'), \quad \operatorname{div} \varphi(r) \mathbf{A}(r) = \frac{\varphi'}{r} (\mathbf{r} \cdot \mathbf{A}) + \frac{\varphi}{r} (\mathbf{r} \cdot \mathbf{A}'),$$

$$\operatorname{curl} \varphi(r) \mathbf{A}(r) = \frac{\varphi'}{r} (\mathbf{r} \times \mathbf{A}) + \frac{\varphi}{r} (\mathbf{r} \times \mathbf{A}'), \quad (\mathbf{l} \cdot \nabla) \varphi(r) \mathbf{A}(r) = \frac{\mathbf{l} \cdot \mathbf{r}}{r} (\varphi' \mathbf{A} + \varphi \mathbf{A}').$$

$$45. \quad -\operatorname{grad} \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right) = \operatorname{curl} \left(\frac{\mathbf{p} \times \mathbf{r}}{r^3} \right); \quad \text{the components of this}$$

vector along the basic unit vectors $\mathbf{e}_r$, $\mathbf{e}_\vartheta$, $\mathbf{e}_\alpha$ are respectively equal to

$$\frac{2p \cos \vartheta}{r^3}, \quad \frac{p \sin \vartheta}{r^3}, \quad 0.$$

The vector lines are formed by the intersection of the two families of surfaces $\alpha = C_1$, $r = C_2 \sin \vartheta$.

$$47. \quad (\Delta \mathbf{A})_r = \Delta A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta A_\vartheta) - \frac{2}{r^2 \sin \vartheta} \frac{\partial A_\alpha}{\partial \alpha},$$

$$(\Delta \mathbf{A})_\vartheta = \Delta A_\vartheta - \frac{A_\vartheta}{r^2 \sin^2 \vartheta} + \frac{2}{r^2} \frac{\partial A_r}{\partial \vartheta} - \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial A_\alpha}{\partial \alpha},$$

$$(\Delta \mathbf{A})_\alpha = \Delta A_\alpha - \frac{A_\alpha}{r^2 \sin^2 \vartheta} + \frac{2}{r^2 \sin \vartheta} \frac{\partial A_r}{\partial \alpha} + \frac{2 \cos \vartheta}{r^2 \sin^2 \vartheta} \frac{\partial A_\vartheta}{\partial \alpha}.$$

$$\begin{aligned}
 48. \quad (\Delta \mathbf{A})_r &= \Delta A_r - \frac{A_r}{r^2} - \frac{2}{r^2} \frac{\partial A_a}{\partial \alpha}, \\
 (\Delta \mathbf{A})_a &= \Delta A_a - \frac{A_a}{r^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \alpha}, \\
 (\Delta \mathbf{A})_z &= \Delta A_z.
 \end{aligned}$$

$$49. \quad \int (\text{grad } \varphi \cdot \text{curl } \mathbf{A}) dV = \oint (\mathbf{A} \times \text{grad } \varphi) dS = \int \varphi \text{curl } \mathbf{A} dS.$$

50. Here, as in a number of other cases, it is convenient to consider the scalar product of the integrals with a constant vector $\mathbf{c}$:

$$\begin{aligned}
 \mathbf{c} \cdot \oint \mathbf{r}(\mathbf{a} \cdot \mathbf{n}) dS &= \oint (\mathbf{c} \cdot \mathbf{r}) a_n dS = \int \text{div} [(\mathbf{c} \cdot \mathbf{r}) \mathbf{a}] dV = \\
 &= (\mathbf{a} \cdot \mathbf{c}) \int dV = (\mathbf{a} \cdot \mathbf{c}) V.
 \end{aligned}$$

Since $\mathbf{c}$ is an arbitrary vector it follows that $\int (\mathbf{a} \cdot \mathbf{n}) \mathbf{r} dS = \mathbf{a}V$.

Similarly, $\oint (\mathbf{a} \cdot \mathbf{r}) \mathbf{n} dS = \mathbf{a}V$.

$$\begin{aligned}
 51. \quad \oint \mathbf{n} \varphi dS &= \int \text{grad } \varphi dV, \quad \oint (\mathbf{n} \times \mathbf{a}) dS = \int \text{curl } \mathbf{a} dV, \\
 \oint (\mathbf{n} \cdot \mathbf{b}) \mathbf{a} dS &= \int (\mathbf{b} \cdot \nabla) \mathbf{a} dV.
 \end{aligned}$$

55. Using the method of Problem 50, it is easy to show that $\oint \varphi d\mathbf{l} = \int (\mathbf{n} \times \text{grad } \varphi) dS$, where $\mathbf{n}$ is a unit normal for the surface.

$$56. \quad \int (\text{grad } u \times \text{grad } f) \cdot \mathbf{n} dS.$$

$$61. \quad \text{a) } A + \frac{B}{r}; \quad \text{b) } A + B \ln \tan \frac{\vartheta}{2}; \quad \text{c) } A + B\alpha.$$

$$62. \quad \text{a) } A + B \ln r; \quad \text{b) } A + B\alpha; \quad \text{c) } A + Bz.$$

$$64. \quad \left. \begin{aligned}
 x &= \pm \left[\frac{(\xi + a^2)(\eta + a^2)(\zeta + a^2)}{(b^2 - a^2)(c^2 - a^2)} \right]^{\frac{1}{2}}, \\
 y &= \pm \left[\frac{(\xi + b^2)(\eta + b^2)(\zeta + b^2)}{(c^2 - b^2)(a^2 - b^2)} \right]^{\frac{1}{2}}, \\
 z &= \pm \left[\frac{(\xi + c^2)(\eta + c^2)(\zeta + c^2)}{(a^2 - c^2)(b^2 - c^2)} \right]^{\frac{1}{2}};
 \end{aligned} \right\} \quad (1)$$

$$h_1 = \frac{V(\xi - \eta)(\xi - \zeta)}{2R_\xi}, \quad h_2 = \frac{V(\eta - \zeta)(\eta - \xi)}{2R_\eta}, \quad h_3 = \frac{V(\zeta - \xi)(\zeta - \eta)}{2R_\zeta},$$

$$\begin{aligned}
 \Delta &= \frac{4}{(\xi - \eta)(\xi - \zeta)(\eta - \zeta)} \left[(\eta - \zeta) R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) + \right. \\
 &\quad \left. + (\zeta - \xi) R_\eta \frac{\partial}{\partial \eta} \left(R_\eta \frac{\partial}{\partial \eta} \right) + (\xi - \eta) R_\zeta \frac{\partial}{\partial \zeta} \left(R_\zeta \frac{\partial}{\partial \zeta} \right) \right],
 \end{aligned}$$

where $R_u = [(u + a^2)(u + b^2)(u + c^2)]^{\frac{1}{2}}$. It is clear from Eqn. (1)

that to each set of values of ξ, η, ζ there correspond eight sets of values of x, y, z .

The orthogonality of the ellipsoidal coordinate system can be proved by finding $\text{grad}\xi, \text{grad}\eta, \text{grad}\zeta$ and evaluating the scalar products $\text{grad}\xi \cdot \text{grad}\eta$ and so on, which all turn out to be equal to zero. The quantities $\text{grad}\xi, \text{grad}\eta, \text{grad}\zeta$ may be found directly from the equations defining ξ, η, ζ by taking the gradient of both sides of these equations and using Eqn. (1).

$$65. \quad z = \pm \left[\frac{(\xi + c^2)(\eta + c^2)}{c^2 - a^2} \right]^{\frac{1}{2}}, \quad r = \left[\frac{(\xi + a^2)(\eta + a^2)}{a^2 - c^2} \right]^{\frac{1}{2}};$$

$$h_1 = \frac{1}{2} \frac{\sqrt{\xi - \eta}}{R_\xi}, \quad h_2 = \frac{1}{2} \frac{\sqrt{\xi - \eta}}{R_\eta}, \quad h_3 = r,$$

where $R_\xi = \sqrt{(\xi + a^2)(\xi + c^2)}$, $R_\eta = \sqrt{(\eta + a^2)(-\eta - c^2)}$;

$$\Delta = \frac{4}{\xi - \eta} \left[R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) + R_\eta \frac{\partial}{\partial \eta} \left(R_\eta \frac{\partial}{\partial \eta} \right) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2}.$$

$$66. \quad x = \pm \left[\frac{(\xi + a^2)(\zeta + a^2)}{a^2 - b^2} \right]^{\frac{1}{2}}, \quad r = \left[\frac{(\xi + b^2)(\zeta + b^2)}{b^2 - a^2} \right]^{\frac{1}{2}};$$

$$h_1 = \frac{1}{2} \frac{\sqrt{\xi - \zeta}}{R_\xi}, \quad h_2 = r, \quad h_3 = \frac{1}{2} \frac{\sqrt{\xi - \zeta}}{R_\zeta},$$

where $R_\xi = \sqrt{(\xi + a^2)(\xi + b^2)}$, $R_\zeta = \sqrt{(\zeta + a^2)(-\zeta - b^2)}$;

$$\Delta = \frac{4}{\xi - \zeta} \left[R_\xi \frac{\partial}{\partial \xi} \left(R_\xi \frac{\partial}{\partial \xi} \right) + R_\zeta \frac{\partial}{\partial \zeta} \left(R_\zeta \frac{\partial}{\partial \zeta} \right) \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2}.$$

$$67. \quad h_\xi = h_\eta = \frac{a}{\cosh \xi - \cos \eta}, \quad h_\alpha = \frac{a \sin \eta}{\cosh \xi - \cos \eta};$$

$$\Delta = \frac{(\cosh \xi - \cos \eta)^3}{a^2} \left[\frac{\partial}{\partial \xi} \left(\frac{1}{\cosh \xi - \cos \eta} \frac{\partial}{\partial \xi} \right) + \right.$$

$$\left. + \frac{1}{\sin \eta} \frac{\partial}{\partial \eta} \left(\frac{\sin \eta}{\cosh \xi - \cos \eta} \frac{\partial}{\partial \eta} \right) + \frac{1}{\sin^2 \eta (\cosh \xi - \cos \eta)} \frac{\partial^2}{\partial \alpha^2} \right].$$

68. The surfaces $\rho = \text{const}$ are the toroids

$$(\sqrt{x^2 + y^2} - a \cot \rho)^2 + z^2 = \left(\frac{a}{\sinh \rho} \right)^2,$$

and the surfaces $\xi = \text{const}$ are the spherical segments

$$(z - \tan^{-1} \xi)^2 + x^2 + y^2 = \left(\frac{a}{\sin \xi} \right)^2,$$

$$h_\rho = h_\xi = \frac{a}{\cosh \rho - \cos \xi}, \quad h_\alpha = \frac{a \sinh \rho}{\cosh \rho - \cos \xi}.$$

Chapter 2

ELECTROSTATICS IN VACUUM

$$69. \quad \varphi_1 = -2\pi\rho z^2, \quad E_1 = -4\pi\rho z e_z \quad (|z| < \frac{a}{2}),$$

$$\varphi_2 = -\frac{1}{2}\pi\rho a(4|z| - a), \quad E_2 = -2\pi\rho a \frac{z}{|z|} e_z \quad (|z| > \frac{a}{2}),$$

where the z -axis is perpendicular to the surface of the plate.

$$70. \quad \varphi(x, y, z) = \frac{4\pi\rho_0}{\alpha^2 + \beta^2 + \gamma^2} \cos \alpha x \cos \beta y \cos \gamma z.$$

$$71. \quad \text{When } z > 0: \varphi = \frac{2\pi\sigma_0}{\lambda} e^{-\lambda z} \sin \alpha x \sin \beta y;$$

$$\text{When } z < 0: \varphi = \frac{2\pi\sigma_0}{\lambda} e^{\lambda z} \sin \alpha x \sin \beta y, \quad \lambda = \sqrt{\alpha^2 + \beta^2}.$$

The exponential decrease in the potential along the z -axis is due to the fact that the plane contains charges of different sign.

72. For the volume charge distribution

$$\varphi_1 = \int_r^R \frac{2\pi r}{R^2} dr = \pi \left(1 - \frac{r^2}{R^2}\right), \quad E_1 = \frac{2\pi r}{R^2} \quad (r \leq R);$$

$$\varphi_2 = \int_r^R \frac{2\pi}{r} dr = -2\pi \ln \frac{r}{R}, \quad E_2 = \frac{2\pi}{r} \quad (r \geq R).$$

The simplest method of solution involves the electrostatic Gauss theorem. In the solution involving integration of the Poisson equation, the Laplace operator must be expressed in terms of cylindrical coordinates. Owing to the symmetry of the system, φ is a function of r only.

In the case of the surface charge distribution $\varphi_1 = 0$, $\varphi_2 = -2\pi \ln(r/R)$.

73. $\varphi = -2\pi \ln r$, $E = 2\pi/r$, where π is the charge per unit length. The arbitrary constant in the potential is chosen so that $\varphi = 0$ when $r = 1$.

$$74. \quad \varphi(x, y, z) = \frac{q}{2a} \ln \left| \frac{z - a + \sqrt{(z - a)^2 + x^2 + y^2}}{z + a + \sqrt{(z + a)^2 + x^2 + y^2}} \right|.$$

75. Substitute

$$z_1 = z + a, \quad z_2 = z - a, \quad r_{1,2} = \sqrt{x^2 + y^2 + z_{1,2}^2}, \quad C = \frac{z_1 + r_1}{z_2 + r_2}.$$

It follows from the solution of the preceding problem that

$$r_1 + r_2 = 2a \frac{C+1}{C-1} = \text{const} \quad (1)$$

(remember that $z_1 - z_2 = 2a$).

Eqn. (1) shows that the equipotential surfaces are ellipsoids of revolution whose foci lie at the ends of the segment.

$$76. \quad \varphi_1(r) = \frac{q}{R} \left(\frac{3}{2} - \frac{r^2}{2R^2} \right), \quad E_1 = \frac{qr}{R^3} \quad (r \leq R);$$

$$\varphi_2(r) = \frac{q}{r}, \quad E_2 = \frac{qr}{r^3} \quad (r \geq R).$$

$$77. \quad \varphi_1(r) = \frac{q}{R}, \quad E_1 = 0 \quad (r < R);$$

$$\varphi_2(r) = \frac{q}{r}, \quad E_2 = \frac{qr}{r^3} \quad (r > R).$$

78. The electric field in the cavity is uniform and is given by

$$\mathbf{E} = \frac{4}{3} \pi \rho \mathbf{r} - \frac{4}{3} \pi \rho (\mathbf{r} - \mathbf{a}) = \frac{4}{3} \pi \rho \mathbf{a}.$$

$$79. \quad q = 4\pi a (R_2 - R_1);$$

$$E_1 = 0, \quad \varphi_1 = \frac{q}{R_2 - R_1} \ln \frac{R_2}{R_1} \quad \text{when } r \leq R_1; \quad E_2 = \frac{q(r - R_1)}{(R_2 - R_1)r^2},$$

$$\varphi_2 = \frac{q}{R_2 - R_1} \left(1 - \ln \frac{r}{R_2} - \frac{R_1}{r} \right) \quad \text{when } R_1 \leq r \leq R_2; \quad E_3 = \frac{q}{r^2}; \quad \varphi_3 = \frac{q}{r}$$

when $r \geq R_2$.

The field of a sphere with a uniform surface charge distribution is obtained by letting $R_2 \rightarrow R_1 \equiv R$ at constant charge q .

80. The energy W for the charge distributions of Problems 76, 77, and 79 is respectively given by

$$W = \frac{3q^2}{5R}, \quad W = \frac{q^2}{2R}, \quad W = \frac{q^2}{R_2 - R_1} - \frac{q^2 R_1}{(R_2 - R_1)^2} \ln \frac{R_2}{R_1}.$$

By comparing the contributions to the energy W given by the integrals $\int_0^R$ and $\int_R^\infty$ it may be shown that most of the energy is

localised outside the charge distribution (83% in the case of a uniform sphere of charge).

$$81. \quad \varphi(r) = \frac{4\pi}{r} \int_0^r \rho(r') r'^2 dr' + 4\pi \int_r^\infty \rho(r') r' dr';$$

$$E(r) = \frac{4\pi r}{r^3} \int_0^r \rho(r') r'^2 dr'.$$

83. The field due to the electron cloud in the atom is given by

$$\varphi_e(r) = -\frac{e_0}{r} \left(1 - e^{-\frac{2r}{a}}\right) + \frac{e_0}{a} e^{-\frac{2r}{a}};$$

$$E_{er} = -\frac{e_0}{r^2} \left[1 - \left(\frac{2r}{a} + 1\right) e^{-\frac{2r}{a}}\right] + \frac{2e_0}{a^2} e^{-\frac{2r}{a}}.$$

The potential of the total electric field in the atom is given by

$$\varphi(r) = \varphi_e(r) + \frac{e_0}{r}.$$

84. The field strength is a maximum on the surface of the nucleus:

$$E_{\max} = \frac{Ze_0}{R^2} = 6.4 \times 10^{18} \frac{Z}{A^{\frac{2}{3}}} \text{ V cm}^{-1}.$$

85. Use the fact that the surface density is given by

$$\rho(r, \theta, a) = \sigma(\theta, a) \delta(r - a).$$

$$86. \quad q_{1,2} = \frac{R \sqrt{R^2 + a^2}}{qa^2} (\sqrt{R^2 + a^2} A_{1,2} - R A_{2,1}).$$

$$87. \quad \varphi = \frac{2q}{R^2} (\sqrt{R^2 + z^2} - |z|),$$

$$E_x = E_y = 0, \quad E_z = \frac{2q}{R^2} \left(\frac{z}{|z|} - \frac{z}{\sqrt{R^2 + z^2}} \right),$$

where z is the coordinate of the point of observation measured from the face of the disc.

88. If the positively charged half of the ring occupies the region $x > 0$ in the xy -plane, then for $x, y \ll (R^2 + z^2)/R$ it is

found, after expanding the integrand of $\int (\mathbf{x}/r_{12}) dl$ into a series, that

$$\varphi = \frac{4qRx}{\pi(R^2 + z^2)^{\frac{3}{2}}},$$

and hence

$$E_x = -\frac{4qR}{\pi(R^2 + z^2)^{\frac{3}{2}}}, \quad E_y = 0, \quad E_z = -\frac{12qRxz}{\pi(R^2 + z^2)^{\frac{5}{2}}}.$$

When $z \gg R$ the field is the same as the field due to an electric dipole which lies along the x -axis and has a moment equal to $4qR/\pi$.

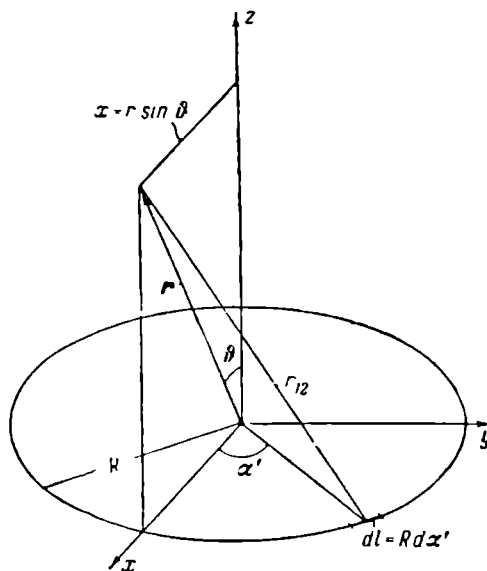


Fig. 30.

89. Owing to the symmetry of the system, the potential φ will be independent of the azimuthal angle α' and hence the xz -plane may be drawn through the point of observation without loss of generality (Fig. 30). Therefore,

$$r_{12} = \sqrt{r^2 + R^2 - 2rR \sin \theta \cos \alpha'}$$

and

$$\varphi(r, \theta) = 2\pi R \int_0^\pi \frac{d\alpha'}{\sqrt{r^2 + R^2 - 2rR \sin \theta \cos \alpha'}},$$

where $\kappa = q/2\pi R$.

Substituting $\alpha' = \pi - 2\beta$ and

$$k^2 = \frac{4rR \sin \vartheta}{r^2 + R^2 + 2rR \sin \vartheta},$$

we have

$$\varphi(r, \vartheta) = \frac{4\pi R}{\sqrt{r^2 + R^2 + 2rR \sin \vartheta}} \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}} = \frac{2k\pi}{\sqrt{rR \sin \vartheta}} K(k).$$

$$90. \quad a) \quad \varphi = \frac{q}{\sqrt{R^2 + z^2}},$$

where z is the distance of the point of observation from the plane of the ring.

b) Let r' be the distance from the point of observation to the ring. For $r' \ll R$ we have

$$1 - k^2 \simeq \frac{r'^2}{4R^2}, \quad K(k) = \ln \frac{8R}{r'} \quad \text{and} \quad \varphi(r) = -2\pi \ln r' + \text{const.}$$

which is the same result as for a linear charge.

$$91. \quad \varphi_1 = \frac{4\pi}{3} \sigma_0 r \cos \vartheta \quad (r \leq R),$$

$$\varphi_2 = \frac{4\pi}{3} \cdot \frac{\sigma_0 R^3}{r^2} \cos \vartheta \quad (r \geq R).$$

Inside the sphere the electric field is uniform and has an intensity $E_{1z} = -4\pi\sigma_0/3$; outside the sphere the field is the same as for a dipole with a moment $4\pi\sigma_0 R^3/3$.

92. Owing to the axial symmetry of the field, the Laplace equation in terms of cylindrical coordinates (the polar axis along the axis of symmetry of the system) is of the form

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \quad (1)$$

We shall seek the solution of Eqn. (1) in the form of the power series

$$\varphi(r, z) = \sum_{n=0}^{\infty} a_n(z) r^n, \quad a_0(z) = \varphi(0, z) \equiv \Phi(z), \quad (2)$$

where $\Phi(z)$ is the potential along the axis of symmetry of the system. Substituting Eqn. (2) into Eqn. (1), rearranging and equating to zero the coefficients of the resulting series, we obtain the

recurrence relations for the coefficients $a_n(z)$, and hence

$$\varphi(r, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \Phi^{(2n)}(z) \left(\frac{r}{2}\right)^{2n} = \Phi(z) - \frac{r^2}{4} \Phi''(z) + \dots$$

$$E_r = -\frac{\partial \varphi}{\partial r} = \frac{r}{2} \Phi''(z) + \dots,$$

$$E_a = 0, \quad E_z = -\frac{\partial \varphi}{\partial z} = -\Phi'(z) + \dots$$

93. The multipole moments to be computed are

$$Q_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int x R^l Y_{lm}^* \left(\frac{\pi}{2}, \alpha\right) R d\alpha,$$

$$Q'_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int \frac{x}{R^{l+1}} Y_{lm}^* \left(\frac{\pi}{2}, \alpha\right) R d\alpha.$$

Using Eqns. (1) and (5) of Appendix II we find that

$$\varphi(r, \vartheta) = \frac{q}{r} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{R}{r}\right)^{2n} P_{2n}(\cos \vartheta) \quad \text{when } r > R,$$

$$\varphi(r, \vartheta) = \frac{q}{R} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} \left(\frac{r}{R}\right)^{2n} P_{2n}(\cos \vartheta) \quad \text{when } r < R.$$

Both formulae hold at $r = R$ ($\vartheta \neq \pi/2$).

$$94. \quad a) \varphi \simeq qa^2 \frac{3z^2 - r^2}{r^5} = 2qa^2 \frac{P_2(\cos \vartheta)}{r^3};$$

$$b) \varphi \simeq \frac{3qa^2 \sin^2 \vartheta \cos \alpha \sin \alpha}{r^3}.$$

$$95. \quad a) \varphi \simeq \frac{6qa^3 P_3(\cos \vartheta)}{r^4} = qa^3 \frac{15 \cos^3 \vartheta - 9 \cos \vartheta}{r^4};$$

$$b) \varphi \simeq \frac{15qabcxyz}{r^7} = \frac{15qabc \sin^2 \vartheta \cos \vartheta \sin \alpha \cos \alpha}{r^4}.$$

$$96. \quad \varphi(r, \vartheta, \alpha) = \sum_{l, m} \frac{4\pi}{2l+1} \frac{r^l}{r_0^{l+1}} Y_{lm}^*(\vartheta_0, \alpha_0) Y_{lm}(\vartheta, \alpha) \quad \text{when } r < r_0;$$

$$\varphi(r, \vartheta, \alpha) = \sum_{l, m} \frac{4\pi}{2l+1} \cdot \frac{r_0^l}{r^{l+1}} Y_{lm}^*(\vartheta_0, \alpha_0) Y_{lm}(\vartheta, \alpha) \quad \text{when } r > r_0.$$

$$97. \quad \varphi(x, y, z) \simeq \frac{q}{r} + q \frac{a^2(3x^2 - r^2) + b^2(3y^2 - r^2) + c^2(3z^2 - r^2)}{10r^5}.$$

In the case of an ellipsoid of revolution $a = b$ and

$$\varphi(r, \vartheta) = \frac{q}{r} + q \frac{c^2 - a^2}{5} \cdot \frac{P_2(\cos \vartheta)}{r^3}.$$

In the case of a sphere $a = b = c$ and $\varphi = q/r$.

98. In spherical coordinates (polar axis along the axis of symmetry of the system and the origin at the centre of the rings) the potential is

$$\varphi(r, \theta) = -\frac{q(a^2 - b^2)}{2} \cdot \frac{P_2(\cos \theta)}{r^3}.$$

This expression represents the potential of a linear quadrupole in which two charges of $-q$ each are at a distance of $\frac{1}{2}(a^2 - b^2)^{\frac{1}{2}}$ from the central charge of $2q$.

99. Since $\delta(\mathbf{r}) = 0$ everywhere except at $\mathbf{r} = 0$, we have

$$q = -\int (\mathbf{p}' \cdot \nabla) \delta(\mathbf{r}) dV = -\oint (\mathbf{p}' \cdot \mathbf{n}) \delta(\mathbf{r}) dS = 0.$$

Also,

$$p_a = -\int x_a (\mathbf{p}' \cdot \nabla) \delta(\mathbf{r}) dV = -\int x_a p'_n \frac{\partial \delta(\mathbf{r})}{\partial x_n} dV = \int p'_n \frac{\partial x_a}{\partial x_n} \delta(\mathbf{r}) dV.$$

The latter transformation involves integration by parts. The repeated subscript n represents summation. The resulting surface integral becomes equal to zero since $\delta(\mathbf{r}) = 0$ when $\mathbf{r} \neq 0$. In view of the definition of the δ -function we have

$$p_a = p'_n \frac{\partial x_a}{\partial x_n} = p'_n \delta_{an} = p'_a.$$

All the multipole moments of higher order are proportional to the components of $\mathbf{r}$ at $\mathbf{r} = 0$ and are therefore equal to zero. For example, in the case of the quadrupole moment

$$Q_{a\beta} = -\int x_a x_\beta p'_n \frac{\partial \delta(\mathbf{r})}{\partial x_n} dV = \int \delta(\mathbf{r}) p'_n \frac{\partial x_a x_\beta}{\partial x_n} dV = p'_a x_\beta + p'_\beta x_a \Big|_{r=0} = 0.$$

100. On integrating by parts n times, we have

$$\varphi(\mathbf{r}) = q(-1)^n \int \delta(\mathbf{r}') \prod_i (\mathbf{a}_i \cdot \nabla') \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV' = q \prod_i (\mathbf{a}_i \cdot \nabla) \frac{1}{r}.$$

101. The simplest method is to use the formula $\varphi = qa^2(3z'^2 - r^2)/r^5$ (cf. solution to Problem 94) and express z' in terms of x , y , and z (Fig. 31). The final result is

$$\varphi = \frac{qa^2}{r^5} [3(x \sin \gamma \cos \beta + y \sin \gamma \sin \beta + z \cos \gamma)^2 - r^2].$$

This result may also be obtained by using the fact that the

components of the quadrupole moment form a tensor of rank 2.

In the coordinate system (x', y', z') the components of the quadrupole moment are

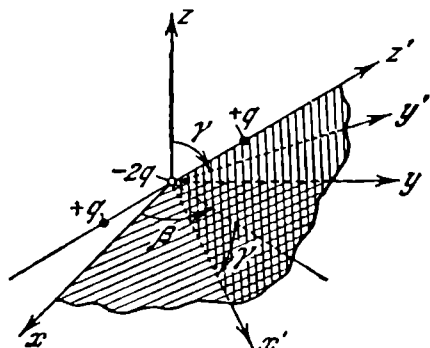


Fig. 31.

$$Q'_{xx} = Q'_{yy} = Q'_{xy} = Q'_{xz} = Q'_{yz} = 0, \\ Q'_{zz} = 2qa^2.$$

The transformation matrix is of the form

$$\hat{a} = \begin{pmatrix} \cos \gamma \cos \beta & -\sin \beta & \sin \gamma \cos \beta \\ \cos \gamma \sin \beta & \cos \beta & \sin \gamma \sin \beta \\ -\sin \gamma & 0 & \cos \gamma \end{pmatrix}.$$

This matrix may then be used to evaluate the components of $Q_{\alpha\beta}$

in the (x, y, z) system using the expression

$$Q_{\alpha\beta} = \sum_{\gamma, \delta} a_{\alpha\gamma} a_{\beta\delta} Q'_{\gamma\delta},$$

and Eqn. (II.8).

$$102. \quad \varphi = \frac{15qabcz}{2r^7} [(y^2 - x^2) \sin 2\beta + 2xy \cos 2\beta].$$

$$103. \quad \varphi = \frac{qa^2}{4r^3} (3 \sin^2 \vartheta \sin 2\alpha - 3 \cos 2\vartheta - 1).$$

104. According to the superposition principle we have

$$\varphi(\mathbf{r}) = \int_V \frac{\mathbf{P} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' = \int \mathbf{P} \cdot \text{grad}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV'.$$

Application of Gauss' theorem to this expression yields

$$\varphi(\mathbf{r}) = \int_S \frac{P_n}{|\mathbf{r} - \mathbf{r}'|} dS, \text{ where } S \text{ is the inner surface of the polarised}$$

sphere and $P_n = P \cos \vartheta$. If the solution of Problem 91 is used it is found that

$$\varphi_1 = \frac{4\pi Pr}{3} \cos \vartheta \quad (r \leq R),$$

$$\varphi_2 = \frac{4\pi PR^3}{3r^2} \cos \vartheta \quad (r \geq R).$$

$$105. \quad \varphi(r) = -2\kappa \ln r + 2 \sum_{n=1}^{\infty} \frac{A_n \cos n\alpha + B_n \sin n\alpha}{nr^n},$$

where $\kappa = \int \rho(\mathbf{r}') dS'$ is the total charge per unit length, $A_n =$

$= \int \rho(\mathbf{r}') \mathbf{r}'^n \cos n\alpha' dS'$ and $B_n = \int \rho(\mathbf{r}') \mathbf{r}'^n \sin n\alpha' dS'$ are the two-dimensional multipole moments of order n . It follows from these formulae that the potential of a dipole in the two-dimensional case is given by $\varphi = 2\mathbf{p} \cdot \mathbf{r}/r^2$ where $\mathbf{p} = \int \rho(\mathbf{r}') \mathbf{r}' dS'$ is the dipole moment per unit length of the distribution and $\mathbf{r}$ is the position vector in the xy -plane.

$$106. \quad \varphi(r, \alpha) = -2\lambda \ln r + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r_0}{r}\right)^n \cos n(\alpha - \alpha_0) \quad \text{when } r > r_0,$$

$$\varphi(r, \alpha) = -2\lambda \ln r + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r_0}\right)^n \cos n(\alpha - \alpha_0) \quad \text{when } r < r_0,$$

$$107. \quad \varphi(r) \simeq \frac{2\lambda a}{r} \cos \alpha = \frac{2\mathbf{p} \cdot \mathbf{r}}{r^2},$$

where $\mathbf{p}$ is the dipole moment per unit length, $\mathbf{r}$ is the position vector in the xy -plane ($r \gg a$), and the z -axis lies along one of the linear charges.

108. Along the axis of symmetry of the disc (the z -axis is drawn from the negative to the positive side of the disc),

$$\begin{aligned} \varphi(z) &= \tau\Omega = 2\pi\tau \left(1 - \frac{|z|}{\sqrt{R^2 + z^2}}\right) \frac{z}{|z|}; \\ E_x &= E_y = 0, \quad E_z = \frac{2\tau a^2 z}{|z|(a^2 + z^2)^{\frac{3}{2}}}. \end{aligned}$$

109. a) In cylindrical coordinates

$$E_a = \frac{2\tau}{r}, \quad E_r = E_z = 0;$$

$$\text{b) } \varphi = 2\tau(\pi - \alpha), \quad E_a = -\frac{1}{r} \frac{\partial \varphi}{\partial \alpha} = \frac{2\tau}{r}; \quad E_r = E_z = 0.$$

The field $\mathbf{E}$ is identical in form with the magnetic field due to a straight line current $\mathcal{J} = \tau c$.

110. The equation of the lines of force is

$$(z+a)[(z+a)^2 + r^2]^{-\frac{1}{2}} \pm (z-a)[(z-a)^2 + r^2]^{-\frac{1}{2}} = C,$$

where C is a constant. The case of charges of different sign is illustrated in Fig. 32a. Fig. 32b illustrates the case of two charges of the same kind. The neutral point occurs at $r = 0$, $z = 0$.

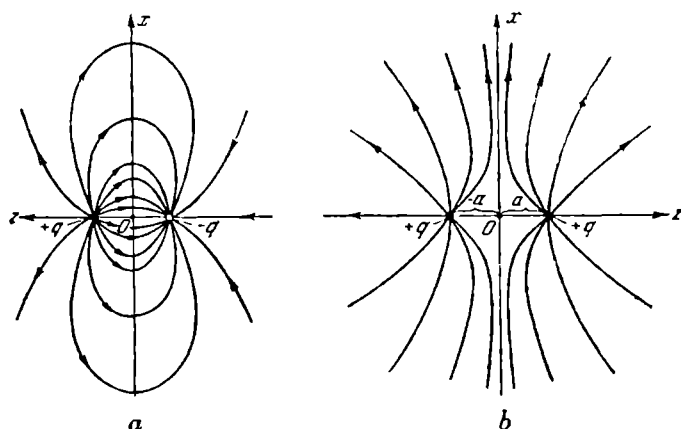


Fig. 32.

111. It is convenient to use spherical coordinates. By letting a tend to zero, expanding into a series, and neglecting terms of the order of a^2 and higher, it is found that $r = C \sin^2 \vartheta$.

$$112. \quad r = C \sqrt{\sin^2 \vartheta |\cos \vartheta|}, \quad C = \text{const.}$$

It should be remembered that in the case of a quadrupole of finite dimensions this formula will only hold at large distances (Fig. 33).

$$114. \quad q_2 = \frac{\Phi + \sqrt{2}(\sqrt{2}-1)\pi q}{\sqrt{2}(\sqrt{2}-1)\pi}.$$

115. Consider a tube of force which is obtained by rotating a line of force about the z -axis. Application of the electrostatic Gauss theorem to the volume bounded by the side surface of this tube and two $z = \text{const}$ planes which do not contain charges, shows that the flux through any cross-section normal to

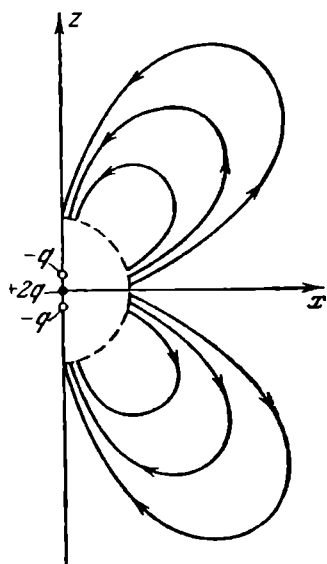


Fig. 33.

$$\text{the axis of the tube } \Phi(z) = \sum_i q_i \Omega_i(z)$$

(cf. Problem 113) is independent of z (when z lies between z_k and z_{k+1}). Here $\Omega_i(z) = 2\pi(\pm 1 - \cos \alpha_i)$ is the solid angle subtended by the negative side of the cross-section at the point z_i at which the charge q_i is located, and α_i is the angle between the z -axis and the position vector of a

point on the periphery of the normal cross-section whose coordinates are r, z . The positive sign should be taken for $z > z_i$ and the negative sign for $z < z_i$. If on varying z , the normal cross-section of the tube passes over a charge q_k , then $\Phi(z)$ will change discontinuously by $\pm 4\pi q_k$, but $\sum_i q_i \cos \alpha_i$ will remain constant. Writing $\cos \alpha_i$ in terms of z, z_i , and r , we obtain the required equation for the family of lines of force:

$$\sum_i \frac{q_i (z - z_i)}{\sqrt{r^2 + (z - z_i)^2}} = C, \quad C = \text{const.}$$

117. Consider the cylindrical coordinate system whose z -axis lies along the axis of the cylinder (Fig. 34). Instead of the condition $\varphi|_S = \text{constant}$ on the surface S of the cylinder, it is more convenient to use the condition $\partial\varphi/\partial\alpha|_S = 0$ which is a consequence of it. After differentiation we have

$$\frac{x_1 x_1}{R^2 + x_1^2 - 2R x_1 \cos \alpha} = \frac{x_2 x_2}{R^2 + x_2^2 - 2R x_2 \cos \alpha}.$$

On rearranging and equating terms with and without $\cos \alpha$, we find that when $x_1 = x_2$, any cylindrical surface whose axis is parallel to the charged filaments and lies in the plane of the filaments will be an equipotential surface, provided its radius satisfies the condition $R^2 = x_1 x_2$.

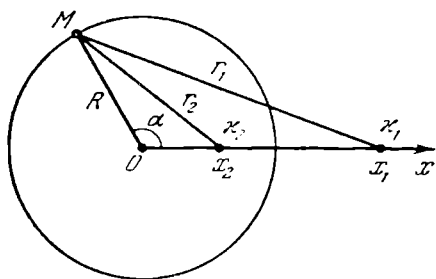


Fig. 34.

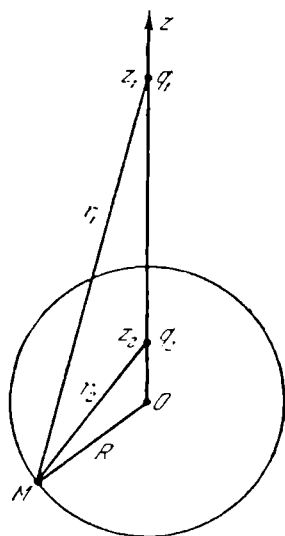


Fig. 35.

When $x_1 = 0$, the solution $x_2 = 0$ will also exist. This corresponds to cylindrical equipotential surfaces in the field due to one of the filaments.

118. Referring to Fig. 35, the radius R of the required sphere and the position of its centre are respectively given by

$$R^2 = z_1 z_2, \quad \frac{z_1}{z_2} = \frac{q_1^2}{q_2^2}.$$

The potential on the surface of this sphere is zero.

$$\begin{aligned} 119. \quad \Delta\varphi = q\Delta \frac{e^{-ar}}{r} &= q\Delta \frac{1}{r} + q\Delta \frac{e^{-ar} - 1}{r} = -4\pi q\delta(r) + \\ &+ \frac{q}{r} \frac{\partial^2}{\partial r^2} \left(r \frac{e^{-ar} - 1}{r} \right) = -4\pi q\delta(r) + \frac{qa^2 e^{-ar}}{r}. \end{aligned}$$

Thus, we have a point charge q at the origin and a spherical symmetric volume distribution with a density

$$\rho = -\frac{qa^2 e^{-ar}}{4\pi r}, \quad \int \rho dV = -q.$$

120. The point charge e_0 at the origin is surrounded by a volume charge with a density

$$\rho(r) = -\frac{e_0}{\pi a^3} e^{-\frac{2r}{a}}.$$

This is the charge distribution in the hydrogen atom (*cf.* Problem 83).

$$121. \quad U = \int \frac{e_0}{r} \rho(r) dV = -\frac{e_0^2}{\pi a^3} \int_0^\infty r e^{-\frac{2r}{a}} 4\pi dr = -\frac{e_0^2}{a}.$$

$$122. \quad U = \frac{5e_0^2}{4a}.$$

$$123. \quad U = \frac{q_1 q_2}{a}, \quad F = \frac{q_1 q_2}{a^2}.$$

$$124. \quad R = \frac{32\pi a}{E_0^2}.$$

$$125. \quad U = \oint_{l_1} \oint_{l_2} \frac{r_1 r_2 dl_1 dl_2}{r_{12}} = \frac{q_1 q_2}{4\pi^2 ab} \int_0^{2\pi} \int_0^{2\pi} \frac{ab dl_1 dl_2}{\sqrt{c^2 + a^2 + b^2 - 2ab \cos(\alpha_1 - \alpha_2)}},$$

where the integration is carried out over all elements dl_1, dl_2 of both rings and α_1, α_2 are the angles which define the position of the elements. Integrating with respect to α_2 and substituting $\alpha_1 = \pi - 2\alpha$ we have

$$U = \frac{q_1 q_2 k}{\pi \sqrt{ab}} K(k),$$

where

$$k = \frac{2\sqrt{ab}}{\sqrt{c^2 + (a+b)^2}}, \text{ and } K(k) = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

The function $K(k)$ is the complete elliptical integral of the first kind.

In evaluating the force $F = -\partial U / \partial c = -(\partial U / \partial k)(\partial k / \partial c)$ it is necessary to use the formula

$$2k^2 \frac{dK(k)}{d(k^2)} = \frac{E(k)}{1 - k^2} - K(k),$$

where $E(k) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 \alpha)^{\frac{1}{2}} d\alpha$ is the complete elliptical integral of the second kind. Finally,

$$F = \frac{q_1 q_2 c k^3}{4\pi (ab)^{\frac{3}{2}}} \cdot \frac{E(k)}{1 - k^2}.$$

$$126. \quad \mathbf{F} = -\frac{3qr(\mathbf{p} \cdot \mathbf{r})}{r^5} + \frac{q\mathbf{p}}{r^3}, \quad \mathbf{N} = \frac{q\mathbf{p} \times \mathbf{r}}{r^3}.$$

$$127. \quad U = p_1 p_2 \frac{\sin \vartheta_1 \sin \vartheta_2 \cos \varphi - 2 \cos \vartheta_1 \cos \vartheta_2}{r^3},$$

where $\vartheta_1 = \angle(\mathbf{r}, \mathbf{p}_1)$, $\vartheta_2 = \angle(\mathbf{r}, \mathbf{p}_2)$, φ is the angle between the planes $(\mathbf{r}, \mathbf{p}_1)$ and $(\mathbf{r}, \mathbf{p}_2)$, and

$$F = 3p_1 p_2 \frac{\sin \vartheta_1 \sin \vartheta_2 \cos \varphi - 2 \cos \vartheta_1 \cos \vartheta_2}{r^4}.$$

The force reaches a maximum when $\vartheta_1 = \vartheta_2 = \varphi = 0$, *i.e.* when the dipoles are parallel.

$$\begin{aligned} 128. \quad U_{21} &= \int \rho(\mathbf{r}') \varphi_1(\mathbf{r}') dV' = \\ &= \sum_{l, m} \sqrt{\frac{4\pi}{2l+1}} a_{lm} \int r'^l Y_{lm}(\vartheta', \alpha') dV' = \sum_{l, m} a_{lm} Q_{lm}^*. \end{aligned}$$

Chapter

3

ELECTROSTATICS OF CONDUCTORS AND DIELECTRICS

1. Basic concepts and methods of electrostatics

$$129. \quad \varphi_1 = \varphi_2 = \frac{2}{\epsilon_1 + \epsilon_2} \cdot \frac{q}{r},$$

$$D_1 = \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2} \cdot \frac{qr}{r^3}, \quad D_2 = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} \cdot \frac{qr}{r^3}.$$

$$130. \quad \varphi_1 = \varphi_2 = \varphi_3 = \frac{2\pi}{\epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3} \cdot \frac{q}{r},$$

$$D_i = \frac{2\pi\epsilon_i}{\epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3} \cdot \frac{qr}{r^3}.$$

131. The boundary conditions ($\varphi = \text{const}$ on the surface of the conductor and $\varphi = 0$ when $r \rightarrow \infty$) may be satisfied by a potential of the form $\varphi = C/r$ where the constant C is determined from the condition $\oint_S D_n dS = 4\pi q$, $C = 2/(\epsilon_1 + \epsilon_2)$. Hence the surface distributions are

$$\sigma_1 = \frac{q\epsilon_1}{2\pi a^2 (\epsilon_1 + \epsilon_2)}, \quad \sigma_2 = \frac{q\epsilon_2}{2\pi a^2 (\epsilon_1 + \epsilon_2)},$$

$$\sigma_{1b} = \frac{q(\epsilon_1 - 1)}{2\pi a^2 (\epsilon_1 + \epsilon_2)}, \quad \sigma_{2b} = \frac{q(\epsilon_2 - 1)}{2\pi a^2 (\epsilon_1 + \epsilon_2)}.$$

$$132. \quad C = \left[\frac{(\epsilon - 1)\Omega}{4\pi} + 1 \right] \frac{ab}{b - a}.$$

$$133. \quad C = \left[\frac{1}{\epsilon_1} \left(\frac{1}{a} - \frac{1}{c} \right) + \frac{1}{\epsilon_2} \left(\frac{1}{c} - \frac{1}{b} \right) \right]^{-1}.$$

The bound charges are located on the spheres with radii a , b , c :

$$\sigma_{a,b} = -\frac{q}{4\pi a^2} \cdot \frac{\epsilon_1 - 1}{\epsilon_1}, \quad \sigma_{b,b} = \frac{q}{4\pi b^2} \cdot \frac{\epsilon_2 - 1}{\epsilon_2},$$

$$\sigma_{c,b} = \frac{q}{4\pi c^2} \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right).$$

where q is the charge on the inner electrode of the capacitor. The total bound charge on the capacitor is zero.

135. The capacitance is

$$C = \frac{\epsilon_0 S}{4\pi a \ln 2}.$$

The surface density of bound charges is given by

$$\sigma_b = -\sigma \left(1 - \frac{1}{\epsilon_0}\right) \quad \text{when } x = 0,$$

$$\sigma_b = \sigma \left(1 - \frac{1}{2\epsilon_0}\right) \quad \text{when } x = a.$$

The volume density is given by

$$\rho_b = \frac{\sigma a}{\epsilon_0 (x + a)^2},$$

where σ is the charge at $x = 0$.

$$136. \quad a) f_0 = \frac{E_0^2}{8\pi} = \frac{V^2}{8\pi d^2};$$

$$b) f = \frac{D^2}{8\pi\epsilon} = \frac{1}{\epsilon} f_0 \quad (\text{liquid dielectric}),$$

$$f = \frac{D^2}{8\pi} = f_0 \quad (\text{solid dielectric});$$

$$c) f = \frac{\epsilon E^2}{8\pi} = \epsilon f_0 \quad (\text{liquid dielectric}),$$

$$f = \frac{(\epsilon E)^2}{8\pi} = \epsilon^2 f_0 \quad (\text{solid dielectric}).$$

$$137. \quad a) F = \frac{(\epsilon - 1) b h_2 V^2}{8\pi\epsilon h_1 \left[h_1 - \frac{h_2(\epsilon - 1)}{\epsilon} \right]};$$

$$b) F = \frac{2\pi(\epsilon - 1) h_1 h_2 q^2 \left[h_1 - \frac{h_2(\epsilon - 1)}{\epsilon} \right]}{b \left\{ a \left[h_1 - \frac{h_2(\epsilon - 1)}{\epsilon} \right] + \frac{(\epsilon - 1) h_2 x}{\epsilon} \right\}}.$$

138. Consider the pressures at the points A and B in the liquid (Fig. 36). At B the pressure is equal to the atmospheric pressure p_{atm} . The pressure at A may be found either from

Eqn. (III.25), according to which $p_A = p_{\text{atm}} + \frac{E^2}{8\pi} \frac{\tau \partial \epsilon}{\partial \tau}$ (where

$p_{\text{atm}} = p_0$, $E = V/d$), or by noting the fact that p_A differs from the pressure at the surface of the liquid in the capacitor by the

hydrostatic pressure ρgh so that $p_A = \rho gh + \tau \frac{E^2}{8\pi} \frac{\partial \epsilon}{\partial \tau} - \frac{\epsilon - 1}{8\pi} E^2 + p_{\text{atm}}$ [cf. (III.23)]. On comparing the two expressions we find that

$$h = \frac{\epsilon - 1}{8\pi g \tau} E^2.$$

139. The Maxwell stress tensor $\mathbf{T}'_{\mathbf{n}}$ is oriented so that the electric field $\mathbf{E}$ bisects the angle between $\mathbf{n}$ and $\mathbf{T}'_{\mathbf{n}}$ (Fig. 37). $|\mathbf{T}'_{\mathbf{n}}| = w = \epsilon E^2/8\pi$ for any orientation of the area. The strictional (Maxwell) stress is given by $\mathbf{T}''_{\mathbf{n}} = \frac{E^2 \tau \mathbf{n}}{8\pi} \frac{\partial \epsilon}{\partial \tau}$ and is always directed along the normal $\mathbf{n}$ ("negative pressure").

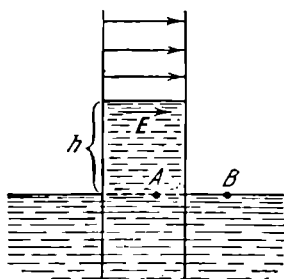


Fig. 36.

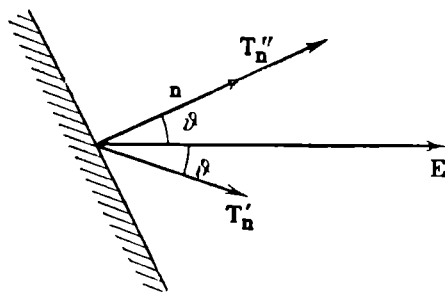


Fig. 37.

140. a) Consider the cylindrical coordinate system illustrated in Fig. 38a. On the xy_3 -plane the field is radial and is given by $E = 2qr/\epsilon(r^2 + a^2/4)^{3/2}$. The force F acting on one of the charges, for example the left-hand charge, is given by

$$F_z = \int T_z dS = -\frac{1}{2\pi} \epsilon q^2 \int_0^\infty \frac{r^2 2\pi r dr}{\epsilon^2 (r^2 + a^2/4)^3} = -\frac{q^2}{\epsilon a^2}.$$

This is in fact the force acting between two charges in a uniform dielectric. However, if this calculation is carried out with the total stress tensor, then the force becomes $F_z + \Delta F_z$ where $\Delta F_z = q^2 \epsilon^{-2} a^{-2} \tau \partial \epsilon / \partial \tau$. The correction ΔF_z is due to the strictional term. However, the theory which takes into account the electrostrictional stresses must also allow for the fact that the liquid will be drawn into the field, and for the associated increase in the hydrostatic pressure, which is given by $\Delta p = \frac{E^2 \tau}{8\pi} \frac{\partial \epsilon}{\partial \tau}$ [Eqn. (III.25)]. The resulting hydrostatic force is $\Delta F_{zh} =$

$= -\frac{q^2\tau}{\epsilon^2 a^2} \frac{\partial \epsilon}{\partial \tau} = -\Delta F_z$. The total force between the charges is thus given by $F_z + \Delta F_z + \Delta F_{zh} = -q^2/\epsilon a^2$ and is, therefore, equal to the force which is obtained without allowance for the strictional forces. Consequently it is the total force due to both electrical and mechanical effects.

b) The same results are obtained by considering the stresses on the surface of a small sphere of radius R whose centre coincides with the point charge q on which the force is acting

(Fig. 38b). Consider the Maxwell stress $\mathbf{T}'_n = \frac{\epsilon}{4\pi} (E_r \mathbf{E} - \frac{1}{2} E^2 \mathbf{e}_r)$,

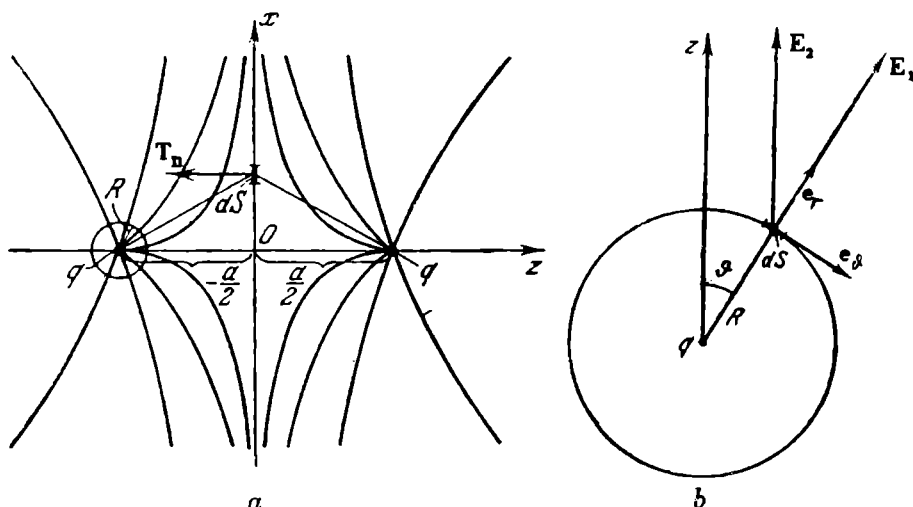


Fig. 38.

where $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and the field due to the charge on which the force is acting is $\mathbf{E}_1 = (q/\epsilon R^2)\mathbf{e}_r$, while the field due to the second charge is $\mathbf{E}_2 = (q/\epsilon a^2)[\sin \theta \mathbf{e}_\theta - \cos \theta \mathbf{e}_r]$ and may be looked upon as uniform, since the distance between the charges is $a \gg R$. Hence the stresses on the surface of the sphere are such that

$$\mathbf{F} = \int \mathbf{T}'_n dS = \frac{q}{\epsilon a^2} \mathbf{e}_z.$$

The inclusion of strictional stresses does not modify this result owing to hydrostatic compensation.

$$141. \quad v_0 = \sqrt{\frac{8mg}{\epsilon - 1}},$$

where g is the acceleration due to gravity.

$$142. \quad \text{When } z \geq 0, \quad \varphi = \varphi_1 = \frac{q}{\epsilon_1 r_1} + \frac{(\epsilon_1 - \epsilon_2)}{\epsilon_1 (\epsilon_1 + \epsilon_2)} \cdot \frac{q}{r_2},$$

$$\text{When } z \leq 0, \quad \varphi = \varphi_2 = \frac{2}{\epsilon_1 + \epsilon_2} \cdot \frac{q}{r_1}.$$

$$143. \quad \sigma_b = \frac{1}{4\pi} \left(\frac{\partial \varphi_2}{\partial z} - \frac{\partial \varphi_1}{\partial z} \right) \Big|_{z=0} = \frac{qa}{2\pi r^3} \cdot \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 (\epsilon_1 + \epsilon_2)},$$

where

$$r = \sqrt{x^2 + y^2 + a^2} = r_1|_{z=0} = r_2|_{z=0}.$$

When $\epsilon_2 \rightarrow \infty$, the problem reduces to that of a point charge q placed in the dielectric ϵ_1 just outside the plane surface of the conductor. The surface density is then $\sigma_b \rightarrow -qa/2\pi r^3 \epsilon_1$. This limiting density is in fact the sum of the density of bound charges on the surface of the dielectric and the density of free charges on the surface of the conductor.

$$144. \quad F = \frac{q^2}{4a^2} \cdot \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 (\epsilon_1 + \epsilon_2)}.$$

When $\epsilon_1 > \epsilon_2$, the charge is repelled from the boundary between the dielectrics, and when $\epsilon_1 < \epsilon_2$ the charge is attracted. A charge which is at first in the medium with the higher permittivity is repelled and moves away from the boundary. A charge which is at first in the medium with the lower permittivity is attracted towards the boundary, passes through it, and having reached the second medium is repelled by it and moves to infinity. All these results will hold provided that frictional effects experienced by the charge while it is moving in the media are neglected.

The above expression for the force F may be obtained in various ways:

(a) by considering the interaction between two point charges q' and q'' ;

(b) by calculating the force on the point charge which is due to bound charges on the separation boundary;

(c) with the aid of the Maxwell stress tensor.

In the latter case it is convenient to discuss stresses on the separation boundary, or on the surface of a small sphere surrounding the charge.

$$145. \quad F_1 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 (\epsilon_1 + \epsilon_2)} \cdot \frac{q_1^2}{4a^2} + \frac{q_1 q_2}{2 (\epsilon_1 + \epsilon_2) a^2},$$

$$F_2 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 (\epsilon_1 + \epsilon_2)} \cdot \frac{q_2^2}{4a^2} + \frac{q_1 q_2}{2 (\epsilon_1 + \epsilon_2) a^2}.$$

The fact that the forces acting on q_1 and q_2 are not equal is due to the fact that the two charges do not by themselves form a closed mechanical system: there are also bound charges on the separation boundary between the two dielectrics. The vector sum of the forces on the boundary and on the charges q_1 and q_2 is equal to zero.

146. If it is assumed that in the metal $\varphi = 0$, then in the dielectric $\varphi = q/\epsilon r_1 - q/\epsilon r_2$ (cf. Fig. 10: charge q at A , charge $-q$ at B ; $\epsilon_1 = \epsilon$, $\epsilon_2 = \infty$). The term $-q/\epsilon r_2$ is due to the charge induced in the conductor and the bound charges on the dielectric. It is equivalent to the potential due to a point charge $-q/\epsilon$ at a distance $-a$ from the boundary. The charge $-q/\epsilon$ is called the image of the charge q/ϵ in the plane $z = 0$ (the factor $1/\epsilon$ represents the effect of the dielectric).

$$\sigma = -\frac{qa}{2\pi r^3}, \quad F = -\frac{q^2}{4a^2\epsilon},$$

where $\mathbf{r}$ is the position vector in the plane $z = 0$.

147. The field between the conducting planes is due to the charge systems shown in Fig. 39.

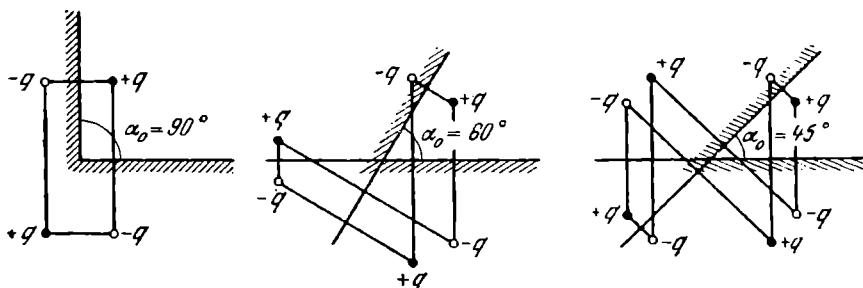


Fig. 39.

148. Suppose that the dipole is located at the point $(0, 0, z)$. If the components of the dipole moment $\mathbf{p}$ along the x -, y -, and z -axes are $(p \sin \alpha, 0, p \cos \alpha)$ then the components of its image $\mathbf{p}'$ along these axes will be $(-p \sin \alpha, 0, p \cos \alpha)$.

$$U = \frac{(\mathbf{p} \cdot \mathbf{p}') r^2 - 3(\mathbf{p} \cdot \mathbf{r})(\mathbf{p}' \cdot \mathbf{r})}{2\epsilon r^5} = -\frac{p^2}{16z^3\epsilon} (1 + \cos^2 \alpha),$$

$$F_z = -\frac{3p^2}{16z^4\epsilon} (1 + \cos^2 \alpha), \quad N_\alpha = -\frac{p^2 \sin^2 \alpha}{16z^3\epsilon}.$$

The factor $1/2$ in the expression for U is due to the fact that the field $\mathbf{E}'$ due to the dipole moment $\mathbf{p}'$ is proportional to $\mathbf{p}$. When $\mathbf{p}$ is increased by $d\mathbf{p}$ at constant orientation, the interaction

energy increases by $dU = -\mathbf{E}' \cdot d\mathbf{p}$ and hence $U = \int_0^p dU = -\frac{1}{2}(\mathbf{E}' \cdot \mathbf{p})$ (cf. solution to Problem 166).

The dipole is attracted to the plane whatever its orientation. The couple $\mathbf{N}$ tends to align the dipole with the z -axis, either in the positive or the negative direction ($\alpha = 0, \pi$). The couple will also vanish when $\alpha = \pi/2$, but this equilibrium position is unstable.

149. Consider a set of polar coordinates with the origin at the centre of the sphere and the z -axis parallel to $\mathbf{E}_0$. The potential may be sought in the form of a series in Legendre polynomials (cf. solution of Problem 153). The final result is

$$\begin{aligned}\varphi_1 &= -\frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2} E_0 r \cos \vartheta \quad \text{when } r < a, \\ \varphi_2 &= -E_0 r \cos \vartheta + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} E_0 a^3 \frac{\cos \vartheta}{r^2} \quad \text{when } r > a.\end{aligned}$$

The electric field inside the sphere is uniform and is given by

$$E_1 = \frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2} E_0 \begin{cases} > E_0 & \text{when } \epsilon_2 > \epsilon_1, \\ < E_0 & \text{when } \epsilon_2 < \epsilon_1. \end{cases}$$

Outside the sphere the resultant field consists of the external uniform field E_0 and the field due to an electric dipole of moment

$$p = E_0 a^3 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}.$$

This secondary field is due to bound charges on the surface of the dielectric sphere with densities

$$\sigma_b = \frac{3}{4\pi} \cdot \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} E_0 \cos \vartheta, \quad \rho_b = 0.$$

It is easy to understand the reason for such a charge distribution by considering each small element of the polarised dielectric as an elementary dipole.

150. For the dielectric with unaltered polarisation $\mathbf{E} = 4\pi\mathbf{P}/3$ (cf. Problem 104). For a normal dielectric

$$\mathbf{E} = \frac{12\pi\epsilon}{(2\epsilon + 1)(\epsilon - 1)} \mathbf{P}.$$

$$151. \quad \varphi = -\mathbf{E}_0 \cdot \mathbf{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \quad (r \gg a),$$

where $\mathbf{p} = R^3 \mathbf{E}_0$, R^3 is the polarisability of the sphere, and

$$\sigma = \frac{3\epsilon_0}{4\pi} E_0 \cos \vartheta.$$

152. The force F on one of the charges may be found by multiplying q_1 by the field strength due to the second charge, q_2 , in the cavity around q_1 . Since the cavity is small, the field inside it is uniform and equal to

$$\frac{3\epsilon E_0}{2\epsilon + 1} = \frac{3q}{(2\epsilon + 1)a^2},$$

where $E_0 = q/\epsilon a^2$ is the uniform field in the vicinity of the cavity. Hence,

$$F = \frac{3q^2}{(2\epsilon + 1)a^2}.$$

This force is different from the force between charges in a uniform liquid dielectric with the same permittivity (*cf.* Problem 140). If by analogy with Problem 140 it were required to find the force on the plane of symmetry, then a calculation involving only the Maxwell stresses would yield $F_1 = q^2/\epsilon a^2$. This differs both from the force F on the charge itself and from the total electric stress force (the strictional term is ignored in view of its complicated form for a solid). An equal force will act on any region of the dielectric surrounding the cavity containing the charge. One part of this force, namely, $3q^2/(2\epsilon + 1)a^2$ acts on the point charge q and the other part, namely, $-(2\epsilon - 1)q^2/(2\epsilon + 1)a^2\epsilon$ acts on the bound charges on the surface of the cavity.

153. Let the origin of spherical polar coordinates be at the centre of the sphere and consider the potential in the form

$$\varphi(r, \vartheta, \alpha) = \frac{q}{\epsilon r_1} + \sum_{l, m} \left(a_{lm} r^l + \frac{b_{lm}}{r^{l+1}} \right) P_{lm}(\cos \vartheta) e^{im\alpha}, \quad (1)$$

where r_1 is the distance between q_1 and the point of observation. The series in Eqn. (1) represents the field due to charges induced on the sphere. This field should vanish at infinity, and therefore $a_{lm} = 0$. Because of the symmetry of the problem, the potential is independent of the angle α , and hence terms with $m \neq 0$ will also vanish. The remaining constants, $b_l \equiv b_{l0}$, will be

determined from the boundary conditions. In case (a) the potential of the sphere is $\varphi(R, \vartheta) = V = \text{const.}$ Using the expansion for q/r_1 given in Problem 96 we have

$$\varphi(R, \vartheta) = \sum_{l=0}^{\infty} \left(\frac{qR^l}{\epsilon a^{l+1}} + \frac{b_l}{R^{l+1}} \right) P_l(\cos \vartheta) = V,$$

and hence $b_l = -qR^{2l+1}/\epsilon a^{l+1}$ when $l \neq 0$, $b_0 = VR - Rq/\epsilon a$, so that the potential outside the sphere is given by

$$\varphi(r, \vartheta) = \frac{q}{\epsilon r_1} + \frac{VR}{r} - \frac{qR}{\epsilon a} \sum_{l=0}^{\infty} \left(\frac{R^2}{a} \right)^l \frac{P_l(\cos \vartheta)}{r^{l+1}}. \quad (2)$$

The density of charges induced on the surface of the sphere can now be found and is given by

$$\sigma(R, \vartheta) = -\frac{\epsilon}{4\pi} \frac{\partial \varphi}{\partial r} \Big|_{r=R} = \frac{\epsilon V}{4\pi R} - \frac{q}{4\pi} \sum_{l=0}^{\infty} (2l+1) \frac{R^{l-1}}{a^{l+1}} P_l(\cos \vartheta). \quad (3)$$

In case (b) the potential V is unknown and should be expressed in terms of the charge Q of the sphere. Clearly,

$$Q = 2\pi \int \sigma(R, \vartheta) R^2 \sin \vartheta d\vartheta = \epsilon VR - \frac{qR}{a},$$

and hence $V = (Q/\epsilon R) + (q/\epsilon a)$. Using the solution of Problem 96, Eqn. (2) may be rewritten in the form

$$\varphi = \frac{q}{\epsilon r_1} + \frac{Q+q'}{\epsilon r} - \frac{q'}{\epsilon r_2}, \quad (4)$$

where

$$q' = q \frac{R}{a}, \quad r_2 = \sqrt{r^2 + a'^2 - 2a'r \cos \vartheta}, \quad a' = \frac{R^2}{a}.$$

Thus, in the region $r > a$, the potential due to the sphere and the point charge is equivalent to the potential due to four point charges lying along the axis of symmetry, namely, a charge q at a distance a from the origin and its three images, *i. e.* charges Q and $q' = qR/a$ at the origin and a charge $-q'$ at the point $a' = R^2/a$. The charge $-q'$ represents the effect of charges induced on the surface of the sphere which is nearest to q (Fig. 40). The sign of these charges is clearly opposite to that of q . The charge $+q'$ represents the effect of charges on the distant side of the sphere and is, of course, of the same sign as q . If the sphere as a whole is neutral, then the term including

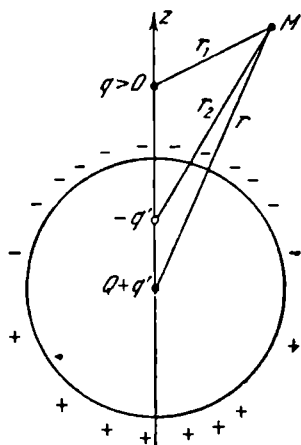


Fig. 40.

Q will vanish. When the sphere is earthed ($V = 0$), the potential is given by

$$\varphi = \frac{q}{\epsilon r_1} - \frac{q'}{\epsilon r_2}. \quad (5)$$

$$154. \quad \varphi(M) = \frac{q}{\epsilon r_1} - \frac{q'}{\epsilon r_2} + V$$

where (cf. Fig. 41)

$$q' = q \frac{R}{a}, \quad a' = \frac{R^2}{a}.$$

$$155. \quad \varphi(M) = \frac{q}{r_1} - \frac{q'}{r_2} + \frac{q'}{r_3} - \frac{q}{r_4}$$

where

$$q' = \frac{qa}{b}, \quad b' = \frac{a^2}{b}$$

(cf. Fig. 42). The charge on the boss is given by

$$Q = -q \left[1 - \frac{b^2 - a^2}{b \sqrt{a^2 + b^2}} \right].$$

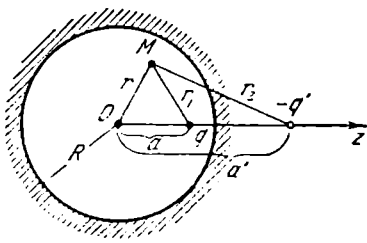


Fig. 41.

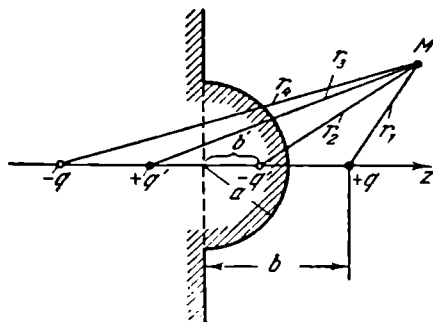


Fig. 42.

156. Outside the sphere

$$\varphi \equiv \varphi_1 = \frac{q}{\epsilon_1 r};$$

in the conductor

$$\varphi \equiv \varphi_3 = \frac{q}{\epsilon_1 R};$$

in the cavity

$$\varphi \equiv \varphi_2 = \frac{q}{\epsilon_2 r_1} - \frac{q'}{\epsilon_2 r_2} + \frac{q}{\epsilon_1 R_1}$$

(cf. Fig. 43) where

$$q' = \frac{q R_2}{a}, \quad a' = \frac{R_2^2}{a}.$$

$$157. \quad \varphi_1(r, \vartheta) = q \sum_{l=0}^{\infty} \frac{2l+1}{l\epsilon_1 + (l+1)\epsilon_2} \cdot \frac{r^l}{a^{l+1}} P_l(\cos \vartheta) \quad \text{when } r \leq R,$$

$$\varphi_2(r, \vartheta) = \frac{q}{\epsilon_2 r_1} + q \frac{\epsilon_2 - \epsilon_1}{\epsilon_2} \sum_{l=0}^{\infty} \frac{l}{l\epsilon_1 + (l+1)\epsilon_2} \cdot \frac{R_2^{2l+1}}{a^{l+1}} \cdot \frac{P_l(\cos \vartheta)}{r^{l+1}} \quad \text{when } r \gg R,$$

where r_1 is the distance between the point charge q and the point of observation. In this case the potential cannot be derived from a simple set of images.

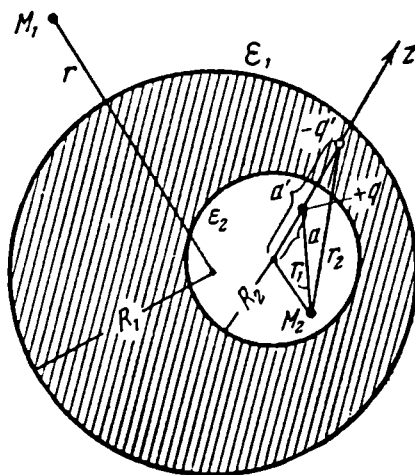


Fig. 43.

158.

$$\varphi_1 = \frac{q}{\epsilon_1 r_1} + q \frac{\epsilon_1 - \epsilon_2}{\epsilon_1} \sum_{l=0}^{\infty} \frac{l+1}{\epsilon_1 l + \epsilon_2 (l+1)} \cdot \frac{a^l r^l}{R_2^{2l+1}} P_l(\cos \vartheta) \quad \text{when } r \leq R,$$

$$\varphi_2 = q \sum_{l=0}^{\infty} \frac{2l+1}{\epsilon_1 l + \epsilon_2 (l+1)} \cdot \frac{a^l}{r^{l+1}} P_l(\cos \vartheta) \quad \text{when } r \gg R,$$

where r_1 is the distance between the point of observation and the

charge q . When $a = 0$,

$$\varphi_1 = \frac{q}{\epsilon_1 r} + \left(1 - \frac{\epsilon_2}{\epsilon_1}\right) \frac{q}{\epsilon_2 R}, \quad \varphi_2 = \frac{q}{\epsilon_2 r}.$$

159. Let the surfaces of the inner and outer spheres be denoted by S_1 and S_2 respectively, and suppose that the potential of the inner sphere is zero. It is convenient to solve the problem in terms of polar coordinates with the polar axis joining the centres of the spheres and the origin at the centre of the inner

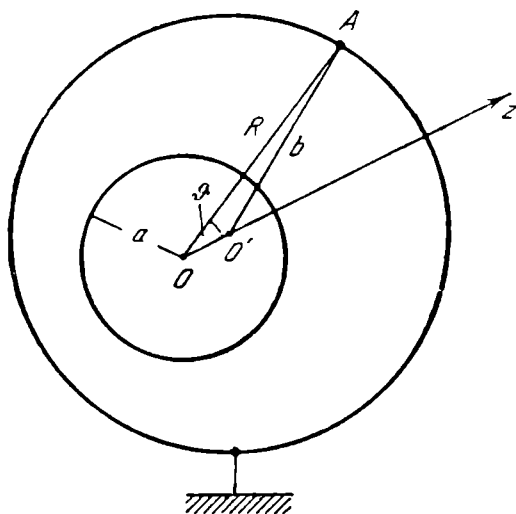


Fig. 44.

sphere. The equation of the surface S_1 is then $r = a$. In order to obtain the equation for S_2 we note that in the triangle $OO'A$ (Fig. 44):

$$\frac{1}{b} = \frac{1}{\sqrt{R^2 + a^2 - 2aR \cos \theta}}. \quad (1)$$

On expanding, and neglecting terms of order higher than c , we find that the equation of S_2 is

$$R(\theta) = b + cP_1(\cos \theta), \quad (2)$$

where

$$P_1(\cos \theta) = \cos \theta.$$

The term $cP_1(\cos \theta) = c \cos \theta$ in Eqn. (2) describes the departure from spherical symmetry and vanishes when $c \rightarrow 0$. It is natural to seek the potential in the form of an expansion in spherical

harmonics (*cf.* Appendix II) and to retain only the first two terms. The second term, which represents the departure from spherical symmetry, should be proportional to c . Thus,

$$\varphi(r, \vartheta) = \left(A_1 + \frac{B_1}{r}\right) + c \left(A_2 r + \frac{B_2}{r^2}\right) \cos \vartheta, \quad (3)$$

where A_i and B_i are determined from the boundary conditions

$$\varphi|_{S_1} = \text{const}, \quad \varphi|_{S_2} = 0, \quad \oint_{S_1} \frac{\partial \varphi}{\partial n} dS_1 = -4\pi q.$$

The final result is

$$\varphi = q \left(\frac{1}{r} - \frac{1}{b} \right) + \frac{qc}{b^3 - a^3} \left(r - \frac{a^3}{r^2} \right) \cos \vartheta,$$

and hence the density of charge on the inner sphere is given by

$$\sigma = \frac{q}{4\pi a^2} - \frac{3qc}{4\pi(b^3 - a^3)} \cos \vartheta;$$

and the force on the inner sphere is given by

$$F = - \frac{qc}{b^3 - a^3}.$$

$$160. \quad \Delta C = \frac{a^2 b^2 c^2}{(b-a)^2 (b^3 - a^3)}.$$

161. When the charge q is increased by dq , the energy U of its interaction with the sphere increases by $dU = \varphi' dq$, where φ' is the potential due to charges induced on the sphere. However, this potential is proportional to q and hence

$$U = \int_0^q dU = \frac{\text{const}}{2} q^2 = \frac{1}{2} \varphi' q. \quad (1)$$

If the magnitude of φ' were independent of q , then the interaction energy would be larger by a factor of two. Using Eqn. (1) and the results of Problem 153, we find that

$$U = - \frac{q^2 R}{2\epsilon (a^2 - R^2)},$$

and hence

$$F = - \frac{q^2 a R}{\epsilon (a^2 - R^2)^2}.$$

$$162. \quad U = \frac{Qq}{\epsilon a} - \frac{q^2 R^3}{2a^2 \epsilon (a^2 - R^2)};$$

$$F = \frac{Qq}{\epsilon a^2} - \frac{q^2 R^3 (2a^2 - R^2)}{\epsilon a^3 (a^2 - R^2)^2}.$$

When the two charges are of the same sign, $Qq > 0$ and the force of interaction may vanish. When q is sufficiently large, or a is very small, the force may even be negative (attraction).

163. The test charge q should be small compared with the charges on other conductors and dielectrics, and should not be too near to any irregularities in the medium, *e.g.* boundaries of conductors and dielectrics, so that the effects due to charges induced by the test body should be small. For example, in measuring the electric field due to a charged conducting sphere, it is necessary that the force due to the electrical image should be small compared with qQ/a^2 , where Q is the charge on the sphere and a is the distance between the test charge and the centre of the sphere. This leads to the condition (*cf.* solution of the preceding problem)

$$\left| \frac{Q}{q} \right| \gg 2 \frac{(2a/R - 1)^2}{(a/R)(a/R - 1)^2}.$$

This condition will be satisfied provided a/R is not too small and q/Q is not too large.

164. The image of the electric dipole $\mathbf{p} = p(\sin \alpha \mathbf{e}_x + \cos \alpha \mathbf{e}_z)$ in the earthed sphere consists of a point charge $q = (pR/r^2) \cos \alpha$ and a dipole $\mathbf{p}' = p(R/r)^3(-\sin \alpha \mathbf{e}_x + \cos \alpha \mathbf{e}_z)$ at the point A' (Fig. 45) which is at a distance $r' = R^2/r$ from the centre of the sphere.

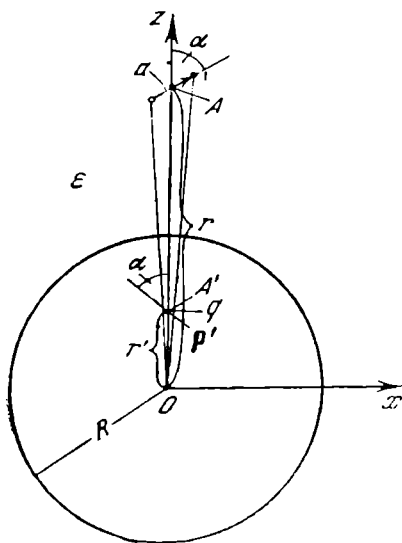


Fig. 45.

$$U = - \frac{p^2 R (r^2 \cos^2 \alpha + R^2)}{2\epsilon (r^2 - R^2)^3},$$

$$F = - \frac{p^2 R r}{\epsilon (r^2 - R^2)^4} [(2r^2 + R^2) \cos^2 \alpha + 3R^2],$$

$$N = - \frac{p^2 R r^2 \sin 2\alpha}{2\epsilon (r^2 - R^2)^3}.$$

In the limit $r \rightarrow R$, we find on substituting $r = R + z$, $R \rightarrow \infty$, $z = \text{const}$, that the results are identical with those of Problem 148 (dipole near a conducting plane).

$$165. \quad \sigma = -\frac{3p}{4\pi R^3} \cos \vartheta,$$

where ϑ is the angle between $\mathbf{p}$ and the line joining the centre and the point of observation. Induced charges give rise to a uniform field $E = p/R^3$ in the cavity.

166. The forces acting on the irregularity may be obtained by differentiating the quantity

$$U' = \sum_{l,m} a_{lm} Q_{lm}^* \quad (1)$$

with Q_{lm}^* held constant. The quantity U' differs from the true energy of interaction of the irregularity with the external field; the latter is defined as the work which must be done in order to produce a potential φ in the presence of the irregularity [cf. Eqn. (III.16)]. It must, however, be remembered that the moments Q_{lm} depend on the external field. In particular, if the irregularity is in the form of an uncharged conductor or dielectric, then the true energy of interaction of the irregularity with the external field is given by

$$U = \frac{1}{2} \sum_{l,m} a_{lm} Q_{lm}^*. \quad (2)$$

The factor $1/2$ may be obtained as in the solution of Problem 161 by recalling that Q_{lm} are proportional to a_{lm} . In the determination of the generalised forces with the aid of Eqn. (2), which involves differentiation with respect to the generalised coordinates, both Q_{lm} and a_{lm} must be looked upon as variable.

$$167. \quad U_0 = q\varphi_0 - \mathbf{p} \cdot \mathbf{E}_0 \text{ and}$$

$$\varphi_1 = \varphi_0 - \mathbf{r} \cdot \mathbf{E}_0, \quad \varphi_2 = \frac{q}{\epsilon r} + \frac{\mathbf{p} \cdot \mathbf{r}}{\epsilon r^3},$$

$$\mathbf{F} = q\mathbf{E}_0 + (\mathbf{p} \cdot \nabla) \mathbf{E}_0, \quad \mathbf{N} = \mathbf{p} \times \mathbf{E}_0$$

(the couple is evaluated about the origin).

169. The body tends to occupy a position in which its potential energy $U = -\frac{1}{2} \mathbf{p} \cdot \mathbf{E}$ is a minimum. Suppose that the coordinate axes lie along the principal axes of the tensor β_{ik} so that $U = -\frac{1}{2} [\beta^{(x)} E_x^2 + \beta^{(y)} E_y^2 + \beta^{(z)} E_z^2]$. It is clear that if $\beta^{(x)} \geq \beta^{(y)} \geq \beta^{(z)} > 0$, then U will be a minimum when $\mathbf{E}$ is parallel to the x -axis. When $\beta^{(x)} \leq \beta^{(y)} \leq \beta^{(z)} < 0$, the minimum occurs when $\mathbf{E}$ is parallel to the z -axis.

170. The axis of the rod and the plane of the disc tend to become parallel to the field when $\epsilon_1 > \epsilon_2$, and perpendicular to the field when $\epsilon_1 < \epsilon_2$.

$$171. \quad F = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2} q^2 \sum_{l=0}^{\infty} \frac{l(l+1)}{l\epsilon_1 + (l+1)\epsilon_2} \cdot \frac{R^{2l+1}}{a^{2l+3}}.$$

Attraction occurs for $\epsilon_2 < \epsilon_1$ and repulsion for $\epsilon_2 > \epsilon_1$. In the case of a conducting sphere $\epsilon_1 \rightarrow \infty$. On summing the geometrical progression we find that the interaction energy is $U = -q^2 R / 2\epsilon_2 (R^2 - a^2)$, and hence

$$F = - \frac{q^2 a R}{\epsilon_2 (a^2 - R^2)^2}$$

(cf. Problem 161).

Consider now the calculation of the force with the aid of Eqn. (III.16). In the integral $U' = \frac{1}{8\pi} \int_{V'} (\epsilon_2 - \epsilon_1) \mathbf{E} \cdot \mathbf{E}_1 dV'$ the volume V' is bounded by a sphere S which is infinitely close to the surface of the dielectric sphere and lies wholly inside the latter. The integral differs from the potential energy of interaction, U , between the point charge and the sphere by an infinitesimal amount. Instead of the total electric field $\mathbf{E}$ and the field of the point charge $\mathbf{E}_1$ in the uniform dielectric ϵ_2 , let us use the corresponding potentials φ and φ_1 and take the constant quantity $\epsilon_2 - \epsilon_1$ outside the integral, so that $U' = \frac{\epsilon_2 - \epsilon_1}{8\pi} \int_{V'} \nabla \varphi \cdot \nabla \varphi_1 dV'$. Using

Green's formula

$$\int \nabla \varphi \cdot \nabla \varphi_1 dV = \oint_S \varphi \frac{\partial \varphi_1}{\partial n} dS + \int \varphi \Delta \varphi_1 dV$$

and the fact that inside the sphere $\Delta \varphi_1 = 0$, we find the following expression for U :

$$U = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2} q^2 \sum_{l=0}^{\infty} \frac{l}{l\epsilon_1 + (l+1)\epsilon_2} \cdot \frac{R^{2l+1}}{a^{2l+3}}.$$

This expression is identical with that obtained from Eqn. (2) of Problem 166. Hence, the formula for F turns out to be identical with that given above.

$$172. \quad C_b = \frac{x}{\varphi_1 - \varphi_2} = \frac{1}{2} \left[\cosh^{-1} \frac{a^2 - R_1^2 - R_2^2}{2R_1 R_2} \right]^{-1}.$$

$$173. \quad \sigma = \pm \frac{|E|}{4\pi} = \pm \frac{4b^2 x^2}{\pi [(x^2 - y^2 - b^2)^2 + 4x^2 y^2]}.$$

where

$$b = \frac{2R^4}{a^2 \sqrt{a^2 - 4R^2}}.$$

The origin is at the centre of a straight line joining the axes of the cylinders and chosen as the x -axis.

$$174. \quad C = \frac{1}{2} \left[\cosh^{-1} \frac{R_1^2 + R_2^2 - a^2}{2R_1 R_2} \right]^{-1}.$$

175. If the x -, y -, and z -axes are parallel to the principal axes of the tensor ε_{ik} , then

$$\varphi(x, y, z) = \frac{e'}{r'} = \frac{e}{\sqrt{\varepsilon(x)\varepsilon(y)\varepsilon(z)}} \left[\frac{x^2}{\varepsilon(x)} + \frac{y^2}{\varepsilon(y)} + \frac{z^2}{\varepsilon(z)} \right]^{-\frac{1}{2}}. \quad (1)$$

For an arbitrary orientation of the coordinate system, Eqn. (1) may be written in the form

$$\varphi(\mathbf{r}) = \frac{e}{\sqrt{|\varepsilon_{ik}| \varepsilon_{ik}^{-1} x_i x_k}},$$

where $|\varepsilon_{ik}|$ is the determinant of the tensor ε_{ik} .

$$176. \quad \mathbf{E} = \mathbf{E}_0 - \frac{(\varepsilon_{ik} - \delta_{ik}) n_i E_{0k}}{\varepsilon_{lm} n_l n_m} \mathbf{n}.$$

177. $C = S\varepsilon^{(z)}/4\pi d$, where z is measured along the normal to the plates of the capacitor.

178. If the x - and z -axes lie in the plane $(\mathbf{E}_0, \mathbf{n})$ and the z -axis is parallel to $\mathbf{n}$, then

$$\tan \vartheta = \frac{E_x}{E_z} = \frac{\varepsilon_{zz} \tan \vartheta_0}{1 - \varepsilon_{zx} \tan \vartheta_0},$$

where $\tan \vartheta_0 = E_{0x}/E_{0z}$. A given line of force in the dielectric will then remain in the plane $(\mathbf{E}_0, \mathbf{n})$.

2. Coefficients of potential and capacitance

180. Suppose that the charge on the first conductor is q_1 and the charge on the outer surface of the second conductor is q' (the charge on the inner surface of the second conductor is $-q_1$;

this follows from the electrostatic Gauss theorem). Eqn. (III.28) then assumes the form

$$\left. \begin{aligned} q_1 &= c_{11}V_1 + c_{12}V_2, \\ -q_1 + q' &= c_{12}V_1 + c_{22}V_2. \end{aligned} \right\} \quad (1)$$

and hence

$$q' = (c_{11} + c_{12})V_1 + (c_{12} + c_{22})V_2. \quad (2)$$

Once q' is specified, the field in the external space is defined, including the potential V_2 of the second conductor. Eqn. (2) should, therefore, hold for all values of V_1 at given q' and V_2 , which can only be true if

$$c_{11} + c_{12} = 0. \quad (3)$$

The first of the equations in (1) then takes the form:

$$q_1 = c_{11}(V_1 - V_2). \quad (4)$$

It follows from (2), (3), and (4) that

$$\begin{aligned} C &= c_{11} = -c_{12} = -c_{21}, \\ C' &= c_{12} + c_{22}. \end{aligned}$$

$$\begin{aligned} 181. \quad s_{11} &= \frac{c_{22}}{c_{11}c_{22} - c_{12}^2}, \quad s_{22} = \frac{c_{11}}{c_{11}c_{22} - c_{12}^2}, \\ s_{22} &= -\frac{c_{12}}{c_{11}c_{22} - c_{12}^2}. \end{aligned}$$

$$183. \quad C = \frac{c_{11}c_{22} - c_{12}^2}{c_{11} + c_{22} + 2c_{12}}.$$

$$\begin{aligned} 184. \quad q_1 &= \frac{s_{11} - 2s_{12} + s_{13}}{s_{11} - s_{12}} \cdot \frac{q}{8}, \quad q_2 = \frac{q}{2}, \\ q_3 &= \frac{q}{4}, \quad q_4 = \frac{s_{11} - s_{13}}{s_{11} - s_{12}} \cdot \frac{q}{8}. \end{aligned}$$

$$185. \quad q_1 = -\frac{2a}{b}q, \quad q_2 = -\frac{a}{b}q, \quad q_3 = \frac{3a^2}{b^2}q.$$

$$186. \quad F = \frac{C_1 C_2 (\epsilon r V_1 - C_2 V_2) (\epsilon r V_2 - C_1 V_1)}{(\epsilon^2 r^2 - C_1 C_2)^2}.$$

$$187. \quad q_1 = q \frac{V_0 - V_P}{V_1 - V_0}, \quad q_0 = q \frac{V_1 - V_P}{V_0 - V_1}.$$

189. The self-capacitance of the joined conductors is given by

$$c_{00} = c_{11} + c_{22} + 2c_{12}.$$

The mutual capacitance between the joined conductors and the i -th

conductor of the system is

$$c_{0l} = c_{1l} + c_{2l}.$$

190. The reduction in the energy is

$$\Delta W = \frac{(q - q')^2}{4} \cdot \frac{r - b}{rb}.$$

191. To within terms of the order of $1/r$, the force F is

$$F = - \frac{bC^2q^2}{r^3 [C + ab(b - a)^{-1}]^2}.$$

192. When the sphere and the conductor touch, they assume the same potential

$$V_1 = qs_{11} + (Q - q)s_{12} = qs_{12} + (Q - q)s_{22} = V_2,$$

and hence

$$\frac{s_{11} - s_{12}}{s_{22} - s_{12}} = \frac{Q}{q} - 1, \quad (1)$$

where s_{ik} are the coefficients of potential and subscripts 1 and 2 refer to the sphere and the conductor respectively. Let q_k be the charge on the conductor after the k -th contact. Since the potentials are equal on contact, it follows that

$$q_k s_{11} + (Q + q_{k-1} - q_k) s_{12} = q_k s_{12} + (Q - q_k + q_{k-1}) s_{22}.$$

Hence, using (1) we obtain the recurrence relation

$$q_k = q + \frac{q}{Q} q_{k-1}. \quad (2)$$

Successive application of (2) with $k \rightarrow \infty$ yields

$$q = \lim_{k \rightarrow \infty} q_k = q \left[1 + \frac{q}{Q} + \left(\frac{q}{Q}\right)^2 + \left(\frac{q}{Q}\right)^3 + \dots \right] = \frac{qQ}{Q - q}.$$

3. Special methods of electrostatics

193. The Laplace equation assumes the form

$$\frac{d}{d\xi} \left(R_\xi \frac{d\varphi}{d\xi} \right) = 0, \quad R_\xi = \sqrt{(\xi + a^2)(\xi + b^2)(\xi + c^2)}.$$

This equation may be integrated subject to the boundary conditions: $\varphi = \text{const}$ when $\xi = 0$ (surface of the ellipsoid), and

$\varphi \rightarrow 0$ when $\xi \rightarrow \infty$. Since $r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \rightarrow \infty$ when $\xi \rightarrow \infty$ we have

$$\varphi(\xi) = \frac{q}{2\epsilon} \int_{\xi}^{\infty} \frac{d\xi}{R_{\xi}}, \quad \frac{1}{C} = \frac{1}{2\epsilon} \int_0^{\infty} \frac{d\xi}{R_{\xi}},$$

and hence

$$\sigma = -\frac{\epsilon}{4\pi} \frac{\partial \varphi}{\partial n} \Big|_{\xi=0} = -\frac{\epsilon}{4\pi} \left(\frac{1}{h_1} \frac{\partial \varphi}{\partial \xi} \right)_{\xi=0} = \frac{q \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}}}{4\pi abc}.$$

The charge densities at the ends of the semi-axes are proportional to the lengths of the semi-axes, so that $\sigma_a : \sigma_b : \sigma_c = a : b : c$.

194. When $a = b > c$ (oblate ellipsoid):

$$\varphi = \frac{q}{\sqrt{a^2 - c^2}} \tan^{-1} \sqrt{\frac{a^2 - c^2}{\xi + c^2}},$$

$$C = \frac{\sqrt{a^2 - c^2}}{\cos^{-1} \frac{c}{a}}.$$

In particular, when $c = 0$ (disc), $C = 2a/\pi$. When $a > b = c$ (prolate ellipsoid)

$$\varphi = \frac{q}{2\sqrt{a^2 - b^2}} \ln \frac{\sqrt{\xi + a^2} + \sqrt{a^2 - b^2}}{\sqrt{\xi + a^2} - \sqrt{a^2 - b^2}},$$

$$C = \frac{\sqrt{a^2 - b^2}}{\ln \frac{a + \sqrt{a^2 - b^2}}{b}}.$$

In particular, when $b \ll a$ (a rod)

$$C = \frac{a}{\ln \frac{2a}{b}}.$$

195. To start with, let us suppose that the ellipsoid is uncharged and, therefore, $q = 0$. If the external uniform field $\mathbf{E}_0$ is parallel to the x -axis, then

$$\varphi_0 = -E_0 x = \mp E_0 \sqrt{\frac{(\xi + a^2)(\eta + a^2)(\xi + a^2)}{(b^2 - a^2)(c^2 - a^2)}}.$$

The negative sign corresponds to $x > 0$ and the positive sign to $x < 0$. Both φ_0 and the potential φ' due to charges induced on the ellipsoid satisfy the Laplace equation. Substituting $\varphi' = \varphi_0 F(\xi)$ into the Laplace equation we obtain the following

differential equation for the unknown function $F(\xi)$:

$$\frac{d^2 F}{d\xi^2} + \frac{dF}{d\xi} \ln [R_\xi (\xi + a^2)] = 0.$$

This equation can easily be integrated. The solution satisfying the boundary conditions is

$$\varphi|_{q=0} = \varphi_0 \left[1 - \frac{\int_{\xi}^{\infty} \frac{d\xi}{(\xi + a^2) R_\xi}}{\int_0^{\infty} \frac{d\xi}{(\xi + a^2) R_\xi}} \right].$$

If the ellipsoid has a total charge q , then the solution satisfying the conditions $\varphi|_{\xi=0} = \text{const}$ and $-\oint_S \frac{\partial \varphi}{\partial n} dS = 4\pi q$ (S is a closed surface surrounding the ellipsoid) may be obtained from the superposition principle (cf. Problem 193):

$$\varphi|_q = \varphi_{q=0} + \frac{1}{2} q \int_{\xi}^{\infty} \frac{d\xi}{R_\xi}.$$

196. The potential is of the same form as in the preceding problem. The integrals entering into the expression for the potential may be expressed in terms of elementary functions whenever the ellipsoid has rotational symmetry. The final result is

$$\varphi = -E_0 x \left[1 - \frac{\ln \frac{\sqrt{1 + \xi/a^2} + e}{\sqrt{1 + \xi/a^2} - e} - \frac{2e}{\sqrt{1 + \xi/a^2}}}{\ln \frac{1+e}{1-e} - 2e} \right],$$

where a and b are the semi-axes, $e = (1 - b^2/a^2)^{\frac{1}{2}}$ is the eccentricity of the ellipsoid, and the x -axis is perpendicular to the plane $x = (a/e)(1 + \xi/a^2)^{\frac{1}{2}}(1 + \xi/a^2)^{\frac{1}{2}}$ (cf. Problem 66). The field strength reaches a maximum at the apex of the ellipsoid:

$$\frac{E_{\max}}{E_0} = - \frac{1}{E_0 h_\xi} \frac{\partial \varphi}{\partial \xi} \Big|_{\xi=0, \zeta=-b^2} = \frac{2e^3 (1-e^2)^{-1}}{\ln \frac{1+e}{1-e} - 2e} = \frac{1}{n^{(x)}},$$

where $n^{(x)}$ are the depolarisation coefficients (cf. Problem 198). For a sphere, $e = 0$ and $E_{\max}/E_0 = 3$. In the case of a long rod

(lightning conductor):

$$\frac{E_{\max}}{E_0} = \frac{a^2}{b^2} \left(\ln \frac{2a}{b} - 1 \right)^{-1}, \quad a \gg b,$$

and therefore a discharge is more likely to occur at the end of the rod rather than at any other point on it.

197. The field at an arbitrary distance from the ellipsoid may be obtained by superimposing three fields of the form established in Problem 195 (the field $\mathbf{E}_0$ will be resolved along the principal axes of the ellipsoid). At large distances from the ellipsoid

$$\varphi = \varphi_0 + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3},$$

$$p_x = \beta^{(x)} E_x, \quad p_y = \beta^{(y)} E_y, \quad p_z = \beta^{(z)} E_z.$$

The principal values of the polarisability tensor for the ellipsoid are

$$\beta^{(x)} = \frac{abc}{3n^{(x)}}, \quad \beta^{(y)} = \frac{abc}{3n^{(y)}}, \quad \beta^{(z)} = \frac{abc}{3n^{(z)}}.$$

$$198. \quad n^{(x)} = \frac{1-e^2}{2e^2} \left(\ln \frac{1+e}{1-e} - 2e \right) \leq \frac{1}{3},$$

$$n^{(y)} = n^{(z)} = \frac{1-n^{(x)}}{2},$$

where $e = (1 - b^2/a^2)^{\frac{1}{2}}$ is the eccentricity of the ellipsoid. When $e \rightarrow 1$ (a rod),

$$n^{(x)} = 0, \quad n^{(y)} = n^{(z)} = \frac{1}{2}.$$

When $e \ll 1$ (nearly spherical shape),

$$n^{(x)} = \frac{1}{3} - \frac{2}{15} e^2, \quad n^{(y)} = n^{(z)} = \frac{1}{3} + \frac{1}{15} e^2.$$

$$199. \quad n^{(z)} = \frac{1+e^2}{e^3} (e - \tan^{-1} e) \geq \frac{1}{3},$$

$$n^{(x)} = n^{(y)} = \frac{1-n^{(z)}}{2}, \quad e = \sqrt{\left(\frac{a}{c}\right)^2 - 1}.$$

For a disc

$$n^{(z)} = 1, \quad n^{(x)} = n^{(y)} = 0.$$

200. $\varphi = \varphi_x + \varphi_y + \varphi_z$. Inside the ellipsoid,

$$\varphi_x = \varphi_{1x} = -E_0 x \left(1 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_2} n^{(x)} \right)^{-1}.$$

Outside the ellipsoid,

$$\varphi_x = \varphi_{2x} = -E_0 x \left\{ 1 - \frac{abc (\epsilon_1 - \epsilon_2) \int_{\xi}^{\infty} \frac{d\xi}{(\xi + a^2) R_{\xi}}}{2 [\epsilon_2 + (\epsilon_1 - \epsilon_2) n^{(x)}]} \right\},$$

where

$$n^{(x)} = \frac{1}{2} abc \int_0^{\infty} \frac{d\xi}{(\xi + a^2) R_{\xi}}.$$

The potentials φ_y and φ_z are given by analogous expressions with x replaced by y and z , and a by b and c respectively. Inside the ellipsoid the field is uniform and is given by

$$\mathbf{E}_1 = \frac{E_0 x \mathbf{e}_x}{1 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_2} n^{(x)}} + \frac{E_0 y \mathbf{e}_y}{1 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_2} n^{(y)}} + \frac{E_0 z \mathbf{e}_z}{1 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_2} n^{(z)}}.$$

At large distances from the ellipsoid

$$\varphi_2 = -\mathbf{E}_0 \cdot \mathbf{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3},$$

where

$$p_x = \beta^{(x)} E_x, \quad \beta^{(x)} = \frac{abc}{3 \left(\frac{\epsilon_2}{\epsilon_1 - \epsilon_2} + n^{(x)} \right)},$$

and so on.

201. Using Eqn. (III.16) we have

$$U = \frac{abc (\epsilon_2 - \epsilon_1) E_0^2 \{ 2 [\epsilon_2 + (\epsilon_1 - \epsilon_2) n] \sin^2 \vartheta + [\epsilon_1 + \epsilon_2 + (\epsilon_2 - \epsilon_1) n] \cos^2 \vartheta \}}{6 [\epsilon_2 + \epsilon_1 + n (\epsilon_2 - \epsilon_1)] [\epsilon_2 + (\epsilon_1 - \epsilon_2) n]},$$

$$N = -\frac{\partial U}{\partial \vartheta} = \frac{abc (\epsilon_2 - \epsilon_1)^2 E_0^2 (3n - 1) \sin 2\vartheta}{6 [\epsilon_2 + \epsilon_1 + n (\epsilon_2 - \epsilon_1)] [\epsilon_2 + (\epsilon_1 - \epsilon_2) n]},$$

where ϑ is the angle between the axis of symmetry and the field $\mathbf{E}_0$, and n is the depolarisation coefficient with respect to the axis of symmetry of the ellipsoid (cf., for example, the solution of the preceding problem). It is clear from the latter formula that the external field tends to turn over the axes of symmetry of prolate ($n < 1/3$) and oblate ($n > 1/3$) ellipsoids into positions respectively

parallel and perpendicular to the field. For a conducting ellipsoid $\epsilon_1 \rightarrow \infty$, and

$$N = \frac{abc(3n-1)E_0^2 \sin 2\vartheta}{6n(1-n)}.$$

202. The potential energy of a charged drop in the form of an ellipsoid of revolution with an eccentricity $e = (1 - b^2/a^2)^{\frac{1}{2}}$ and volume equal to that of a sphere of radius R and charge q may be written down in the form

$$U(e) = \frac{q^2}{2C} + \alpha S = \frac{q^2 \sqrt[3]{1-e^2}}{4Re} \ln \frac{1+e}{1-e} + 2\pi R^2 \alpha \left(\sqrt[3]{1-e^2} + \frac{\sin^{-1} e}{e \sqrt{1-e^2}} \right), \quad (1)$$

where we have used the expression for the capacitance of a prolate ellipsoid of revolution given in the solution of Problem 194.

In order to investigate the stability of a charged spherical drop, consider the function $U(e)$ for small values of e . The expansion for U up to terms of the order of e^4 is

$$U(e) = \frac{q^2}{2R} + 4\pi R^2 \alpha + \frac{e^4}{45} \left(8\pi R^2 \alpha - \frac{q^2}{2R} \right).$$

It is clear from this formula that if the charge on the drop is such that $q < q_c = (16\pi R^3 \alpha)^{\frac{1}{2}}$, then for small deformations the drop will tend to return to the spherical state and will therefore be stable. When $q > q_c$, the drop will be unstable because the deformation will continue to increase. The process ends with the division of the unstable drop into two or more parts. It is easy to verify that when the charged drop divides into two equal spherical drops the energy is reduced by a factor of $2^{\frac{2}{3}}$. The process finally ends with the formation of stable drops. This is clear from the expression for the critical charge. As the dimensions of the drop are reduced, the critical charge decreases in proportion to the square root of the volume, while the total charge on the drop decreases in proportion to the volume. It follows that the stability conditions are satisfied for sufficiently small drop radii.

203.

$$\varphi = -\frac{E_0 z}{\pi} \left(\tan^{-1} \frac{a}{\sqrt{\xi}} - \frac{a}{\sqrt{\xi}} \right) = -\frac{E_0}{\pi} \sqrt{-\eta} \left(\frac{\sqrt{\xi}}{a} \tan^{-1} \frac{a}{\sqrt{\xi}} - 1 \right),$$

where $\xi^{\frac{1}{2}}$ should be taken with the positive sign for $z > 0$ and the negative sign for $z < 0$. At large distances from the aperture

$\xi \simeq r^2$ and the field becomes

$$\varphi \simeq \frac{E_0 a^3 z}{3\pi r^3} \text{ when } z > 0.$$

This is the field of an electric dipole lying along the z -axis and having a moment $p = E_0 a^3/3\pi$. Hence it is clear that the lines of force passing through the aperture close on the other side of the metal screen.

$$204. \quad \sigma = -\frac{E_0}{4\pi^2} \left(\pi - \sin^{-1} \frac{a}{r_1} + \frac{a}{\sqrt{r_1^2 - a^2}} \right) \text{ when } z = -0,$$

$$\sigma = -\frac{E_0}{4\pi^2} \left(\frac{a}{\sqrt{r_1^2 - a^2}} - \sin^{-1} \frac{a}{r_1} \right) \text{ when } z = +0,$$

where $r_1 = (\xi + a^2)^{\frac{1}{2}}$ is the distance between the centre of the aperture and the point of observation on the plane.

205. The equation to be solved is $\Delta\varphi = -4\pi q\delta(\mathbf{r} - \mathbf{r}_0)$. In cylindrical coordinates the δ -function is given by

$$\delta(\mathbf{r} - \mathbf{r}_0) = \frac{1}{r_0} \delta(r - r_0) \delta(\alpha - \gamma) \delta(z).$$

The Fourier component of the potential $\varphi(r, \alpha, z)$

$$\varphi_k(r, \alpha) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \varphi(r, \alpha, z) \cos kz \, dz \quad (1)$$

satisfies the equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi_k}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi_k}{\partial \alpha^2} - k^2 \varphi_k = -\frac{4q}{r_0} \delta(r - r_0) \delta(\alpha - \gamma) \quad (2)$$

and the boundary conditions (*cf.* Fig. 11)

$$\varphi_k(r, 0) = \varphi_k(r, \beta) = 0, \quad (3)$$

$$\varphi_k(\infty, \alpha) = 0. \quad (4)$$

Consider the homogeneous equation corresponding to (2). The particular solutions satisfying (3) are the products $R_n(r) \sin(n\pi\alpha/\beta)$ ($n = 1, 2, 3, \dots$) where $R_n(r)$ is equal (apart from a constant factor) either to $I_{n\pi/\beta}(kr)$ or to $K_{n\pi/\beta}(kr)$. The solution of the non-homogeneous Eqn. (2) will be sought in the form of a superposition of these particular solutions:

$$\varphi_k = \begin{cases} \sum_{n=1}^{\infty} A_n I_{\frac{n\pi}{\beta}}(kr) \sin \frac{n\pi\alpha}{\beta} & \text{when } r < a, \\ \sum_{n=1}^{\infty} B_n K_{\frac{n\pi}{\beta}}(kr) \sin \frac{n\pi\alpha}{\beta} & \text{when } r > a. \end{cases} \quad (5)$$

This solution satisfies (4) and is bounded at $r = 0$ (cf. Appendix III). In order to determine the constants A_n and B_n we shall first use the fact that the potential is continuous at $r = r_0$. This gives

$$\frac{B_n}{A_n} = \frac{\frac{I_{n\pi}(kr_0)}{\beta}}{K_{\frac{n\pi}{\beta}}(kr_0)}. \quad (6)$$

Secondly, we shall require that the potential given by (5) should satisfy (2). Substitute Eqn. (5) into Eqn. (2), multiply both sides by $\sin(m\pi\alpha/\beta)$ ($m = 1, 2, \dots$) and integrate with respect to α between 0 and β . Since the functions $\sin(n\pi\alpha/\beta)$ are orthogonal in the above range, we have

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR_m}{dr} \right) - \left(k^2 + \frac{m^2\pi^2}{\beta^2 r^2} \right) R_m = -\frac{8q}{\beta r_0} \delta(r - r_0) \sin \frac{m\pi\gamma}{\beta}, \quad (7)$$

where

$$R_m(r) = \begin{cases} A_m \frac{I_{m\pi}(kr)}{\beta} & \text{when } r < a, \\ B_m K_{\frac{m\pi}{\beta}}(kr) & \text{when } r > a. \end{cases}$$

The function $R_m(r)$ is continuous at $r = r_0$, but its first derivative with respect to r has a discontinuity at that point and changes by

$$b \equiv R'_m(r_0 + 0) - R'_m(r_0 - 0) = k B_m K'_{\frac{m\pi}{\beta}}(kr_0) - k A_m I'_{m\pi}(kr_0).$$

The second derivative is $R''_m(r) = b\delta(r - r_0)$.

Substituting this expression into (7) and rejecting terms which are bounded at $r = r_0$ we obtain the second equation for A_n , B_n :

$$k B_n K'_{\frac{n\pi}{\beta}}(kr_0) - k A_n I'_{n\pi}(kr_0) = -\frac{8q}{\beta r_0} \sin \frac{n\pi\gamma}{\beta}. \quad (8)$$

In simplifying the expressions for A_n and B_n it is useful to employ the relation

$$K_\nu(x) I'_\nu(x) - K'_\nu(x) I_\nu(x) = \frac{1}{x}.$$

207.

$$\varphi(r, \alpha, z) = \frac{2q}{\pi} \left[\frac{1}{R_0} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \frac{\alpha - \gamma}{2}}{\cosh \frac{\eta}{2} - \cos \frac{\alpha - \gamma}{2}}} - \frac{1}{R_0} \tan^{-1} \sqrt{\frac{\cosh \frac{\eta}{2} + \cos \frac{\alpha + \gamma}{2}}{\cosh \frac{\eta}{2} - \cos \frac{\alpha + \gamma}{2}}} \right],$$

where

$$R_0 = \sqrt{r_0^2 + r^2 + z^2 - 2rr_0 \cos(\gamma - \alpha)} = \sqrt{2rr_0} \cdot \sqrt{\cosh \eta - \cos(\gamma - \alpha)},$$

$$R'_0 = \sqrt{r_0^2 + r^2 + z^2 - 2rr_0 \cos(\gamma + \alpha)} = \sqrt{2rr_0} \cdot \sqrt{\cosh \eta - \cos(\gamma + \alpha)}.$$

$$208. \quad \sigma = \text{const} \cdot r^{\left(\frac{\pi}{\beta} - 1\right)},$$

where r is the distance from the line of intersection. In the special case of two intersecting surfaces in the field of a point charge (cf. Problem 205), we have

$$-\frac{q\sqrt{\pi}r_0^{\frac{\pi}{\beta}} \sin \frac{\pi\gamma}{\beta}}{\beta^2(r_0^2 + z^2)^{\frac{\pi}{\beta} + \frac{1}{2}}} \cdot \frac{\Gamma\left(\frac{\pi}{\beta} + \frac{1}{2}\right)}{\Gamma\left(\frac{\pi}{\beta} + 1\right)} = \text{const}.$$

From this it is clear that $\sigma \rightarrow 0$ when $r \rightarrow 0$ and $\beta < \pi$; $\sigma \rightarrow \infty$ when $r \rightarrow 0$ and $\beta > \pi$. In the special case where the charge lies on the line of intersection, $\sigma \propto r^{-\frac{1}{2}}$.

209. Suppose that the charge q is at the origin and the z -axis is perpendicular to the surface of the plate. The equations of the front and rear surfaces will then be $z = a$ and $z = a + c$ respectively. The potential will be sought in the form

$$\left. \begin{aligned} \varphi_1 &= q \int_0^\infty J_0(kr_1) e^{-k|z|} dk + \int_0^\infty A_1(k) J_0(kr_1) e^{kz} dk \quad (-\infty < z < a), \\ \varphi_2 &= \int_0^\infty B_1(k) J_0(kr_1) e^{-kz} dk + \int_0^\infty B_2(k) J_0(kr_1) e^{kz} dk \quad (a < z < b), \\ \varphi_3 &= \int_0^\infty A_2(k) J_0(kr_1) e^{-kz} dk \quad (b < z < \infty, \text{ where } b = a + c). \end{aligned} \right\} \quad (1)$$

The boundary conditions on the surfaces of the plate are equivalent to four algebraic equations for the coefficients A_1 , A_2 , B_1 , and B_2 . On solving these equations we have

$$\begin{aligned} A_1 &= q\beta \frac{e^{-2kb} - e^{-2ka}}{1 - \beta^2 e^{-2kc}}, & A_2 &= q \frac{1 - \beta^2}{1 - \beta^2 e^{-2kc}}, \\ B_1 &= \frac{q(1 - \beta)}{1 - \beta^2 e^{-2kc}}, & B_2 &= \frac{q\beta(1 - \beta) e^{-2kb}}{1 - \beta^2 e^{-2kc}}, \end{aligned} \quad (2)$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$, $b = a + c$. At large distances from

the plate ($z > 0$), the field is of the form

$$\varphi(r_1, z) \simeq \frac{q}{\sqrt{r_1^2 + z^2}} + \frac{pz}{(r_1^2 + z^2)^{3/2}},$$

where

$$r_1 = \sqrt{x^2 + y^2}, \quad p = -\frac{(\epsilon - 1)^2}{2\epsilon} cq.$$

210.

$$\varphi(M) = \frac{2q}{\epsilon + 1} \int_0^\infty \frac{\sinh k(a - |z|)}{\cosh ka} J_0(kr_1) dk,$$

where $r_1 = (x^2 + y^2)^{1/2}$ (Fig. 46). When $(z^2 + r_1^2)^{1/2} \rightarrow 0$ (near the charge), $\varphi \rightarrow 2q/(\epsilon + 1)(r_1^2 + z^2)^{1/2}$ (cf. Problem 129). The potential φ may be written down in the form

$$\varphi = \frac{2q}{\epsilon + 1} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{r_1^2 + (z - 2an)^2}}.$$

The corresponding system of images is shown in Fig. 48b.

211. The use of bispherical coordinates will ensure that the surfaces of the inner and outer electrodes will have the equations $\xi = \xi_1$ and $\xi = \xi_2$ respectively. To achieve this, the z -axis must be drawn through the centres of the spheres as shown in Fig. 47.

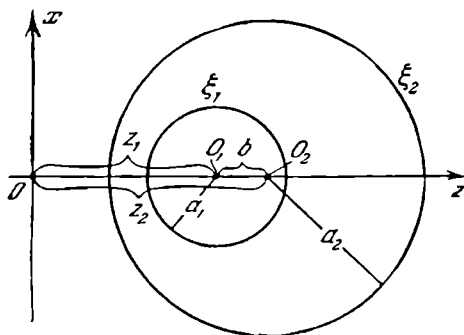


Fig. 47.

The coordinates of the centres will then be $z_1 = a \cotanh \xi_1$ and $z_2 = a \cotanh \xi_2$. The relations between the radii of the electrodes

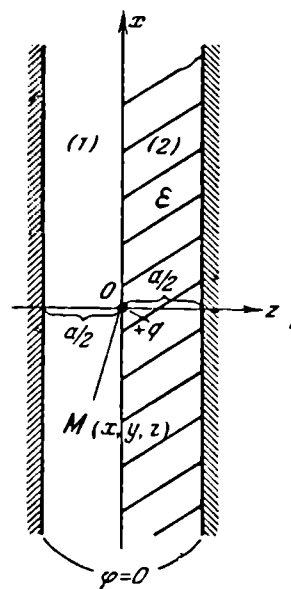


Fig. 46.

and the quantities a , ξ_1 , ξ_2 are $a = a_1 \sinh \xi_1$, $a = a_2 \sinh \xi_2$,
 $b = z_2 - z_1 = a (\coth \xi_2 - \coth \xi_1)$, and hence

$$\cosh \xi_1 = \frac{a_2^2 - a_1^2 - b^2}{2a_1 b}, \quad \cosh \xi_2 = \frac{a_2^2 + b^2 - a_1^2}{2a_2 b}. \quad (1)$$

In the space between the electrodes, the function ψ satisfies the equation

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{\sin \eta} \frac{\partial}{\partial \eta} \left(\sin \eta \frac{\partial \psi}{\partial \eta} \right) + \frac{1}{\sin^2 \eta} \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{1}{4} \psi = 0. \quad (2)$$

Separating the variables in (2) and recalling the fact that ψ is independent of the azimuthal angle α , we find the following special solution which is bounded at $\eta = 0, \pi$:

$$\psi_l(\xi, \eta) = \left[A_l \cosh \left(l + \frac{1}{2} \right) \xi + B_l \sinh \left(l + \frac{1}{2} \right) \xi \right] P_l(\cos \eta),$$

where $l = 0, 1, 2, 3, \dots$

The function ψ will be sought in the form of the series
 $\psi(\xi, \eta) = \sum_{l=0}^{\infty} \psi_l(\xi, \eta)$. The coefficients A_l and B_l will be determined from the boundary conditions $\psi(\xi_2, \eta) = 0$, $\psi(\xi_1, \eta) =$
 $= V (2 \cosh \xi_1 - 2 \cos \eta)^{-\frac{1}{2}} = V \sum_{l=0}^{\infty} \exp \left[- \left(l + \frac{1}{2} \right) \xi_1 \right] P_l(\cos \eta)$. The
 final result is

$$\varphi(\xi, \eta) = V \sqrt{2 \cosh \xi - 2 \cos \eta} \sum_{l=0}^{\infty} \frac{e^{-\left(l + \frac{1}{2} \right) \xi_1} \sinh \left(l + \frac{1}{2} \right) (\xi - \xi_2)}{\sinh \left(l + \frac{1}{2} \right) (\xi_1 - \xi_2)} P_l(\cos \eta). \quad (4)$$

The capacitance is given by

$$C = \frac{q_1}{V} = \frac{1}{4\pi V} \int_0^\pi \int_0^{2\pi} \frac{1}{h_\xi} \frac{\partial \varphi}{\partial \xi} h_\eta h_\alpha \bigg|_{\xi=\xi_1} d\eta d\alpha.$$

Substituting (4) into this expression, and remembering that Legendre polynomials are orthogonal, we have

$$C = \frac{a_1}{2} + a_1 \sinh \xi_1 \sum_{l=0}^{\infty} e^{-(2l+1)\xi_1 \coth \xi_1} \left(l + \frac{1}{2} \right) (\xi_1 - \xi_2).$$

213.

$$c_{11} = \frac{a_1}{2} + a_1 \sinh \xi_1 \sum_{l=0}^{\infty} e^{-(l+\frac{1}{2})\xi_1} \coth \left(l + \frac{1}{2}\right) (\xi_1 + \xi_2),$$

$$c_{22} = \frac{a_2}{2} + a_2 \sinh \xi_2 \sum_{l=0}^{\infty} e^{-(l+\frac{1}{2})\xi_2} \coth \left(l + \frac{1}{2}\right) (\xi_1 + \xi_2),$$

$$c_{12} = -a_1 \sinh \xi_1 \sum_{l=0}^{\infty} \frac{e^{-(l+\frac{1}{2})(\xi_1+\xi_2)}}{\sinh \left(l + \frac{1}{2}\right) (\xi_1 + \xi_2)},$$

where

$$\cosh \xi_1 = \frac{b^2 + a_1^2 - a_2^2}{2ba_1}, \quad \cosh \xi_2 = \frac{b^2 - a_1^2 + a_2^2}{2ba_2}.$$

The surfaces of the first and second conductor are given by $\xi = -\xi_1$ and $\xi = \xi_2$ respectively, where $a_1 \sinh \xi_1 = a_2 \sinh \xi_2$.

214.

$$c_{11} = a_1 (1 + mn + mn^3 + m^2 n^2),$$

$$c_{12} = -a_1 n (1 + mn),$$

$$c_{22} = a_2 (1 + mn + m^3 n + m^2 n^2),$$

where $m = a_1/b$ and $n = a_2/b$.

215. Suppose that the potential of the spheres is zero and that the potential at infinity is $-V$. Inversion of the system with respect to a sphere of radius $R = 2a$, whose centre lies at the point of contact of the conducting spheres, yields a plane parallel capacitor with the earthed electrodes at a distance $2R$ from each other. (In Fig. 48 the spheres of inversion are shown by the dashed curves.) The inner regions of the spheres are then found

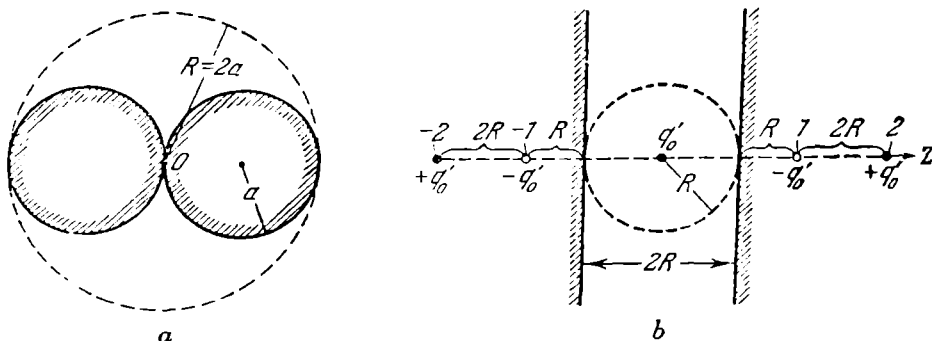


Fig. 48.

to correspond to the outer regions of the capacitor. The centre of inversion in the capacitor corresponds to an infinitely distant point of the original system. A point charge $q'_0 = -RV$ is, therefore, located at the centre of inversion. In the inverted system the field may be looked upon as being due to the following infinite system of images (*cf.* Problem 210, $\epsilon = 1$): point charges $(-1)^n q'_0$ which lie at the points $z'_n = 2Rn$ on the z' -axis. The latter passes through the centre of inversion, at right angles to the capacitor plates. Since we are interested in the capacitance, the total charge of the original system must now be determined:

$$q = 2 \sum_{n=1}^{\infty} q_n = 2 \sum_{n=1}^{\infty} \frac{q'_n R}{z'_n} = q'_0 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -q'_0 \ln 2 = RV \ln 2.$$

In summing up the series, we have made use of the well-known expansion for $\ln 2$. Hence, the capacitance is given by

$$C = \frac{q}{V} = 2a \ln 2.$$

In order to determine the potential from (III.32) and (III.33), consider a cylindrical coordinate system with the z -axis lying along the axis of symmetry of the system and the origin at the point of contact of the spheres. In these coordinates $z' = R^2 z / r^2$, $r'_1 = R^2 r_1 / r^2$, $r^2 = r_1^2 + z^2$ and the potential is given by

$$\varphi(r) = \frac{q}{C} - \frac{R^2 q}{Cr} \int_0^{\infty} \frac{\sinh k \left(R - \frac{R^2 |z|}{r^2} \right)}{\cosh kR} J_0 \left(\frac{k R^2 r_1}{r^2} \right) dk.$$

The term q/C is added in order to ensure that $\varphi(\mathbf{r})$ will vanish when $r \rightarrow \infty$.

217. The angle β at which the spherical surfaces intersect is given by $\beta = 2\pi - |\xi_2 - \xi_1|$ if ξ_1 and ξ_2 are of the same sign and by $\beta = 2\pi - |\xi_1 + \xi_2|$ if ξ_1 and ξ_2 are of different sign. If we choose the centre of inversion O on the line of intersection of the spheres, and let the radius of inversion be equal to $2a$, then the result of inversion will be a set of two planes intersecting at an angle β with the line of intersection (z' -axis) and perpendicular to the plane of symmetry ($\alpha = 0, \pi$) of the conductor under investigation. Fig. 49 illustrates the case $\xi_1 > 0$, $\xi_2 < 0$. After inversion, a charge $q'_0 = -2aV$ appears at O . It can easily be shown that $\gamma = \xi_1$ if γ is measured from the plane into which the spherical surface $\xi = \xi_1$ is transformed. Upon inversion, the

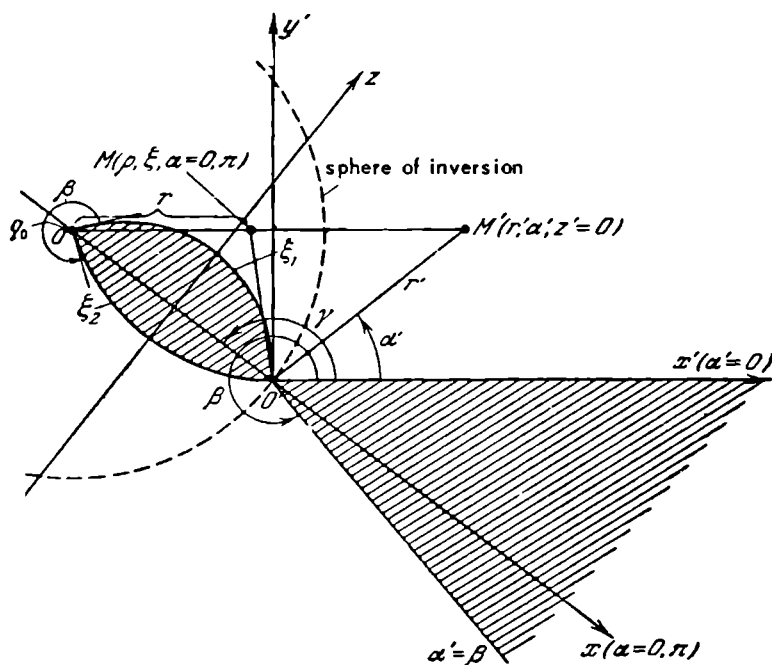


Fig. 49.

surfaces $\xi = \text{const}$ become the half-planes $\alpha' = \text{const}$, where

$$\xi = \begin{cases} \gamma - \alpha' & 0 \leq \alpha' \leq \pi + \gamma, \\ \gamma - \alpha' + 2\pi & \pi + \gamma < \alpha' < \beta \text{ (when } \beta > \pi + \gamma). \end{cases} \quad (1)$$

The distances r and r' may be expressed in terms of the coordinates ρ , ξ of the point of observation M :

$$r = \frac{2ae^{\frac{\rho}{2}}}{\sqrt{2(\cosh \rho - \cos \xi)}}, \quad r' = 2ae^{-\rho}. \quad (2)$$

The above equations are obtained by using the relations between cartesian and toroidal coordinates given in Problem 68, and the congruent triangles $OO'M'$ and $OO'M$. From the expression for the potential of a wedge given in Problem 206, and also Eqns. (1) and (2), the following expression for the capacitance is found:

$$C = \frac{q}{V} = \lim_{r \rightarrow \infty} \frac{r(\varphi + V)}{V} = \frac{a}{\pi} \int_0^\infty \frac{d\zeta}{\sinh \frac{\zeta}{2}} \left(\frac{\pi}{\beta} \cdot \frac{\sinh \frac{\pi\zeta}{\beta}}{\cosh \frac{\pi\zeta}{\beta} - \cos \frac{2\pi\gamma}{\beta}} - \right. \\ \left. - \frac{\pi}{\beta} \cdot \frac{\sinh \frac{\pi\zeta}{\beta}}{\cosh \frac{\pi\zeta}{\beta} - 1} + \frac{\sinh \zeta}{\cosh \zeta - 1} \right).$$

218.

$$\text{a) } C = \frac{R}{\pi} (\sin \theta + \theta),$$

$$\text{b) } C = 2R \left(1 - \frac{1}{\sqrt{3}} \right) \simeq \frac{11}{13} R,$$

where the integral is evaluated with the aid of the substitution
 $x = \exp(-\xi/2)$.

219.

$$C = \frac{a}{2} \left(5 - \frac{4}{\sqrt{3}} \right).$$

Chapter

4

STEADY CURRENTS

220. $\frac{i}{\mathcal{J}} = \frac{V_1 - V_2}{V_2 - \mathcal{E}}.$

221. The resistance of the galvanometer coil should be equal to the external resistance R .

222.

$$R = \frac{3}{2}r \quad \text{when } n = 2,$$

$$R = \frac{13}{7}r \quad \text{when } n = 3,$$

$$R = \frac{47}{22}r \quad \text{when } n = 4.$$

The number of circulating currents which must be used may be reduced by making use of the symmetry of the circuit. For example, for $n = 3$, the number of circulating currents may be reduced to three.

223. Consider the circulating currents indicated in Fig. 12. Kirchhoff's equation for the circuit $B_k A_k A_{k+1} B_{k+1}$ is

$$\mathcal{J}_{k-1} + \mathcal{J}_{k+1} = \left(2 + \frac{R}{r}\right) \mathcal{J}_k. \quad (1)$$

This second order linear difference equation has two linearly independent solutions, namely, $\exp(k\alpha)$ and $\exp(-k\alpha)$, where

$$\sinh \frac{\alpha}{2} = \frac{1}{2} \sqrt{\frac{R}{r}}. \quad (2)$$

The general solution of (1) is of the form $\mathcal{J}_k = A' \exp(k\alpha) + B' \exp(-k\alpha)$. It will be convenient to rearrange the terms and rewrite (1) in the form

$$\mathcal{J}_k = A \cosh(\beta - k)\alpha, \quad (3)$$

where A and β are constants. These constants can be determined from the boundary conditions at the ends of the line. The Kirchhoff equation for the last cell is

$$\mathcal{J}_n(R + R_a + r) - \mathcal{J}_{n-1}r = 0. \quad (4)$$

Substituting (3) into (4) and using (2), we obtain, after some rearrangement, the following expression for β :

$$\tanh \beta \alpha = \frac{R_a \cosh n \alpha + \sqrt{Rr} \sinh \left(n + \frac{1}{2} \right) \alpha}{R_a \sinh n \alpha + \sqrt{Rr} \cosh \left(n + \frac{1}{2} \right) \alpha}. \quad (5)$$

The constant A may be determined from Kirchhoff's equation for the first cell:

$$\mathcal{G}_0(R + R_l + r) - \mathcal{G}_1 r = \mathcal{E}. \quad (6)$$

After rearrangement the latter equation yields

$$A = \frac{\mathcal{E}}{R_l \cosh \beta \alpha + \sqrt{Rr} \sinh \left(\beta + \frac{1}{2} \right) \alpha}.$$

The final expression for the current in the section $A_k A_{k+1}$ is

$$\mathcal{G}_k = \frac{\mathcal{E} \cosh(\beta - k) \alpha}{R_l \cosh \beta \alpha + \sqrt{Rr} \sinh \left(\beta + \frac{1}{2} \right) \alpha}. \quad (7)$$

The constants α and β in this relation are given by Eqns. (2) and (5).

For dry insulation when $r \rightarrow \infty$, $\alpha \rightarrow 0$, Eqn. (7) assumes the expected form

$$\mathcal{G}_k = \frac{\mathcal{E}}{R_l + R_a + (n+1)R}. \quad (8)$$

The ratio of the e.m.f.'s ensuring the same current through the load for dry and poor insulation can be obtained from (7) and (8), and is given by

$$\frac{\mathcal{E}}{\mathcal{E}_0} = \frac{R_l \cosh \beta \alpha + \sqrt{Rr} \sinh \left(\beta + \frac{1}{2} \right) \alpha}{[R_l + R_a + (n+1)R] \cosh \left(\beta - n + \frac{1}{2} \right) \alpha}. \quad (9)$$

When the load resistance is zero ($R_a = 0$), we have from (5) the simple result

$$\beta = n + \frac{1}{2}. \quad (10)$$

224. Let $\mathcal{G}(x)$ and $\varphi(x)$ be the current and potential of the central conductor (relative to ground) at the point x . It follows that

$$\varphi(x) = -\rho' \frac{d\mathcal{G}}{dx}, \quad \mathcal{G} = -\rho \frac{d\varphi}{dx},$$

and

$$\frac{d^2\mathcal{G}}{dx^2} = \frac{\rho}{\rho'} \mathcal{G}.$$

225.

$$\mathcal{G}(x) = \frac{\mathcal{E} \cosh s(x - x_0)}{R_l \cosh s x_0 + \sqrt{\rho\rho'} \sinh s x_0}, \quad (1)$$

where $s = (\rho/\rho')^{\frac{1}{2}}$. The constant x_0 can be determined from

$$\tanh s(x_0 - a) = \frac{R_a}{\sqrt{\rho\rho'}}. \quad (2)$$

When $R_l = R_a = 0$, we have

$$\mathcal{G}(x) = \frac{\mathcal{E} \cosh s(x - a)}{\sqrt{\rho\rho'} \sinh s a}. \quad (3)$$

In the absence of leakage $\rho' \rightarrow \infty$, $x_0 \rightarrow a$, $s \rightarrow 0$, and the current along the cable is

$$\mathcal{G}_0 = \frac{\mathcal{E}}{R_l + \rho a + R_a}.$$

The following substitutions must be made when using Eqn. (7) of the solution of Problem 223:

$$R = \rho dx, \quad r = \frac{\rho'}{dx}, \quad k = \frac{x}{dx}, \quad n = \frac{a}{dx}.$$

It then follows from Eqn. (2) of the solution of Problem 223 that $\alpha = s dx$. The quantity β in that solution is related to x_0 by $\beta = x_0/dx$ so that $\beta\alpha = x_0 s$. Substitution of these expressions into Eqns. (5) and (7) of the solution of Problem 223 leads to Eqns. (1) and (2) given above.

226.

$$\begin{aligned} E_1 &= \frac{\tau_2 V}{\tau_1 h_2 + \tau_2 h_1}, & D_1 &= \frac{\varepsilon_1 \tau_2 V}{\tau_1 h_2 + \tau_2 h_1}, \\ E_2 &= \frac{\tau_1 V}{\tau_1 h_2 + \tau_2 h_1}, & D_2 &= \frac{\varepsilon_2 \tau_1 V}{\tau_1 h_2 + \tau_2 h_1}, \\ j_1 &= j_2 = \frac{\tau_1 \tau_2 V}{\tau_1 h_2 + \tau_2 h_1}. \end{aligned}$$

At the boundary of separation between the plates

$$\sigma_b = \frac{E_2 - E_1}{4\pi} - \sigma = \frac{x_2(\epsilon_1 - 1) - x_1(\epsilon_2 - 1)}{4\pi(x_1 h_2 + x_2 h_1)} V,$$

$$\sigma = \frac{D_2 - D_1}{4\pi} = \frac{(\epsilon_2 x_1 - \epsilon_1 x_2) V}{4\pi(x_1 h_2 + x_2 h_1)}.$$

The quantity V is greater than zero if the first plate is in contact with the positively charged electrode.

At the boundary between the electrode and the first plate

$$\sigma = \frac{D_1}{4\pi}, \quad \sigma_b = \frac{E_1 - D_1}{4\pi}.$$

At the boundary between the electrode and the second plate

$$\sigma = -\frac{D_2}{4\pi}, \quad \sigma_b = -\frac{E_2 - D_2}{4\pi}.$$

227.

$$\frac{\tan \beta_1}{\tan \beta_2} = \frac{x_1}{x_2},$$

where β_1 and β_2 are respectively the angles between the current flow-lines and the normal to the separation boundary in the first and second media.

228.

$$\varphi = \frac{\mathcal{E}z \ln \frac{r}{b}}{\pi a^2 x \ln \frac{a}{b}}.$$

It is clear from this formula that the electric field in the space between the conductors is not parallel to the z -axis. The absence of a finite radial component of the electric field E_r shows that the surface charges on the cylindrical surfaces of the conductors are given by

$$\sigma_1 = \frac{\epsilon E_r}{4\pi} \Big|_{r=a} = \frac{\epsilon \mathcal{E}z}{4\pi^2 a^3 x \ln \frac{a}{b}},$$

$$\sigma_2 = \frac{\epsilon E_r}{4\pi} \Big|_{r=b} = -\frac{\epsilon \mathcal{E}z}{4\pi^2 a^2 b x \ln \frac{a}{b}}.$$

When $z = 0$, the surface densities σ_1 and σ_2 become equal to zero. The position of the section on which $\sigma_1 = \sigma_2 = 0$ is

indeterminate. This section may be displaced if an additional constant charge is placed on the conductor. The relation between the charges $q_1 = 2\pi a\sigma_1$ and $q_2 = 2\pi b\sigma_2 = -q_1$ per unit length of the conductor and envelope (at the same value of z) and the potential difference between them

$$V = \int_a^b E_r dr = -\frac{\mathcal{J}z}{a^2\kappa}$$

is

$$\frac{q_1}{V} = \frac{1}{2\ln\frac{b}{a}} = \text{const.}$$

The ratio q_1/V is equal to the capacitance per unit length in the corresponding electrostatic problem (cylindrical capacitor). The magnetic field is, of course, of the same form as for an infinitely long straight conductor carrying a current $\mathcal{J}$.

$$229. \quad E_0 = -k(x_2l_1 + x_1l_2)\mathcal{E}_0, \quad E_1 = kx_2\mathcal{E}_0, \quad E_2 = kx_1\mathcal{E}_0.$$

where $k = x_0/l_0(x_0x_1l_2 + x_0x_2l_1 + x_1x_2l_0)$ and $\mathcal{E}_0 = E_e l_0$ is the e.m.f. of the source. Inside the source the electric field acts in the opposite direction to the direction of flow of the current ($E_0 < 0$).

The charges responsible for this electric field are located on the separation boundaries between conductors of different conductivities, and may be determined with the aid of the proper boundary conditions. For example, the charge on the boundary 01 is given by

$$q_{01} = \frac{r^2}{4}(E_1 - E_0).$$

230. Consider, for example, the energy flux through the surface of conductor 0. The magnetic field near the surface is the same as the field due to an infinitely long straight conductor ($H = 2\mathcal{J}/cr$). The Poynting vector $\boldsymbol{\gamma} = (c/4\pi)(\mathbf{E}_0 \times \mathbf{H})$ can be easily shown to be parallel to the outward normal to the surface of the conductor ($\mathbf{E}_0$ is the field strength in conductor 0 and its direction is opposite to the direction of the current; cf. Problem 229). The energy flux through the surface of this conductor is therefore equal to $2\pi r l_0 \gamma = \mathcal{J}V$, where $V = E_0 l_0$ is the potential difference between the ends of the conductor. The quantity $\mathcal{J}V$ is the difference between the work done by the e.m.f. and the

Joule losses per unit time in the source. The former is equal to $\mathcal{E}\mathcal{J}$ where $\mathcal{E} = E_e l_0$.

The energy $\mathcal{J}V$ leaves the source, flows in the surrounding space (mainly outside the conductors), and enters conductors 1 and 2 through their surface and is converted into Joule heat. The total amount of energy entering conductors 1 and 2 per unit time is $\mathcal{J}V_1$ and $\mathcal{J}V_2$ respectively. This may be easily proved with the aid of the Poynting vector, as indicated above.

231. The resistance is given by $R = \int_1^2 \frac{dl}{Sx}$ where dl is

parallel to the normal to the equipotential surface of area S . The limits 1 and 2 represent the bounding surfaces of the conductor.

$$232. \quad a) \quad R = \frac{1}{4\pi\kappa} \left(\frac{1}{a} - \frac{1}{b} \right);$$

$$b) \quad R = \frac{1}{4\pi\kappa_1} \left(\frac{1}{a} - \frac{1}{c} \right) + \frac{1}{4\pi\kappa_2} \left(\frac{1}{c} - \frac{1}{b} \right);$$

$$c) \quad R = \frac{1}{2\pi l\kappa} \ln \frac{b}{a}.$$

$$233. \quad R = \frac{1}{2\pi\kappa_2} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{2\pi\kappa_1} \cdot \frac{1}{b}.$$

$$234. \quad \mathcal{J}_k = \frac{4\pi\kappa}{\epsilon} q_k, \quad R_{lk} = \frac{\epsilon}{4\pi\kappa} s_{lk}.$$

$$235. \quad C = \frac{\epsilon}{4\pi\kappa R}.$$

$$236. \quad Q = \sum_{i,k} R_{ik} \mathcal{J}_i \mathcal{J}_k.$$

$$237. \quad R = \frac{\epsilon}{4\pi\kappa} (s_{11} - 2s_{12} + s_{22}) = \frac{\epsilon}{4\pi\kappa} \frac{c_{11} + 2c_{12} + c_{22}}{c_{12}^2 - c_{11}c_{22}}.$$

$$238. \quad R = \frac{V_1 - V_2}{\mathcal{J}} = R_1 + R_2 - \frac{1}{\pi\kappa l} \simeq R_1 + R_2,$$

where $R_1 = 1/2\pi\kappa a_1$ and $R_2 = 1/2\pi\kappa a_2$ are the resistances of the joined conductors (cf. Problem 233).

239. Let $e_0 = (1 - b^2/a^2)^{1/2}$ be the eccentricity of the ellipsoid of revolution, where b/a is the ratio of the semi-minor to semi-major axes. It follows that for an oblate ellipsoid

$$R = \frac{1}{\kappa} \left(\frac{4}{3\pi^2 V} \right)^{\frac{1}{3}} \frac{(1 - e_0^2)^{\frac{1}{6}}}{e_0} \cos^{-1} \sqrt{1 - e_0^2}$$

while for a prolate ellipsoid

$$R = \frac{1}{\pi} \cdot \frac{(1 - e_0^2)^{\frac{1}{3}}}{(6\pi^2 V)^{\frac{1}{3}} e_0} \ln \frac{1 + e_0}{1 - e_0}.$$

For a given volume V , the most efficient shape is either a very elongated or a very flattened conductor.

240. The current density in the space between the electrodes

$$j = \rho v \quad (1)$$

is independent of x [$v(x)$ is the particle velocity at the given point x]. The relation between the potential $\varphi(x)$ and the velocity is

$$v = \sqrt{-\frac{2e\varphi}{m}} \quad (2)$$

($\varphi = 0$ when $x = 0$).

It follows from (1) and (2) that $\rho = j(-m/2e\varphi)^{\frac{1}{2}}$, since the Poisson equation is

$$\frac{d^2\varphi}{dx^2} = -4\pi j \sqrt{-\frac{m}{2e\varphi}}. \quad (3)$$

Integration of Eqn. (3) subject to the boundary conditions

$d\varphi/dx|_{x=0} = 0$ and $\varphi|_{x=a} = \varphi_0$ yields

$$j = \frac{1}{9\pi a^2} \sqrt{\frac{2|e|}{m}} |\varphi_0|^{\frac{3}{2}},$$

which is the so-called three-halves law (Child-Langmuir law).

Chapter

5

MAGNETOSTATICS

$$241. \quad H_r = H_z = 0, \quad H_a = \begin{cases} \frac{2\mathcal{G}r}{ca^2} & \text{when } r < a, \\ \frac{2\mathcal{G}}{cr} & \text{when } a \leq r \leq b, \\ 0 & \text{when } r > b. \end{cases}$$

242. Let the z -axis lie along the axis of the cylinder so that the cartesian components of the vector potential $\mathbf{A}$ satisfy the following equation:

$$\Delta A_x = 0, \quad \Delta A_y = 0, \quad \Delta A_z = -\frac{4\pi\mu_0}{c} j_z, \quad (1)$$

where $j_z = 0$ when $r > a$ and $j_z = \mathcal{G}/\pi a^2$ when $r \leq a$.

Since the current $\mathcal{G}$ does not enter into the equations for A_x and A_y , these components may be put equal to zero. A_z will then depend only on the distance r from the z -axis. Integrating the equation for A_z and using the continuity conditions for A_z and H_a on the boundary $r = a$, and also the condition that H must be finite at $r = 0$, we have for $r < a$,

$$A_z = C - \frac{\mu_0 \mathcal{G}}{c} \left(\frac{r}{a} \right)^2, \quad B_a = \frac{2\mu_0 \mathcal{G}}{ca^2} r, \quad H_a = \frac{2\mathcal{G}}{ca^2} r. \quad (2)$$

When $r > a$

$$A_z = C - \frac{\mathcal{G}}{c} \left(\mu_0 + 2\mu \ln \frac{r}{a} \right), \quad B_a = \frac{2\mu \mathcal{G}}{cr}, \quad H_a = \frac{2\mathcal{G}}{cr}, \quad (2')$$

where C is an arbitrary constant.

243. When $r < a$

$$A_z = C_1, \quad \mathbf{B} = 0,$$

while when $a \leq r \leq b$

$$A_z = \frac{2\mu_0 \mathcal{G} a^2}{c(b^2 - a^2)} \left(\ln \frac{r}{a} - \frac{r^2}{2a^2} \right) + C_2, \quad B_a = \frac{2\mu_0 \mathcal{G}}{c(b^2 - a^2)} \left(r - \frac{a^2}{r} \right).$$

When $r > b$

$$A_z = \frac{2\mu\mathcal{I}}{c} \ln \frac{b}{r} + C_3, \quad B_a = \frac{2\mu\mathcal{I}}{cr}.$$

The remaining components of **A** and **B** are all equal to zero. Any two of the constants entering into the expression for A_z may be expressed in terms of the third constant by using the continuity of the vector potential at the boundaries.

$$\begin{aligned} 244. \quad H_x &= \frac{2\mathcal{I}}{ca} \left(\tan^{-1} \frac{a+2x}{2y} + \tan^{-1} \frac{a-2x}{2y} \right), \\ H_y &= \frac{\mathcal{I}}{ca} \ln \frac{\left(x - \frac{a}{2}\right)^2 + y^2}{\left(x + \frac{a}{2}\right)^2 + y^2}, \quad H_z = 0. \end{aligned}$$

The y -axis is perpendicular to the strip and passes through its axis.

245. The repulsive force on each of the plates is given by

$$f = \frac{4\mathcal{I}^2}{c^2 a^2} \left(a \tan^{-1} \frac{a}{b} - \frac{1}{2} b \ln \frac{a^2 + b^2}{b^2} \right).$$

$$\begin{aligned} 246. \quad A_z &= \frac{2\mathcal{I}}{c} \ln \frac{r_2}{r_1} = \frac{\mathcal{I}}{c} \ln \frac{(a+x)^2 + y^2}{(a-x)^2 + y^2}, \\ H_x &= \frac{\partial A_z}{\partial y} = -\frac{8\mathcal{I}}{c} \cdot \frac{axy}{r_1^2 r_2^2}, \\ H_y &= -\frac{\partial A_z}{\partial x} = -\frac{2\mathcal{I}}{c} \left(\frac{a-x}{r_1^2} + \frac{a+x}{r_2^2} \right), \end{aligned}$$

where the position of the two currents in the plane perpendicular to them is defined by the coordinates $(a, 0)$ and $(-a, 0)$ respectively; r_1 and r_2 are the distances from the points $(a, 0)$ and $(-a, 0)$ to the point of observation.

247. (a) In the space between the planes $H = 4\pi i/c$, in the remaining space $H = 0$; (b) in the space between the planes $H = 0$, in the remaining space $H = 4\pi i/c$. In both cases the magnetic field is perpendicular to the currents and parallel to the planes.

248. $H_y = 2\mathcal{I}d/c(b^2 - a^2)$, $H_x = H_z = 0$ where the y -axis is perpendicular to the plane containing the axes of the cylinders.

249. In the cylindrical system of coordinates whose z -axis is perpendicular to the plane of the ring and passes through its centre

$$A_a = \frac{2\mu\mathcal{I}}{c} \left(\frac{a}{r} \right)^{\frac{1}{2}} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right], \quad A_z = A_r = 0,$$

where $K(k)$ and $E(k)$ are the complete elliptic Legendre integrals and $k^2 = 4ar/[(a+r)^2 + z^2]$. The components of the magnetic field are

$$H_r = \frac{2\mathcal{G}}{c} \cdot \frac{z}{r\sqrt{(a+r)^2 + z^2}} \left[-K(k) + \frac{a^2 + r^2 + z^2}{(a-r)^2 + z^2} E(k) \right],$$

$$H_z = \frac{2\mathcal{G}}{c} \cdot \frac{1}{\sqrt{(a+r)^2 + z^2}} \left[K(k) + \frac{a^2 - r^2 - z^2}{(a-r)^2 + z^2} E(k) \right], \quad H_a = 0.$$

On the axis of the ring ($r = 0$) these expressions become

$$H_r = 0, \quad H_z = \frac{2\pi a^2 \mathcal{G}}{c(a^2 + z^2)^{\frac{3}{2}}}.$$

250. The flux of magnetic induction remains constant along any such tube. Hence the equation of the surface of the tube is

$$N = \int_S \mathbf{B} \cdot d\mathbf{S} = f(r, z) = \text{const.}$$

where the surface of integration S has a circular cross-section of radius r in the plane perpendicular to the axis of symmetry (the centre of the circle lies on the axis of symmetry). Since A_α is independent of α , Stokes' theorem gives

$$\int \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{l} = 2\pi r A_\alpha(r, z) = \text{const.}$$

The lines of intersection of these surfaces with the planes $\alpha = \text{const}$ are the required lines of magnetic induction.

251. The components of the magnetic field are given by

$$H_z = -\frac{\partial \psi}{\partial z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} H^{(2n)}(z) \left(\frac{r}{2}\right)^{2n} = H(z) - \frac{r^2}{4} H''(z) + \dots,$$

$$H_r = -\frac{\partial \psi}{\partial r} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)! n!} H^{(2n-1)}(z) \left(\frac{r}{2}\right)^{2n-1} = -\frac{r}{2} H'(z) + \dots,$$

$$H_a = 0.$$

The vector potential can be expressed in terms of the magnetic field with the aid of Stokes' theorem and the relation $\mathbf{H} = \text{curl } \mathbf{A}$. The result is

$$A_\alpha(r, z) = \frac{1}{r} \int_0^r H_z r' dr' = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n+1)!} H^{(2n)}(z) \left(\frac{r}{2}\right)^{2n+1} = \frac{r}{2} H(z) - \dots$$

252.

$$H_z = \frac{2\pi n\mathcal{I}}{c} (\cos \theta_1 + \cos \theta_2),$$

where (cf. Fig. 50):

$$\cos \theta_1 = \frac{h-z}{\sqrt{a^2 + (h-z)^2}}, \quad \cos \theta_2 = \frac{z}{\sqrt{a^2 + z^2}}.$$

253. The density of the surface current which is produced as a result of the rotation of the sphere is

$$\mathbf{i} = \frac{e\omega}{4\pi a} \sin \vartheta \mathbf{e}_\alpha,$$

where the polar axis lies along the vector ω . The vector potential at all points outside the surface of the sphere satisfies the Laplace equation. In view of the symmetry of the system, the vector potential may be chosen so that A_α is the only finite component. A_α will then be independent of α . It follows that the equation for the vector potential is

$$\Delta A_\alpha - \frac{1}{r^2 \sin^2 \vartheta} A_\alpha = 0 \quad (1)$$

(cf. solution of Problem 47). Since the current density is proportional to $\sin \vartheta$, it is natural to look for the solution of Eqn. (1) in the form

$$A_\alpha(r, \vartheta) = F(r) \sin \vartheta. \quad (2)$$

It will be seen below that $F(r)$ can be chosen so that it will

satisfy both the above equation and the boundary conditions. We note that the vector potential (2) satisfies the condition $\text{div } \mathbf{A} = 0$, which is necessary if (1) is to hold. A_α and $\mathbf{H} = \text{curl } \mathbf{A}$ can be obtained by determining $F(r)$ with the aid of Eqn. (1) and the boundary conditions. The magnetic field inside the sphere ($r < a$) is $\mathbf{H} = (2e/3c)\omega$. The field outside the sphere ($r > a$) is

$$\mathbf{H} = \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{m}}{r^3},$$

where $\mathbf{m} = (ea^2/3c)\omega$ is the magnetic moment of the system.

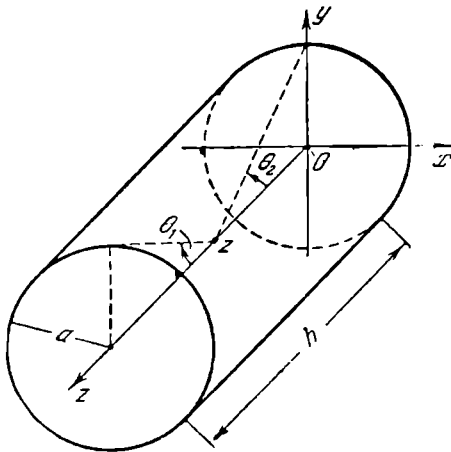


Fig. 50.

254. At all points where $j = 0$ it may be assumed that $\mathbf{H} = -\text{grad } \psi$. The equation $\text{curl } \mathbf{H} = 0$ is then satisfied for all ψ and the equation $\text{div } \mathbf{H} = 0$ yields $\Delta\psi = 0$. The latter equation may be solved subject to the additional condition

$$\int_l \mathbf{H} \cdot d\mathbf{l} = \frac{4\pi}{c} \mathcal{J},$$

where l is a contour drawn round the current $\mathcal{J}$. Consider the cylindrical coordinates r, α, z and suppose that the solution is of the form $\psi = \psi(\alpha)$. The final result is

$$\psi = -\frac{2\mathcal{J}}{c} \alpha, \quad H_\alpha = \frac{2\mathcal{J}}{cr}, \quad H_r = H_z = 0.$$

255. (a) In order to ensure that the scalar potential ψ of the magnetic field is a single-valued function, let us choose a surface S drawn on the linear current and assume that the change in ψ on passing through this surface is given by

$$\psi(2) - \psi(1) = \frac{4\pi}{c} \mathcal{J}. \quad (1)$$

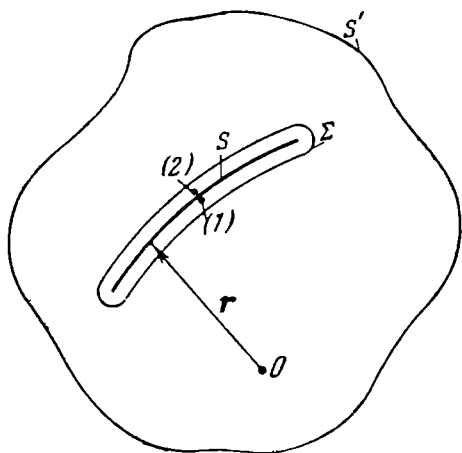


Fig. 51.

The points 1 and 2 lie infinitely near to each other while still remaining on opposite sides of the surface. The direction of the line drawn from 1 to 2 forms a right-handed system with the direction of the current (cf. Fig. 51). The solution of the

Laplace equation may be written in the form

$$\psi = \frac{1}{4\pi} \oint \left[\frac{1}{r} \cdot \frac{\partial \psi}{\partial n} - \psi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] dS, \quad (2)$$

where the integration may be carried out over an infinitely distant closed surface S' and over all closed surfaces Σ_i at a finite distance from the origin, within which ψ and $\partial\psi/\partial n$ have discontinuities. In the example under consideration, the integral over the infinitely distant surface will vanish because the field source (linear current) has finite dimensions. There are no surfaces across which the normal derivative $\partial\psi/\partial n = -H_n$ is discontinuous

since H_n is a continuous function. It follows that the integral in Eqn. (2) should be taken over a single surface Σ surrounding S .

Let us shrink Σ until it coincides with S . Since r^{-1} , $\frac{\partial \psi}{\partial n}$, and $\frac{\partial}{\partial n} \left(\frac{1}{r} \right)$ are all continuous across S , Eqn. (2) will assume the form

$$\psi = -\frac{1}{4\pi} \int [\psi(1) - \psi(2)] \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS, \quad (3)$$

where the integration is now carried out over the open surface S .

Using Eqn. (1) we have

$$\psi = \frac{\mathcal{J}}{c} \int \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS = -\frac{\mathcal{J}}{c} \int \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3}. \quad (4)$$

The integral $\int \mathbf{r} \cdot d\mathbf{S}/r^3$ represents the solid angle Ω subtended by the linear current at the point of observation, and hence Eqn. (4) may be rewritten in the form

$$\psi = -\frac{\mathcal{J}}{c} \Omega.$$

Ω is positive if the radius vector $\mathbf{r}$ drawn from the point of observation to any point on the surface S and the direction of the current form a right-handed system.

(b) Transform the line integral into a surface integral using the result of Problem 55. Thus,

$$\mathbf{A} = \frac{\mathcal{J}}{c} \int d\mathbf{S} \times \nabla \left(\frac{1}{r} \right) = \frac{\mathcal{J}}{c} \int \nabla_M \left(\frac{1}{r} \right) \times d\mathbf{S},$$

where ∇_M represents differentiation with respect to the coordinates of the point of observation M . Using the relation $\mathbf{H} = \text{curl } \mathbf{A}$ we have

$$\mathbf{H} = \frac{\mathcal{J}}{c} \int (d\mathbf{S} \cdot \nabla_M) \nabla_M \left(\frac{1}{r} \right) = \frac{\mathcal{J}}{c} \nabla_M \int d\mathbf{S} \cdot \nabla_M \left(\frac{1}{r} \right). \quad (5)$$

[In carrying out the transformation, use was made of the result $\Delta \left(\frac{1}{r} \right) = 0$, and it was assumed that the point $r = 0$ did not lie on the surface of integration.] Comparing (5) with $\mathbf{H} = -\text{grad } \psi$ we have

$$\psi = -\frac{\mathcal{J}}{c} \int d\mathbf{S} \cdot \nabla_M \left(\frac{1}{r} \right) = -\frac{\mathcal{J}}{c} \int \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = -\frac{\mathcal{J}}{c} \Omega.$$

256. $\mathbf{F} = 0$, $\mathbf{N} = \mathbf{m} \times \mathbf{H}$ where

$$\mathbf{m} = \frac{\mathcal{I}}{c} \int \mathbf{n} \cdot d\mathbf{S}$$

is the magnetic moment of the current-carrying conductor.

$$257. \quad U = \frac{\mathbf{m}_1 \cdot \mathbf{m}_2}{r^3} - \frac{3(\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \cdot \mathbf{r})}{r^5},$$

$$\begin{aligned} \mathbf{F}_2 = -\mathbf{F}_1 = \frac{3}{r^5} [(\mathbf{m}_1 \cdot \mathbf{r}) \mathbf{m}_2 + (\mathbf{m}_2 \cdot \mathbf{r}) \mathbf{m}_1 + (\mathbf{m}_1 \cdot \mathbf{m}_2) \mathbf{r}] - \\ - \frac{15}{r^7} (\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \cdot \mathbf{r}) \mathbf{r}, \end{aligned}$$

where $\mathbf{r}$ is the position vector of the second loop relative to the first, $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces on the two loops, and the couples acting on them are given by

$$\begin{aligned} \mathbf{N}_1 &= \frac{3(\mathbf{m}_2 \cdot \mathbf{r})(\mathbf{m}_1 \times \mathbf{r})}{r^5} + \frac{\mathbf{m}_2 \times \mathbf{m}_1}{r^3}, \\ \mathbf{N}_2 &= \frac{3(\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \times \mathbf{r})}{r^5} + \frac{\mathbf{m}_1 \times \mathbf{m}_2}{r^3}. \end{aligned}$$

It should be noted that $\mathbf{N}_1 \neq -\mathbf{N}_2$ and

$$\mathbf{N}_1 + \mathbf{N}_2 + (\mathbf{r} \times \mathbf{F}_2) = 0.$$

If the magnetic moments are parallel ($\mathbf{m}_1 = m_1 \mathbf{n}$, $\mathbf{m}_2 = m_2 \mathbf{n}$, $\mathbf{r} = r \mathbf{r}_0$ where $\mathbf{n}$ and $\mathbf{r}_0$ are unit vectors) then

$$\mathbf{F}_2 = \frac{3m_1 m_2 [2\mathbf{n} \cos \vartheta - \mathbf{r}_0 (5 \cos^2 \vartheta - 1)]}{r^4},$$

where ϑ is the angle between $\mathbf{n}$ and $\mathbf{r}_0$.

259. The potential function of the current $\mathcal{I}_2$ in the field of $\mathcal{I}_1$ is

$$u_{21} = \frac{2\mathcal{I}_1 \mathcal{I}_2}{c^2} \ln a + \text{const.},$$

where a is the distance between the currents. The force per unit length of the second current is given by

$$f = -\frac{\partial u_{21}}{\partial a} = -\frac{2\mathcal{I}_1 \mathcal{I}_2}{c^2 a}.$$

When the currents are of the same sign they attract each other.

260. The force $\mathbf{F}$ and the couple $\mathbf{N}$ can be obtained by differentiating the potential function

$$U(r, a) = -\frac{\mathcal{I}_1 \mathcal{I}_2 a}{c^2} \ln \frac{4r^2 + a^2 + 4ar \cos a}{4r^2 + a^2 - 4ar \cos a}.$$

$$261. \quad N = \frac{4\mathcal{G}\mathcal{I}'a}{c} (\sin \varphi - \varphi \cos \varphi).$$

$$262. \quad \mathcal{L} = \frac{1}{2} \mu_0 + 2\mu \ln \frac{b}{a}.$$

$$263. \quad \mathcal{L} = 2\mu \ln \frac{b}{a}.$$

$$264.$$

$$L_{12} = 4\pi (b - \sqrt{b^2 - a^2}); \quad F = \frac{\mathcal{I}_1 \mathcal{I}_2}{c^2} \cdot \frac{\partial L_{21}}{\partial b} = \frac{4\pi \mathcal{I}_1 \mathcal{I}_2}{c^2} \left(1 - \frac{b}{\sqrt{b^2 - a^2}}\right).$$

265. It is convenient to use Eqn. (V.23). As in the solution of Problem 89, we have

$$L_{12} = 4\pi \sqrt{ab} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right],$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}, \quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \psi} d\psi,$$

$$k^2 = \frac{4ab}{(a+b)^2 + l^2}.$$

When $l \gg a, b$ the parameter k is small so that

$$k^2 \simeq \frac{4ab}{l^2}, \quad k \simeq \frac{2\sqrt{ab}}{l} \ll 1.$$

It follows that the following approximate formulae for E and K can be used:

$$K(k) = \frac{\pi}{2} \left(1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 \right), \quad E(k) = \frac{\pi}{2} \left(1 - \frac{1}{4} k^2 - \frac{3}{64} k^4 \right).$$

Hence the first non-vanishing approximation for L_{12} is

$$L_{12} = \frac{2\pi^2 a^2 b^2}{l^3}.$$

The latter result can also be obtained from the equation $L_{12} = c\Phi_{12}/\mathcal{I}_1$ by regarding the two rings as magnetic dipoles.

266. In the notation of the preceding problem

$$F = \frac{4\pi \mathcal{I}_1 \mathcal{I}_2}{c^2} \cdot \frac{l}{\sqrt{(a+b)^2 + l^2}} \left[-K(k) + \frac{a^2 + b^2 + l^2}{(a+b)^2 + l^2} E(k) \right].$$

267. $\mathcal{L} = 4\pi n^2 S$. For a solenoid having a large though finite length h , the total self-inductance (neglecting edge effects) is $L = 4\pi n^2 S h$.

268. The magnetic energy is given by

$$W = \frac{1}{2c^2} \int \frac{\mathbf{i}_1 \cdot \mathbf{i}_2}{R} dS_1 dS_2,$$

where dS_1 and dS_2 are elements of the surface of the solenoid, R is the distance between them, $\mathbf{i}(i_1 = i_2 = i = n\mathcal{J})$ is the density of the surface current by which the actual current flowing in the solenoid may be replaced, and n is the number of turns per unit length. The integral can conveniently be evaluated in terms of cylindrical coordinates:

$$W = \frac{\pi n^2 a^2 \mathcal{J}^2}{c^2} \int_0^h dz_1 \int_0^h dz_2 \oint \frac{\cos \alpha \, d\alpha}{\sqrt{(z_1 - z_2)^2 + 4a^2 \sin^2 \frac{\alpha}{2}}} = \frac{2\pi^2 a^2 n^2 \mathcal{J}^2 h \left(1 - \frac{8a}{3\pi h}\right)}{c^2},$$

where terms of the order of $(a/h)^2$ or higher are neglected. Hence

$$L = 4\pi^2 a^2 n^2 h \left(1 - \frac{8a}{3\pi h}\right).$$

When $a/h \ll 1$, this result will be identical with that obtained in the preceding solution:

$$L = 4\pi^2 a^2 n^2 h = 4\pi n^2 S h.$$

269. For a circular cross-section

$$L = 4\pi N^2 (b - \sqrt{b^2 - a^2}).$$

The self-inductance per unit length of an infinite solenoid $\mathcal{L} = L/2\pi b$ may be obtained by letting $b \rightarrow \infty$ at a constant number of turns per unit length $n = N/2\pi b$ (cf. Problem 267):

$$\mathcal{L} = 4\pi^2 n^2 a^2 = 4\pi n^2 S.$$

For a rectangular cross-section

$$L = 2N^2 h \ln \frac{2b + a}{2b - a}.$$

When $b \gg a$, we have $\mathcal{L} = 4\pi n^2 S$ as before.

270. The magnetic energy per unit length of the system can be calculated with the aid of Eqn. (V.16). The vector potential for a straight conductor was obtained in the solution of Problem 242. For conductor 1 (Fig. 52) this potential will be written in the form

$$A_{1z} = C - \frac{\mathcal{J} r_1^2}{ca^2} \quad \text{when } r_1 < a, \quad (1)$$

$$A_{1z} = C - \frac{\mathcal{J}}{c} \left(1 + 2 \ln \frac{r_1}{a}\right) \quad \text{when } r_1 > a.$$

The vector potential due to conductor 2 is obtained by making the following substitutions in Eqn. (1): $\mathcal{J} \rightarrow -\mathcal{J}$, $a \rightarrow b$, $r_1 \rightarrow r_2$.

The magnetic energy is given by

$$W = \frac{\mathcal{J}}{2\pi ca^2} \int_1 (A_{1z} + A_{2z}) dS_1 - \frac{\mathcal{J}}{2\pi cb^2} \int_2 (A_{1z} + A_{2z}) dS_2. \quad (2)$$

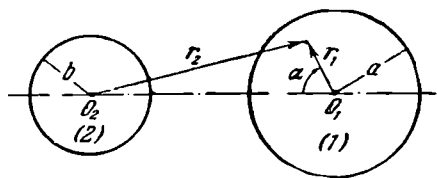


Fig. 52.

Recalling the relation between the inductance and magnetic energy, we find that

$$\mathcal{L} = 1 + 2 \ln \frac{h^2}{ab}.$$

271. The total magnetic energy of the current flowing in the conductor consists of two parts:

$$W = W_1 + W_2, \quad (1)$$

where

$$W_1 = \frac{\mu_0}{8\pi} \int H_1^2 dV$$

is the total energy stored inside the conductor (the integration is carried out over the volume of the conductor), and

$$W_2 = \frac{\mu}{8\pi} \int H_2^2 dV$$

is the energy stored in the remaining space. Let us suppose that we may introduce a parameter r_0 which has the dimensions of length and satisfies the condition

$$a \ll r_0 \ll R, \quad (2)$$

where a is the radius of the conductor and R is the radius of curvature of the axial line of the conductor (which in general will not be a constant). It follows that at distances smaller than r_0 the magnetic field can be considered as identical with that due to an infinitely long straight conductor. In particular, inside the conductor

$$H_1 = \frac{2\mathcal{J}r}{ca^2}$$

(cf. Problem 242). This may be used to find the "internal" energy

W_1 which is given by

$$W_1 = \frac{\mu_0 I^2}{4c^2}. \quad (3)$$

In order to determine the "external" energy W_2 , consider a surface S on an arbitrary contour lying on the surface of the conductor and introduce a scalar potential ψ . This potential will exhibit a discontinuity across S which is given by

$$\psi_+ - \psi_- = \frac{4\pi}{c} \mathcal{I}. \quad (4)$$

The integral in the expression for W_2 may be transformed as follows:

$$\int (\mathbf{B} \cdot \mathbf{H}) dV = - \int \mathbf{B} \operatorname{grad} \psi dV = - \int \operatorname{div} (\psi \mathbf{B}) dV = \oint \psi B_n dS,$$

where the subscript 2 has been omitted and $\operatorname{div} \mathbf{B} = 0$. In the final integral the integration should be carried out over both sides of the surface S and the surface of the conductor S' (cf. Fig. 53, which shows a cross-section of the conductor). The integral over an infinitely distant surface will be zero in view of the finite dimensions of the conductor. Thus

$$W_2 = -\frac{1}{8\pi} \int_{S'} \psi B_n dS + \frac{1}{8\pi} \int_S \psi_+ B_n dS - \frac{1}{8\pi} \int_S \psi_- B_n dS. \quad (5)$$

The first of these integrals is zero because in view of Eqn. (2) the magnetic field on S' is identical with the field due to a straight conductor and has, therefore, only a tangential component. The

remaining integrals may be transformed with the aid of Eqn. (4) and the fact that B_n is continuous. Thus

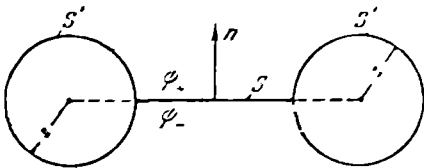


Fig. 53.

$$W_2 = \frac{\mathcal{I}}{2c} \int_S B_n dS. \quad (6)$$

At large distances from the conductor ($r > r_0$) the magnetic field is independent of the current distribution across the conductor and hence it may be assumed that the current flows along the axis. At small distances ($a \leq r < r_0$) the field is identical with the field due to an infinitely long circular cylinder, and again it may be

considered that the current flows along the axis. Thus, the integral (6) represents the flux of magnetic induction due to a current flowing along the axis of the conductor through a surface drawn on a closed contour lying on the surface of the conductor. Using (V.22) we have

$$W_2 = \frac{\mathcal{I}^2}{2c^2} L'. \quad (7)$$

Finally, using Eqns. (1), (3), and (7) and the relation between self-inductance and the magnetic energy, we obtain the required formula for the self-inductance:

$$L = \frac{\mu_0 l}{2} + L'. \quad (8)$$

272. Using the result of the preceding problem we have

$$L' = 4\pi\mu b \left(\ln \frac{8b}{a} - 2 \right),$$

where μ is the magnetic permeability of the medium in which the conductor is located. The total self-inductance is

$$L = 4\pi b \left(\mu \ln \frac{8b}{a} - 2\mu + \frac{1}{4}\mu_0 \right),$$

or

$$L = 4\pi b \left(\ln \frac{8b}{a} - \frac{7}{4} \right)$$

when $\mu_0 = \mu = 1$.

$$273. \quad L_{12} = 2l - 2\sqrt{a^2 + l^2} + 2a \ln \frac{a + \sqrt{a^2 + l^2}}{l}.$$

274. Using the result of Problem 273 we have

$$L_{12} = 8 \left[l - 2\sqrt{a^2 + l^2} + \sqrt{2a^2 + l^2} + a \ln \frac{a + \sqrt{a^2 + l^2}}{l} - a \ln \frac{a + \sqrt{2a^2 + l^2}}{\sqrt{a^2 + l^2}} \right],$$

$$F = \frac{8\mathcal{I}_1\mathcal{I}_2}{c^2} \left[\frac{a^2 + 2l^2}{l\sqrt{a^2 + l^2}} - \frac{l\sqrt{2a^2 + l^2}}{a^2 + l^2} - 1 \right].$$

$$275. \quad L = 2\mu_0 b + 8\mu b \left[\ln \frac{2b}{a(1 + \sqrt{2})} + \sqrt{2} - 2 \right].$$

276. Using Eqn. (V.13) in the integration with respect to the angles, and the relation $\overline{n_i n_k} = \frac{1}{3}\delta_{ik}$ (cf. Problem 32), we

have:

(1) for a uniform volume distribution of charge

$$\mathbf{m} = \frac{ea^2}{5c} \boldsymbol{\omega};$$

(2) for a uniform surface distribution of charge

$$\mathbf{m} = \frac{ea^2}{3c} \boldsymbol{\omega}.$$

For a sphere whose radius is equal to the classical radius of the electron (2.8×10^{-13} cm) and whose magnetic moment is equal to the magnetic moment of the electron as measured experimentally (0.9×10^{-20} erg gauss $^{-1}$), it turns out that the linear velocity at the equator of the sphere is $v = a\omega \simeq 10^{13}$ cm sec $^{-1}$ and is therefore greater than the velocity of light in a vacuum. This indicates that classical ideas cannot be used to describe electron spin.

278. For the secondary field, $\text{curl } \mathbf{H}' = 0$, *i.e.* it may be described by a scalar potential defined by $\mathbf{H}' = -\text{grad } \psi$. Hence the equation for the potential is the same as in electrostatics in the case of a non-uniform medium, *i.e.*

$$\text{div} (\mu \text{grad } \psi) = -4\pi\rho_m,$$

where

$$\rho_m = -\frac{1}{4\pi} \mathbf{H}_0 \cdot \text{grad } \mu$$

represents the density of magnetic charges. On the boundary of separation between two media the tangential components must satisfy the conditions

$$H'_{1\tau} = H'_{2\tau} \quad \text{or} \quad \frac{\partial \psi_1}{\partial \tau} = \frac{\partial \psi_2}{\partial \tau}$$

while the normal components must satisfy the conditions

$$\mu_2 H'_{2n} - \mu_1 H'_{1n} = (\mu_1 - \mu_2) H_{0n} \quad \text{or} \quad \mu_1 \frac{\partial \psi_1}{\partial n} - \mu_2 \frac{\partial \psi_2}{\partial n} = 4\pi\sigma_m.$$

In the above expression

$$\sigma_m = \frac{1}{4\pi} (\mu_1 - \mu_2) H_{0n}$$

is the surface density of magnetic charge. We note that the

expression for σ_m may be obtained from the volume density of charge by means of the following formula

$$\sigma_m = \lim_{h \rightarrow 0} \rho_m h.$$

Let us replace the separation boundary by a thin layer of thickness h . The vector $\text{grad } \mu$ will then lie along the normal to the layer and will be equal to $(\mu_2 - \mu_1)/h$, and hence

$$\rho_m = -\frac{1}{4\pi} \cdot \frac{\mu_2 - \mu_1}{h} H_{0n}, \quad \sigma_m = \lim_{h \rightarrow 0} \rho_m h = \frac{1}{4\pi} (\mu_1 - \mu_2) H_{0n}.$$

$$279. \quad \mathbf{H}_1 = \frac{\mu_2}{\mu_1 + \mu_2} \mathbf{H}_0, \quad \mathbf{H}_2 = \frac{\mu_1}{\mu_1 + \mu_2} \mathbf{H}_0,$$

where $\mathbf{H}_0$ is the field due to the circuit in a vacuum and $\mathbf{H}_1$, $\mathbf{H}_2$ are the fields in the media with permeabilities μ_1 , μ_2 .

280. The magnetic field in medium 1 is the same as the field produced in a vacuum by two straight currents

$$\mathcal{J}_1 = \mu \mathcal{J} \quad \text{and} \quad \mathcal{J}_2 = \frac{\mu_1 (\mu_2 - \mu_1)}{\mu_1 + \mu_2} \mathcal{J},$$

where $\mathcal{J}_1$ flows along the same conductor as the original current $\mathcal{J}$, while $\mathcal{J}_2$ flows along a conductor which is a mirror image of the first conductor with respect to the separation boundary between the two media. The magnetic field in medium 2 is identical with the field produced in a vacuum by a current

$$\mathcal{J}_1 = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} \mathcal{J}$$

which flows along the same conductor as the original current $\mathcal{J}$.

281. The field vectors satisfy the relations $\text{curl } \mathbf{H} = 0 = \text{div } \mathbf{B}$ in all space, and hence it is possible to introduce a scalar potential ψ which is defined by $\mathbf{H} = -\text{grad } \psi$ and satisfies the Laplace equation. This reduces the magnetostatic problem to the corresponding electrostatic problem. The solution inside the sphere is (cf. Problem 149)

$$\mathbf{H}_1 = \frac{3}{\mu + 2} \mathbf{H}_0.$$

Outside the sphere the solution is

$$\mathbf{H}_2 = \mathbf{H}_0 + \mathbf{H}_d,$$

where $\mathbf{H}_d$ is the field due to a magnetic dipole whose moment is

given by

$$\mathbf{m} = \frac{\mu - 1}{\mu + 2} a^3 \mathbf{H}_0.$$

Since the field inside the sphere is uniform, the magnetisation is constant and is given by

$$\mathbf{M} = \frac{\mathbf{m}}{\frac{4\pi}{3} a^3} = \frac{3(\mu - 1)}{4\pi(\mu + 2)} \mathbf{H}_0.$$

The equivalent volume current density is therefore zero, *i.e.*

$$\mathbf{j}_{\text{mol}} = c \operatorname{curl} \mathbf{M} = 0.$$

The surface current density is given by

$$\mathbf{i}_{\text{mol}} = c [\mathbf{n} \times (\mathbf{M}_2 - \mathbf{M}_1)].$$

This expression may be obtained from (V.3) by taking it to the limit (*cf.* the derivation of the boundary condition for the tangential component $\mathbf{H}_\tau$ from Maxwell's equations). Substituting $\mathbf{M}_2 = 0$ and $\mathbf{M}_1 = \mathbf{M}$ we have

$$\mathbf{i}_{\text{mol}} = \frac{3c(\mu - 1)}{4\pi(\mu + 2)} H_0 \sin \vartheta \mathbf{e}_\alpha.$$

It is interesting to note that the surface current may be obtained by rotating a sphere with a uniform surface charge about one of its diameters (*cf.* Problem 253).

282. Consider a coordinate system whose axes lie along the principal axes of the permeability tensor. The field components inside the sphere will then be equal to $[3/(\mu^{(k)} + 2)]\mathbf{H}_{0k}$ where $\mathbf{H}_0$ is the external field. Inside the sphere $\mathbf{H}_2 = \mathbf{H}_0 + \mathbf{H}_d$ where $\mathbf{H}_d$ is the field due to a magnetic dipole having a moment $\mathbf{m}$ where

$$m_k = \frac{\mu^{(k)} - 1}{\mu^{(k)} + 2} a^3 H_{0k}.$$

The couple acting on the sphere is $\mathbf{N} = \mathbf{m} \times \mathbf{H}_0$.

$$283. \quad H = \left[1 - \frac{1 - \left(\frac{a}{b}\right)^2}{\left(\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}\right)^2 - \left(\frac{a}{b}\right)^2} \right] H_0.$$

When $\mu_1 \gg \mu_2$ the field in the cavity is considerably reduced (magnetic screening).

$$284. \quad H = \left[1 - \frac{1 - \left(\frac{a}{b}\right)^3}{\frac{(\mu_1 + 2\mu_2)(2\mu_1 + \mu_2)}{2(\mu_1 - \mu_2)^2} - \left(\frac{a}{b}\right)^3} \right] H_0.$$

When $\mu_1 \gg \mu_2$ the field is considerably reduced ($H \ll H_0$).

285. The magnetic field is given by $\mathbf{H} = \text{curl } \mathbf{A}$ where

$$A_z = -\frac{\mu_1 \mathcal{I}}{4\pi a^2} r^2 + \frac{2\mu_1}{\mu_1 + \mu_2} B_0 r \sin \alpha \quad \text{when } r < a,$$

$$A_z = \frac{\mu_2 \mathcal{I}}{2\pi} \ln \frac{a}{r} + \left(1 + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{a^2}{r^2} \right) B_0 r \sin \alpha \quad \text{when } r > a.$$

The z -axis lies along the axis of the cylinder; the remaining components of $\mathbf{A}$ are zero.

$$288. \quad f = \frac{2\mathcal{I}^2 a^2 (\mu - 1)}{c^2 b (b^2 - a^2) (\mu + 1)}.$$

$$289. \quad f = \frac{2\mathcal{I}^2 b (\mu - 1)}{c^2 (a^2 - b^2) (\mu + 1)}.$$

$$290. \quad \mathbf{H}_l = \frac{1}{\mu_l} \cdot \frac{2\pi\mu_1\mu_2\mu_3}{\mu_1\mu_2a_3 + \mu_2\mu_3a_1 + \mu_1\mu_3a_2} \mathbf{H}_0,$$

where $\mathbf{H}_0$ is the field produced by the current in a vacuum.

291. Outside the sphere the magnetic induction $\mathbf{B}$ and the magnetic field $\mathbf{H}$ are related by the usual formula $\mathbf{B}_2 = \mu_2 \mathbf{H}_2$. According to (V.27), the magnetic induction inside the sphere is given by $\mathbf{B}_1 = \mu_1 \mathbf{H}_1 + 4\pi \mathbf{M}_0$ where $\mathbf{M}_0$ is the constant magnetisation. Using the scalar potential as in Problem 281, we have

$$\psi_1 = -\mathbf{H}_1 \cdot \mathbf{r}, \quad \psi_2 = \frac{\mathbf{m} \cdot \mathbf{r}}{r^3},$$

where

$$\mathbf{H}_1 = -\frac{4\pi \mathbf{M}_0}{2\mu_2 + \mu_1}, \quad \mathbf{m} = \frac{4\pi a^3 \mathbf{M}_0}{2\mu_2 + \mu_1}.$$

Thus the field inside the sphere is uniform while the field outside the sphere is the same as that due to a magnetic dipole of moment $\mathbf{m}$.

292. The field inside the cylinder is

$$\mathbf{H}_1 = -\frac{4\pi \mathbf{M}_0}{\mu_2 + \mu_1}.$$

The field outside the cylinder is

$$\mathbf{H}_2 = \frac{2\mathbf{r} (\mathbf{m} \cdot \mathbf{r})}{r^4} - \frac{\mathbf{m}}{r^2},$$

where $\mathbf{M}_0$ is the constant magnetisation and

$$\mathbf{m} = \frac{4\pi a^2 \mathbf{M}_0}{\mu_2 + \mu_1}.$$

293. The field inside the sphere is

$$\mathbf{H}_1 = \frac{3}{\mu + 2} \mathbf{H}_0 - \frac{4\pi \mathbf{M}_0}{\mu + 2}.$$

Outside the sphere the field is

$$\mathbf{H}_2 = \mathbf{H}_0 + \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{m}}{r^3}, \quad (1)$$

where

$$\mathbf{m} = \frac{4\pi a^2 \mathbf{M}_0}{\mu + 2} + \frac{\mu - 1}{\mu + 2} a^3 \mathbf{H}_0.$$

Since the external field is uniform, it follows that the resultant force on the sphere is zero. However, the sphere will experience a couple if the directions of $\mathbf{M}_0$ and $\mathbf{H}_0$ are different. This couple may be computed with the aid of the magnetic field stress tensor. The couple is given by

$$N_l = \oint_S e_{lkl} x_k T_{lm} dS_m, \quad (2)$$

where T_{lm} is the stress tensor given by (V.26) and e_{ikl} is the skew-symmetric unit tensor. The integration is carried out over the surface of the magnet. Substituting (V.26) into (2) we have (in vector notation)

$$\mathbf{N} = \frac{1}{4\pi} \oint (\mathbf{r} \times \mathbf{H}_2)(\mathbf{H}_2 \cdot d\mathbf{S}) - \frac{1}{8\pi} \oint H_2^2 (\mathbf{r} \times d\mathbf{S}). \quad (3)$$

Since the centre of this sphere is taken as the origin, it follows that $\mathbf{r}$ and $d\mathbf{S}$ are parallel and the second integral in (3) is equal to zero. In order to evaluate the first integral let $d\mathbf{S} = \mathbf{n}dS = \mathbf{n}a^2 d\Omega$, $\mathbf{r} = a\mathbf{n}$, and substitute for $\mathbf{H}_2$ from (1). This gives

$$\mathbf{N} = \frac{1}{4\pi} \int [a^3 (\mathbf{n} \times \mathbf{H}_0) + \mathbf{m} \times \mathbf{n}] \left(\mathbf{H}_0 \cdot \mathbf{n} + \frac{2}{a^3} \mathbf{m} \cdot \mathbf{n} \right) d\Omega. \quad (4)$$

The components of the couple are therefore given by

$$N_l = a^3 e_{lkl} H_{0l} H_{0m} \overline{n_k n_m} + 2e_{lkl} H_{0l} m_s \overline{n_k n_s} + e_{lkl} m_k H_{0m} \overline{n_l n_m} + \frac{2}{a^3} e_{lkl} m_k m_s \overline{n_l n_s}. \quad (5)$$

Using the relation $\overline{n_k n_m} = \frac{1}{3} \delta_{km}$ (cf. Problem 32) we find that two of the four terms on the right-hand side of (5) are equal to zero, while the remaining terms lead to the following expression for the couple

$$\mathbf{N} = \mathbf{m} \times \mathbf{H}_0.$$

Finally, when $\mathbf{m}$ is expressed in terms of the constant magnetisation we have

$$\mathbf{N} = \frac{4\pi a^3}{\mu + 2} \mathbf{M}_0 \times \mathbf{H}_0. \quad (7)$$

As can be seen from this formula, the induced part of the magnetic moment $[(\mu - 1)/(\mu + 2)] a^3 \mathbf{H}_0$ does not contribute to the resultant couple.

$$\begin{aligned} 294. \quad F &= \frac{3}{16} \cdot \frac{\mu - 1}{\mu + 2} \cdot \frac{m^2 (1 + \cos^2 \theta)}{a^4}, \\ N &= \frac{\mu - 1}{\mu + 2} \cdot \frac{m^2 \sin \theta \cos \theta}{8a^3}, \end{aligned}$$

where a is the distance of the magnet from the plane and θ is the angle between $\mathbf{m}$ and the normal to the plane. When $\mu \gg 1$ (soft iron in a weak magnetic field), the result becomes identical to that for an electric dipole in the vicinity of a metal surface (cf. Problem 148).

295. The required parameters may be obtained by replacing the electrical quantities by the corresponding magnetic quantities in the solution of Problem 201. In particular, for arbitrary co-ordinate axes, the internal field $\mathbf{H}_1$ in the ellipsoid is of the form

$$H_{1k} = H_{0k} - 4\pi N_{kl} M_l,$$

where $\mathbf{M}$ is the magnetisation and N_{kl} are the demagnetisation coefficients. The principal values of the demagnetisation tensor were denoted by $n^{(i)}$ in Problem 197 and were referred to as the depolarisation coefficients.

296. The formula obtained in the preceding solution will also hold for an anisotropic magnetic material. A further relation connecting $\mathbf{M}$ and $\mathbf{H}_1$ is

$$H_{1k} + 4\pi M_k = \mu_{kl} H_{1l}.$$

Combining these formulae we have

$$H_{0k} = b_{km} H_{1m},$$

where

$$b_{km} = \delta_{km} - N_{km} + N_{kl} \mu_{lm},$$

and hence

$$H_{ik} = b_{km}^{-1} H_{0m},$$

where b_{km}^{-1} are the components of the reciprocal tensor. They may be determined with the aid of the formulae obtained in Problem 11.

Consider a special case. Suppose that the coordinate axes lie along the principal axes of the ellipsoid. If the tensor μ_{ik} referred to these axes has the diagonal form

$$\mu_{ik} = \begin{pmatrix} \mu^{(x)} & 0 & 0 \\ 0 & \mu^{(y)} & 0 \\ 0 & 0 & \mu^{(z)} \end{pmatrix},$$

then the tensor b_{ik} and its reciprocal b_{ik}^{-1} will also be diagonal:

$$b_{ik}^{-1} = \begin{pmatrix} [1 + N^{(x)}(\mu^{(x)} - 1)]^{-1} & 0 & 0 \\ 0 & [1 + N^{(y)}(\mu^{(y)} - 1)]^{-1} & 0 \\ 0 & 0 & [1 + N^{(z)}(\mu^{(z)} - 1)]^{-1} \end{pmatrix}.$$

Chapter

6

ELECTRICAL AND MAGNETIC PROPERTIES OF MATTER

1. Polarisation of matter in a constant field

297. $\beta = 3a_0^3/4$. If the charge of the electron is distributed uniformly within a sphere of radius a_0 then $\beta = a_0^3$. The model considered in this problem is very approximate and yields only the order of magnitude. Accurate quantum mechanical calculations show that for hydrogen $\beta = 9a_0^3/2$.

299. It is clear from the symmetry of the molecule that one of the principal axes of the polarisability tensor will lie along the axis of the molecule, while the two other axes may be chosen arbitrarily in the plane perpendicular to the axis of the molecule. It follows that of the three principal values of the polarisability tensor, only two will be distinct. In order to determine these principal values, it is necessary to consider the following situations separately.

(a) The external field is parallel to the axis of the molecule. It is clear that the induced dipole moment of each of the atoms will be parallel to the external field. The dipole moments are given by

$$\mathbf{p}' = \beta' (\mathbf{E} + \mathbf{E}'), \quad \mathbf{p}'' = \beta'' (\mathbf{E} + \mathbf{E}''), \quad (1)$$

where $\mathbf{E}$ is the external field and $\mathbf{E}'$ and $\mathbf{E}''$ are the additional fields produced at the centre of each atom by the other atom. The fields $\mathbf{E}'$ and $\mathbf{E}''$ may be expressed in terms of the dipole moments of the corresponding atoms by means of the formula for the field strength due to a dipole having a moment $\mathbf{p}$ and the fact that all the vectors are parallel to the axis of the molecule. If p' and p'' are determined from (1), we have

$$\beta^{(1)} = \frac{1}{\frac{1}{\beta'} - \frac{2(a^3 + 2\beta')}{a^3(a^3 + 2\beta')}} + \frac{1}{\frac{1}{\beta''} - \frac{2(a^3 + 2\beta'')}{a^3(a^3 + 2\beta'')}},$$

where $p = p' + p'' = \beta^{(1)}E$.

(b) The external field is perpendicular to the axis of the molecule. Using a similar method, we have

$$\beta^{(2)} = \beta^{(3)} = \frac{1}{\frac{1}{\beta'} + \frac{a^3 - \beta'}{a^3(a^3 - \beta')}} + \frac{1}{\frac{1}{\beta''} + \frac{a^3 - \beta''}{a^3(a^3 - \beta')}}.$$

When $\beta' = \beta''$ the above expressions assume the simpler form

$$\beta^{(1)} = \frac{2\beta'}{1 - \frac{2\beta'}{a^3}}, \quad \beta^{(2)} = \frac{2\beta'}{1 + \frac{\beta'}{a^3}}.$$

The average polarisability is given by

$$\bar{\beta} = \frac{1}{3} (\beta^{(1)} + 2\beta^{(2)}) = \frac{2}{3} \beta' \left(\frac{1}{1 - \frac{2\beta'}{a^3}} + \frac{2}{1 + \frac{\beta'}{a^3}} \right).$$

301. (a) The dielectric as a whole will be anisotropic. The principal values of the polarisability tensor are [cf. (VI.4')]

$$\alpha^{(i)} = \frac{N\beta^{(i)}}{1 - \frac{4\pi}{3} N\beta^{(i)}}.$$

(b) When the molecules are randomly oriented there will be no preferred macroscopic directions other than the direction of the external field. Hence, the average dipole moment of a molecule $\bar{\mathbf{p}}$ will be proportional to the field $\mathcal{E}$ acting on the molecule:

$$\bar{\mathbf{p}} = \beta \mathcal{E}.$$

On the other hand,

$$\bar{p}_i = \overline{\beta_{ik} \mathcal{E}_k} = \bar{\beta}_{ik} \mathcal{E}_k,$$

where the averaging process is carried out over a macroscopically small volume. It follows from the last two expressions that

$$\beta = \bar{\beta}_{11} = \bar{\beta}_{22} = \bar{\beta}_{33}, \quad \bar{\beta}_{ik} = 0 \quad (\text{when } i \neq k),$$

and hence

$$\beta = \frac{1}{3} (\bar{\beta}_{11} + \bar{\beta}_{22} + \bar{\beta}_{33}).$$

However, the sum of the diagonal components of the tensor is invariant and is equal to the sum of the principal values

$\beta^{(1)} + \beta^{(2)} + \beta^{(3)}$ (cf. Problem 9). Hence,

$$\beta = \frac{1}{3} (\beta^{(1)} + \beta^{(2)} + \beta^{(3)}).$$

The polarisation coefficient α of the dielectric is related to β by the usual formula (VI.4').

302. If the axis of the molecule is at an angle θ to the direction of the external field, $\mathbf{E}_0$, then the energy of the molecule is given by

$$W = -\frac{1}{2} \mathbf{p} \cdot \mathbf{E}_0 = -\frac{1}{2} (\beta_1 \cos^2 \theta + \beta_2 \sin^2 \theta) E_0^2.$$

The number of particles per unit volume whose axes lie at the angle θ to $\mathbf{E}_0$ is given by the Boltzmann formula (VI.6). In the normalisation condition (VI.7), the quantity N should then represent the number of particles per unit volume. The polarisation vector is given by $\mathbf{P} = N\bar{\mathbf{p}}$ where $\bar{\mathbf{p}}$ is the dipole moment of a molecule averaged over the Boltzmann distribution. Since, in the absence of the field, the molecules are randomly oriented, $\bar{\mathbf{p}}$ will be parallel to the direction of the external field.

In view of the above considerations, $\bar{p}$ may be calculated from the formula

$$\bar{p} = \frac{1}{N} \int p_{\parallel} dN = \frac{E_0 \int_0^{\pi} \exp\left(-\frac{W(\theta)}{kT}\right) (\beta_1 \cos^2 \theta + \beta_2 \sin^2 \theta) \sin \theta d\theta}{\int_0^{\pi} \exp\left(-\frac{W(\theta)}{kT}\right) \sin \theta d\theta},$$

where $p_{\parallel}$ is the component of the dipole moment of the molecule which is parallel to the field. Since by definition the field is weak, it is sufficient to retain only those terms which are linear in $a = [(\beta_1 - \beta_2)E_0^2/2kT] \ll 1$. Moreover, $P = N\bar{p} = \alpha E_0$, and hence

$$\alpha = N\beta_2 + \frac{1}{3} N(\beta_1 - \beta_2) \left[1 + \frac{2}{15} \frac{(\beta_1 - \beta_2) E_0^2}{kT} \right].$$

As can be seen from this formula, the relation between P and E_0 is not linear and α is a function of E_0 . Let us estimate the magnitude of the correction term at ordinary temperatures ($T = 300^\circ\text{K}$). Assuming that $\beta_1 - \beta_2$ is of the order of 10^{-24} cm^3 , we have $kT/(\beta_1 - \beta_2) \simeq 10^6$. Thus, the correction term is small

provided $E_0 \ll 1000 \text{ V cm}^{-1}$. Neglecting the correction term, we have the same expression for α as before:

$$\alpha = \frac{1}{3} N (\beta_1 + 2\beta_2)$$

(cf. Problem 301).

305. The additional potential due to the quadrupole polarisation of the dielectric is given by

$$\varphi = \frac{1}{2} \int \frac{\partial^2 (1/R)}{\partial x_i \partial x_k} Q_{ik} dV, \quad (1)$$

where R is the distance of the point of observation from the volume element dV and the integration is carried out over the volume of the dielectric. On the other hand, the volume and surface charges will in general give rise to a potential

$$\varphi = \int \frac{\rho'}{R} dV + \int \frac{\sigma'}{R} dS + \int \boldsymbol{\tau}' \cdot \nabla \left(\frac{1}{R} \right) dS, \quad (2)$$

where ρ' is the volume density of charges, σ' is the surface density of charges, and $\boldsymbol{\tau}'$ is the strength of the double layer. Comparison of (1) with (2) yields

$$\rho' = \frac{1}{2} \cdot \frac{\partial^2 Q_{ik}}{\partial x_i \partial x_k}, \quad \sigma' = -\frac{1}{2} \cdot \frac{\partial Q_{in}}{\partial x_i}, \quad \tau'_k = \frac{1}{2} Q_{ki} n_i. \quad (3)$$

Thus, the quadrupole polarisation is equivalent to a volume density ρ' inside the dielectric, a surface density σ' , and an electric double layer of strength $\boldsymbol{\tau}'$ on the surface of the dielectric. Since the volume and surface densities in the dielectric are related to the polarisation vector by the formulae $\rho' = -\text{div } \mathbf{P}'$, $\sigma' = P'_n$, it follows from Eqn. (3) that the quadrupole polarisation is equivalent to an additional dipole polarisation given by

$$P'_k = -\frac{1}{2} \cdot \frac{\partial Q_{ik}}{\partial x_i}$$

and a double layer of strength τ'_k .

The formulae given by (3) can also be obtained from a consideration of the energy of the dielectric which is due to the quadrupole polarisation.

306.

$$\epsilon = \frac{1}{4} \left[1 + 3x + 3 \left(1 + \frac{2}{3}x + x^2 \right)^{\frac{1}{2}} \right],$$

where $x = 4\pi N\beta$. The polarisability β for polar substances in weak fields is given by $\beta = p^2/3kT$, where p is the molecular dipole moment, k is the Boltzmann constant, and T is the temperature. When $x \ll 1$ the difference between the field acting on the molecule and the average field is very small and $\epsilon = 1 + x = 1 + 4\pi N\beta$.

307. The total magnetic susceptibility is equal to the sum of the paramagnetic and diamagnetic susceptibilities

$$\chi = \frac{Nm^2}{3kT} - \frac{Ne^2}{6mc^2} \bar{r}^2. \quad (1)$$

This formula includes the magnetic moment of a rotator m which may be calculated as follows. In general,

$$m = \frac{e}{2mc} K, \quad (2)$$

where K is the angular momentum of the particle. In the case of the rotator, K is related to the kinetic energy by the formula

$$W_k = \frac{K^2}{2ma^2}. \quad (3)$$

It follows that the average angular momentum is given by

$$\bar{K}^2 = 2ma^2 \bar{W}_k. \quad (4)$$

However, the average kinetic energy $\bar{W}_k$ may be found from the theorem of equipartition of energy. Since the rotator has two degrees of freedom, $\bar{W}_k = kT$. Substituting (4) and (2) into (1) we find that $\chi = 0$. This result is in accordance with the general theorem which states that the total magnetic moment of a body obeying classical statistics is zero. A finite magnetic moment will be obtained only when it is assumed that there are discrete electronic orbits in atoms. This assumption lies outside the framework of classical theories.

308. The concentration of ions N and electrons n is given by the Boltzmann formula (VI.6):

$$N = N_0 e^{-\frac{Ze\varphi}{kT}}, \quad n = n_0 e^{\frac{e\varphi}{kT}}, \quad (1)$$

where $\varphi(x, y, z)$ is the electrostatic potential. The pre-exponential factors are chosen so that as $T \rightarrow \infty$, when the interaction between the particles becomes negligible, N and n become

equal to N_0 and n_0 respectively. Using (1), the charge density is given by

$$\rho = ZeN - en = e \left(ZN_0 e^{-\frac{Ze\varphi}{kT}} - n_0 e^{\frac{e\varphi}{kT}} \right). \quad (2)$$

The potential φ can be determined by solving the Poisson equation

$$\Delta\varphi = -4\pi\rho = -4\pi e \left(ZN_0 e^{-\frac{Ze\varphi}{kT}} - n_0 e^{\frac{e\varphi}{kT}} \right). \quad (3)$$

In order to solve this equation, we shall assume that the interaction energy is small in comparison with thermal energy, so that

$$\left| \frac{Ze\varphi}{kT} \right| \ll 1, \quad \left| \frac{e\varphi}{kT} \right| \ll 1.$$

Expanding the exponentials into series, and retaining only those terms which are linear in φ , we have

$$\rho = -\frac{\kappa^2}{4\pi} \varphi, \quad \kappa^2 = \frac{4\pi e^2 (Z^2 N_0 + n_0)}{kT}, \quad (4)$$

where the gas is assumed to be electrically neutral, so that $ZN_0 = n_0$. Hence, Eqn. (3) may be written in the form

$$\Delta\varphi = \kappa^2 \varphi. \quad (5)$$

The potential φ can only be a function of the distance r to the ion under consideration. The spherically symmetric solution of (5) is of the form

$$\varphi = C_1 \frac{e^{-\kappa r}}{r} + C_2 \frac{e^{\kappa r}}{r}.$$

$C_2 = 0$ because the potential must be finite at infinity. C_1 is determined from the condition that when $r \ll \kappa^{-1}$, the potential should become identical with the Coulomb potential of the ion under consideration:

$$\varphi|_{r \ll 1/\kappa} = \frac{Ze}{r} = \frac{C_1}{r}, \quad C_1 = Ze.$$

Thus, the ion is surrounded by a cloud of electrons and other ions whose density decreases exponentially and whose mean radius κ^{-1} decreases with decreasing temperature.

The above method of calculating the potential is due to Debye and Hückel and has been used in their theory of strong electrolytes.

The constant κ^{-1} is known as the Debye-Hückel radius.

309. The electric induction inside the plate is given by

$$D(x) = E_0 \frac{\cosh \kappa x}{\cosh \kappa h},$$

where $\kappa = (4\pi e^2 n_0 / \epsilon kT)^{\frac{1}{2}}$. When $\kappa h \gg 1$, near the surfaces $x = \pm h$

$$D(x) = E_0 e^{-\kappa(h-|x|)},$$

and hence it follows that when $|x - h| \gg \kappa^{-1}$, $D(x) = 0$, *i.e.* the field penetrates into the conductor to a depth κ^{-1} . A charge

$$\rho = \frac{1}{4\pi} \cdot \frac{\partial D}{\partial x} = \pm \frac{\kappa E_0}{4\pi} e^{-\kappa(h-|x|)}$$

is concentrated in a layer of this thickness. The density of the "surface" charge which is considered in the macroscopic theory is obtained by integrating ρ . At $x = h$ we have

$$\sigma = \int \rho dx = -\frac{\kappa E_0}{4\pi} \int_0^\infty e^{-\kappa x'} dx' = \frac{E_0}{4\pi}$$

which is the same as the usual boundary condition at the surface of a conductor.

310.

$$\varphi = \varphi_0 \frac{\sinh \kappa x}{\sinh \kappa h}, \quad \kappa = \sqrt{\frac{8\pi e^2 n_0}{\epsilon kT}}.$$

The magnitude of κ^2 is found to be twice as large as in the preceding problem, since there are two kinds of moving ions.

2. Polarisation of matter in a variable field

311.

$$\epsilon = 1 + 4\pi N a^3; \quad \mu = 1 - 2\pi N a^3 < 1.$$

This type of dielectric is diamagnetic. The permittivity ϵ and the permeability μ are independent of frequency, since it is assumed that the spheres are perfect conductors. In order that the artificial dielectric should behave as a continuous medium, the wavelength must be much greater than both the average distance between the spheres and the radius of each sphere. The difference between the effective field and the average field can only be neglected when the polarisability of the medium is small, *i.e.* when $4\pi N a^3 \ll 1$.

312. The equation of motion for an electron is of the form

$$m\ddot{\mathbf{r}} + \eta\dot{\mathbf{r}} = e\mathbf{E}_0 e^{-i\omega t}. \quad (1)$$

The special solution of this equation corresponding to forced oscillations is of the form

$$\mathbf{r} = - \frac{e\mathbf{E}_0 e^{-i\omega t}}{m(\omega^2 + i\gamma\omega)},$$

where $\gamma = \eta/m$.

The dipole moment per unit volume is obtained by multiplying $\mathbf{r}$ by the charge of the electron e and the number of particles per unit volume N . The polarisability of the medium $\alpha(\omega)$ and the permittivity $\epsilon(\omega)$ can then be determined from

$$\epsilon(\omega) = 1 + 4\pi\alpha(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}, \quad \omega_p^2 = \frac{4\pi e^2 N}{m}. \quad (2)$$

The relation between the resistivity ρ and the coefficient η may be found with the aid of Eqn. (1) and Ohm's law:

$$\rho \equiv \frac{1}{\sigma} = \frac{\eta}{Ne^2}. \quad (3)$$

This result can also be obtained by comparing the permittivity (2) with the complex permittivity (VIII.8) expressed in terms of the conductivity:

$$\epsilon(\omega) = \epsilon' + i \frac{4\pi\sigma}{\omega}. \quad (4)$$

Separating the real and imaginary parts in (2) we find that

$$\epsilon' = 1 - \frac{\omega_p^2}{\omega^2 + \gamma^2}, \quad \sigma = \frac{e^2 N \gamma}{m(\omega^2 + \gamma^2)}. \quad (5)$$

It follows from (5) that ϵ' and σ are functions of frequency. When $\omega \ll \gamma$ they assume their static values

$$\epsilon' = 1 - \frac{\omega_p^2}{\gamma^2} < 1, \quad \sigma = \frac{e^2 N}{m\gamma}.$$

It follows from (4) and (5) that the complex permittivity of a conducting medium becomes infinite at low frequencies ($\omega \rightarrow 0$). At high frequencies it is of the form

$$\epsilon(\omega) = \epsilon'(\omega) = 1 - \frac{\omega_p^2}{\omega^2}.$$

This dependence of the permittivity on frequency at high frequencies will also hold for dielectrics.

Let us estimate the order of magnitude of $\gamma = \eta/m$ for copper for which the static conductivity is $\sigma = 5 \times 10^{17} \text{ sec}^{-1}$. Eqn. (3) gives

$$\gamma = \frac{Ne^2}{\sigma m} = \frac{N_0 e^2 d}{\sigma m A},$$

where $N_0 \simeq 6 \times 10^{23}$ (Avogadro's number), $A \simeq 63.5$ (atomic weight), and $d \simeq 8.9$ (density of copper). It follows that $\gamma \simeq 10^{14} \text{ sec}^{-1}$. We recall that in the visible region of the spectrum the frequencies are of the order of 10^{15} sec^{-1} .

Thus, it may be concluded that in this example the conductivity remains equal to the static value right up to frequencies lying in the infra-red region of the spectrum. However, it must be remembered that at high frequencies, when the mean free path of electrons becomes comparable with the depth of penetration of the field into the metal, spatial non-uniformity of the field and the macroscopic permittivity ϵ lose their meaning.

The results obtained in this problem can be applied, within a limited range, to a metal and also to a semiconductor or an ionised gas (plasma), provided the motion of positive ions can be neglected. Calculations of the permittivity of a plasma with allowance for the motion of the positive ions are given in the solution of Problem 321.

313. Since the molecules of the dielectric are not spherically symmetric, the external field $\mathbf{E}_0$ will partially orient them and the dielectric as a whole will become anisotropic. Since $\mathcal{E} \ll E_0$, the orienting effect of the alternating field may be neglected. Since the external electric field $\mathbf{E}_0$ is responsible for the anisotropy, one of the principal axes of the permittivity tensor will lie in the direction of this field, while the other two principal axes will be perpendicular to $\mathbf{E}_0$.

Let β'_{ik} represent the components of the polarisability of the molecule along these axes ($i, k = 1$ correspond to the axis parallel to $\mathbf{E}_0$). The components β'_{ik} are given by the usual formula

$$\beta'_{ik} = \alpha_{il} \alpha_{km} \beta_{lm} = (\beta - \beta') \alpha_{i1} \alpha_{k1} + \beta' \delta_{ik},$$

where α_{il} are the cosines of the angles between the axis of symmetry of the molecule and the principal axes of the permittivity tensor ($\alpha_{il} \alpha_{kl} = \delta_{ik}$ in view of the orthogonality of the matrix α_{ik}).

In order to calculate the permittivity tensor per unit volume of the dielectric, it is necessary to find the statistical average of the quantities β'_{ik} with the aid of the Boltzmann formula, *i.e.* it is necessary to average the product $\alpha_{i1}\alpha_{k1}$.

The quantities α_{1i} can be expressed in terms of the polar angles of the axis of symmetry of the molecule in the dashed system ϑ, φ :

$$\alpha_{11} = \cos \vartheta, \quad \alpha_{12} = \sin \vartheta \cos \varphi, \quad \alpha_{13} = \sin \vartheta \sin \varphi.$$

Using the same averaging procedure as in Problem 302, we have, to within terms which are linear in $a = [(\beta_0 - \beta'_0)E_0^2/2kT]$:

$$\overline{\alpha_{11}^2} = \frac{1}{3} \left(1 + \frac{4}{15} a \right), \quad \overline{\alpha_{12}^2} = \overline{\alpha_{13}^2} = \frac{1}{3} \left(1 - \frac{2}{15} a \right),$$

$$\overline{\alpha_{i1}\alpha_{k1}} = 0 \text{ when } i \neq k$$

where β_0 and β'_0 are the static values of the polarisability tensor. Hence,

$$\overline{\beta'_{11}} = \frac{1}{3} (\beta - \beta') \left(1 + \frac{4}{15} a \right) + \beta',$$

$$\overline{\beta'_{22}} = \overline{\beta'_{33}} = \frac{1}{3} (\beta - \beta') \left(1 - \frac{2}{15} a \right) + \beta'.$$

Neglecting the difference between the field acting on the molecule and the average field, we obtain the principal values of the permittivity tensor:

$$\epsilon^{(1)} = 1 + 4\pi N \overline{\beta'_{11}}, \quad \epsilon^{(2)} = \epsilon^{(3)} = 1 + 4\pi N \overline{\beta'_{22}}.$$

This result shows that in a strong constant electric field, the dielectric will become anisotropic with respect to high frequency oscillations (for example, visible radiation). The appearance of anisotropy under the action of an electric field is known as the Kerr effect. This effect is a very rapid one: the relaxation time is of the order of 10^{-10} sec. It is determined by the time necessary to establish statistical equilibrium in the dielectric. The Kerr effect is widely used in technology for high frequency modulation of the intensity of light.

The above problem was first solved by Langevin (1910).

314. Assuming that the parameter $pE_0/kT = a$ is small, we have, to within terms of the order of a^2 ,

$$\overline{\beta'_{11}} = \frac{1}{3} (\beta - \beta') \left(1 + \frac{2}{15} a^2 \right) + \beta',$$

$$\overline{\beta'_{22}} = \overline{\beta'_{33}} = \frac{1}{3} (\beta - \beta') \left(1 - \frac{1}{15} a^2 \right) + \beta',$$

$$\epsilon^{(1)} = 1 + 4\pi N \overline{\beta'_{11}}, \quad \epsilon^{(2)} = \epsilon^{(3)} = 1 + 4\pi N \overline{\beta'_{22}}.$$

The symbols have the same meaning as in the preceding problem.

315. Let the field amplitude $\mathcal{E}$ increase by $d\mathcal{E}$ ($d\mathcal{E}_x$, $d\mathcal{E}_y$, $d\mathcal{E}_z$). The work done on a molecule is then given by

$$dA = \frac{1}{2} \operatorname{Re} (\mathbf{p} \cdot d\mathcal{E}^*) = \frac{1}{4} (\mathbf{p} \cdot d\mathcal{E}^* + \mathbf{p}^* \cdot d\mathcal{E}), \quad (1)$$

where $p_i = \beta_{ik} E_k$ is the component of the dipole moment of the system. Since there is no absorption of energy, this work is used entirely to increase the average potential energy of the molecule in the external field:

$$dA = d\overline{W}.$$

Hence, dA should be a perfect differential of a function of the field amplitude. $d\overline{W}$ may be re-written in the form

$$d\overline{W} = \frac{1}{4} \sum_{i,k} (\beta_{ik} \mathcal{E}_k d\mathcal{E}_i^* + \beta_{ki}^* \mathcal{E}_i^* d\mathcal{E}_k). \quad (2)$$

It is clear that this quantity is a perfect differential, provided $\beta_{ik} = \beta_{ki}^*$. When this is so,

$$\begin{aligned} d\overline{W} &= \frac{1}{4} \sum_{i,k} \beta_{ik} (\mathcal{E}_k d\mathcal{E}_i^* + \mathcal{E}_i^* d\mathcal{E}_k) = \frac{1}{4} \sum_{i,k} \beta_{ik} d(\mathcal{E}_i^* \mathcal{E}_k) = d\left(\frac{1}{4} \mathbf{p} \cdot \mathcal{E}^*\right), \\ \overline{W} &= \frac{1}{4} \mathbf{p} \cdot \mathcal{E}^*. \end{aligned}$$

Similarly, it can be shown that the magnetic polarisability tensor for a system in which there is no dissipation of energy is Hermitian.

317. The equation of motion for an atomic electron which is bound to the nucleus by an elastic force is

$$\ddot{\mathbf{r}} + \omega_0^2 \mathbf{r} = \frac{e}{m} \left[\mathbf{E}_0 e^{-i\omega t} + \left(\frac{\mathbf{v}}{c} \times \mathbf{H}_0 \right) \right],$$

where ω_0 is the frequency of natural oscillations. On solving this equation by the method of successive approximations we obtain the following approximate expression:

$$\mathbf{r} = \frac{e\mathbf{E}}{m(\omega_0^2 - \omega^2)} - i \frac{e\omega}{m^2 c (\omega_0^2 - \omega^2)^2} (\mathbf{E} \times \mathbf{H}_0).$$

In order to obtain the polarisability tensor for the atom we shall use the representation of the vector product involving the

skew-symmetric tensor e_{ikl} (cf. Problem 26). This gives

$$\beta_{ik} = \frac{e^2}{m(\omega_0^2 - \omega^2)} \delta_{ik} - i \frac{e^2 \omega H_0}{m^2 c (\omega_0^2 - \omega^2)^2} e_{ikl}.$$

This tensor is Hermitian, in accordance with the general result established in Problem 315. The gyration vector (cf. Problem 316) is now of the form

$$\mathbf{g} = - \frac{e^2 \omega}{m^2 c (\omega_0^2 - \omega^2)^2} \mathbf{H}_0 = - \frac{2e\omega}{m(\omega_0^2 - \omega^2)^2} \omega_L,$$

where $\omega_L = e\mathbf{H}_0/2mc$ is the Larmor frequency.

318.

$$\varepsilon_{ik} = \begin{pmatrix} \frac{1}{2}(\varepsilon^+ + \varepsilon^-) & \frac{i}{2}(\varepsilon^+ - \varepsilon^-) & 0 \\ -\frac{i}{2}(\varepsilon^+ - \varepsilon^-) & \frac{1}{2}(\varepsilon^+ + \varepsilon^-) & 0 \\ 0 & 0 & \varepsilon^0 \end{pmatrix},$$

where

$$\varepsilon^\pm = 1 - \frac{\omega_p^2}{\omega(\omega \pm 2\omega_L) - \omega_0^2}, \quad \varepsilon^0 = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}, \quad \omega_p^2 = \frac{4\pi e^2 N}{m}.$$

The magnitude of the gyration vector is given by

$$g = \frac{1}{2}(\varepsilon^+ - \varepsilon^-)$$

and its direction is parallel to the z -axis. The result established in the solution of the preceding problem can be obtained from the above exact solution by assuming that $2\omega_L \omega \ll |\omega_0^2 - \omega^2|$.

319. The tensor ε_{ik} is of the same form as in the preceding problem. However, its components $\varepsilon^\pm$ and ε^0 are given by the following expressions

$$\varepsilon^\pm = 1 - \frac{\omega_p^2}{\omega^2 + \omega(l\gamma \pm 2\omega_L)},$$

$$\varepsilon^0 = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma},$$

$$\gamma = \frac{\eta}{m}, \quad \omega_L = -\frac{eH_0}{2mc} > 0, \quad \omega_p^2 = \frac{4\pi e^2 N}{m}.$$

Owing to the presence of "friction" ($\eta \neq 0$) in the electron gas there is dissipation of energy and the tensor ε_{ik} is not Hermitian.

320.

$$\mathbf{j} = \sigma \mathbf{E} + (\mathbf{E} \times \mathbf{a}),$$

where

$$\sigma = \frac{e^2 N}{m \gamma}, \quad \mathbf{a} = \frac{e^3 N}{m^2 \gamma^2 c} \mathbf{H}_0, \quad \gamma = \frac{\eta}{m}.$$

The magnetic field gives rise to a current whose direction is perpendicular to the electric field (Hall current). The reciprocal relation (on the same approximation) is

$$\mathbf{E} = \frac{1}{\sigma} \mathbf{j} + R (\mathbf{j} \times \mathbf{H}_0),$$

where $R = (ceN)^{-1}$ is the Hall constant. The electrical conductivity tensor is

$$\sigma_{ik} = \sigma \delta_{ik} - e_{ikl} a_l.$$

321. Let m , $\dot{\mathbf{r}}$, and $-e$ be the mass, velocity, and charge of an electron, and let the corresponding quantities for an ion be M , $\dot{\mathbf{R}}$, and $+e$. The equations of motion are

$$\left. \begin{aligned} m\ddot{\mathbf{r}} &= -e\mathbf{E}_0 e^{-i\omega t} - \frac{e}{c} (\dot{\mathbf{r}} \times \mathbf{H}_0) - m\gamma (\dot{\mathbf{r}} - \dot{\mathbf{R}}), \\ M\ddot{\mathbf{R}} &= e\mathbf{E}_0 e^{-i\omega t} + \frac{e}{c} (\dot{\mathbf{R}} \times \mathbf{H}_0) - m\gamma (\dot{\mathbf{R}} - \dot{\mathbf{r}}), \end{aligned} \right\} \quad (1)$$

where $\mathbf{H}_0$ is the constant uniform magnetic field and $m\gamma$ is the coefficient of "friction". The frictional force is proportional to the relative velocity of the electrons and ions, *i.e.* to the differences $(\dot{\mathbf{r}} - \dot{\mathbf{R}})$ and $(\dot{\mathbf{R}} - \dot{\mathbf{r}})$ for electrons and ions respectively. The electric field $\mathbf{E}$ is a harmonic function of time, so that $\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t)$.

The solution of (1) will be sought in the form

$$\mathbf{r} = \mathbf{r}_0 e^{-i\omega t}, \quad \mathbf{R} = \mathbf{R}_0 e^{-i\omega t}. \quad (2)$$

Let the z -axis be in the direction of $\mathbf{H}_0$ and introduce the cyclic components of the vectors $\mathbf{r}_0$ and $\mathbf{R}_0$:

$$r_{0\pm 1} = \mp \frac{1}{\sqrt{2}} (r_{0x} \pm ir_{0y}), \quad R_{0\pm 1} = \mp \frac{1}{\sqrt{2}} (R_{0x} + iR_{0y}).$$

Substituting (2) into (1) we have, after addition

$$-i\omega (m\mathbf{r}_0 + M\mathbf{R}_0) = \frac{e}{c} [(\mathbf{R}_0 - \mathbf{r}_0) \times \mathbf{H}_0].$$

The left-hand side of the latter equation may be re-written in the form

$$-i\omega[(M+m)\mathbf{R}_0 + m(\mathbf{r}_0 - \mathbf{R}_0)].$$

Neglecting m in comparison with M , we have

$$\omega R_{0\pm 1} = \left(\pm \Omega_H + \omega \frac{m}{M} \right) s_{\pm 1}, \quad (3)$$

where

$$\Omega_H = \frac{eH_0}{Mc} \quad \text{and} \quad \mathbf{s} = \mathbf{R}_0 - \mathbf{r}_0.$$

Let us now divide the first of the equations in (1) by m , and the second by M , and subtract one from the other. Neglecting eE/M in comparison with eE/m , $m\gamma s/M$ in comparison with γs , and $e(\dot{\mathbf{R}} \times \mathbf{H}_0)/Mc$ in comparison with $e(\dot{\mathbf{r}} \times \mathbf{H}_0)/mc$, we have

$$\left. \begin{aligned} (-i\omega + \gamma \mp i\omega_H) s_{\pm 1} \mp i\omega_H R_{0\pm 1} &= \frac{e}{m} E_{0\pm 1}, \\ -\omega^2 s_z &= \frac{eE_{0z}}{m} + i\omega \gamma s_z, \end{aligned} \right\} \quad (4)$$

where $\omega_H = eH_0/mc$. Eqns. (3) and (4) can then be used to determine $\mathbf{s}$.

The polarisation vector $\mathbf{P}$ may be calculated from the formula $\mathbf{P} = Ne\mathbf{s} \exp(-i\omega t)$ where N is the number of ions per unit volume, which is equal to the number of electrons per unit volume.

The components of the permittivity tensor are

$$\begin{aligned} \epsilon^{\pm} &= 1 - \frac{\omega_p^2}{\omega \left(\omega + i\gamma \pm \omega_H - \frac{\omega_H \Omega_H}{\omega} \right)}, \\ \epsilon^{(z)} &= 1 - \frac{\omega_p^2}{\omega (\omega + i\gamma)}. \end{aligned}$$

The component $\epsilon^{(z)}$ is of the same form as the scalar permittivity in the absence of a magnetic field, which was obtained in Problem 312. The permittivity becomes infinite when $\omega \rightarrow 0$. The components $\epsilon^{\pm}$ will contain the extra term $\omega_H \Omega_H$ in the denominator when the motion of ions is taken into account. It may be neglected when $\Omega_H/\omega \ll 1$, *i.e.* at high frequencies ω . However, this term becomes important at low frequencies. When $\omega \rightarrow 0$ it is responsible for the fact that $\epsilon^{\pm}$ remains finite:

$\epsilon^{\pm} = 1 + \omega_p^2 / \omega_H \Omega_H$. This means that very low frequency waves may exist in a plasma (magnetohydrodynamic waves). The theory of propagation of electromagnetic waves in a plasma with allowance for the oscillation of positive ions is considered below in Problem 457.

322. In the system of coordinates in which the x_3 -axis is parallel to the special direction, the tensor T_{ik} should be of the form

$$T_{ik} = \begin{pmatrix} T & T_a & 0 \\ -T_a & T & 0 \\ 0 & 0 & T_{\parallel} \end{pmatrix}.$$

This is in agreement with the results obtained in Problems 318, 319, etc.

324. Since the field is switched on at $t = 0$, it follows from the principle of causality that $\mathbf{P}(t) = 0$ when $t < 0$. Denoting the permittivity by $\alpha = \alpha' + i\alpha''$, we have

$$\mathbf{P}(t) = \int_{-\infty}^{\infty} \alpha(\omega') \mathbf{E}(\omega') e^{-i\omega' t} d\omega' = \frac{\mathbf{E}_0}{2\pi} \int_{-\infty}^{\infty} \alpha(\omega') e^{-i\omega' t} d\omega', \quad (1)$$

where $\mathbf{E}(\omega')$ is the Fourier component of the field $\mathbf{E}(t) = \mathbf{E}_0 \delta(t)$. Next, multiply (1) by $\exp(i\omega t)$ and integrate with respect to t between $-\infty$ and 0. Since $\mathbf{P}(t) = 0$ for $t < 0$, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \alpha(\omega') \int_{-\infty}^0 e^{-i(\omega' - \omega)t} dt = 0. \quad (2)$$

Using Eqn. (17) of Appendix I, and separating the real and imaginary parts, we have

$$\alpha'(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha''(\omega')}{\omega' - \omega} d\omega', \quad \alpha''(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha'(\omega')}{\omega' - \omega} d\omega'. \quad (3)$$

The Kramers-Krönig relation follows immediately.

325.

$$\epsilon'(\omega) = 1 + \frac{\epsilon_0 - 1}{1 + \omega^2 \tau^2}.$$

3. Ferromagnetic resonance

326.

$$M_x = A \sin(\omega_0 t + \alpha), \quad M_y = A \cos(\omega_0 t + \alpha), \quad M_z = C,$$

where $\omega_0 = \gamma H_0$, α is the initial phase, A and C are constants related by the condition $M^2 = M_0^2$, i.e. $A^2 + C^2 = M_0^2$, where M_0 is the saturation magnetisation. The motion of the magnetisation vector takes the form of the usual Larmor precession.

327. The solution of the equation

$$\frac{d\mathbf{M}}{dt} = -\gamma \mathbf{M} \times \mathbf{H}_0 + \omega_r (\chi_0 \mathbf{H}_0 - \mathbf{M}) \quad (1)$$

will be sought in the form $M_x = m_x \exp(-i\omega t)$, $M_y = m_y \exp(-i\omega t)$, $M_z = M_0 + m_z \exp(-i\omega t)$, where ω is the unknown frequency and the z -axis is parallel to $\mathbf{H}_0$.

On taking the components of (1) along the coordinate axes, and substituting for $\mathbf{M}$, we obtain a system of algebraic equations. The condition that these equations should be consistent is

$$\omega_0^2 - (\omega + i\omega_r)^2 = 0.$$

The frequency ω turns out to be complex:

$$\omega = \omega_0 - i\omega_r.$$

The presence of losses gives rise to damped motion. The components m_x and m_y exhibit a phase difference of $\pi/2$. The vector $\mathbf{M}$ executes a damped precession about $\mathbf{H}_0$.

328. Let the z -axis be parallel to $\mathbf{H}$. The resultant magnetic field will then have the components $h_x \exp(-i\omega t)$, $h_y \exp(-i\omega t)$, and $H_0 + h_z \exp(-i\omega t)$. The solution of the Landau-Lifshits equation (VI.15) will be sought in the form

$$M_x = m_x e^{-i\omega t}, \quad M_y = m_y e^{-i\omega t}, \quad M_z = M_0 + m_z e^{-i\omega t}, \quad (1)$$

where M_0 is the saturation magnetisation. This form of the solution corresponds to the assumption that the Larmor precession has been damped out and the oscillations are maintained by the high frequency (forcing) field. Hence, the quantities m_x , m_y , m_z should be regarded as small (of the order of h or lower). Substituting (1) into the Landau-Lifshits equation, and rejecting terms which are quadratic in h and m , we obtain

the components of $\mathbf{m}$:

$$\left. \begin{aligned} m_x &= \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2} h_x - \chi_0 \frac{i\omega\omega_0}{\omega_0^2 - \omega^2} h_y, \\ m_y &= \chi_0 \frac{i\omega\omega_0}{\omega_0^2 - \omega^2} h_x + \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2} h_y, \quad m_z = 0. \end{aligned} \right\} \quad (2)$$

It is clear from these formulae that ferromagnetic resonance will occur at $\omega = \omega_0$. The infinite magnitude of $\mathbf{m}$ at resonance is due to the approximate method used in the solution of the Landau-Lifshits equation. The exact solution (*cf.* Problem 330) should ensure that $|\mathbf{M}|$ remains constant, since it follows from the Landau-Lifshits equation that $M^2 = \text{const.}$ When the problem is solved by the method of successive approximations with allowance for losses, $\mathbf{M}$ will also remain finite.

329.

$$\chi_{ik} = \begin{pmatrix} \chi_{\perp} & -i\chi_a & 0 \\ i\chi_a & \chi_{\perp} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\chi_{\perp} = \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2}, \quad \chi_a = \chi_0 \frac{\omega\omega_0}{\omega_0^2 - \omega^2},$$

$$\mu_{ik} = \begin{pmatrix} \mu_{\perp} & -i\mu_a & 0 \\ i\mu_a & \mu_{\perp} & 0 \\ 0 & 0 & \mu_{\parallel} \end{pmatrix},$$

and

$$\mu_{\perp} = 1 + 4\pi\chi_{\perp}, \quad \mu_a = 4\pi\chi_a, \quad \mu_{\parallel} = 1.$$

As can be seen from the above formulae, χ_{ik} and μ_{ik} are Hermitian tensors ($\mu_{ik} = \mu_{ki}^*$). This means that the medium is

gyrotropic and there are no losses. Fig. 54 shows the components μ_{ik} as functions of frequency for $H_0 \text{ res} \approx 3400$ Oe.

330.

$$M_x = \frac{\omega_1}{\Delta\omega} C \cos \omega t,$$

$$M_y = \frac{\omega_1}{\Delta\omega} C \sin \omega t, \quad M_z = C,$$

where $\Delta\omega = \omega_0 - \omega$, $\omega_0 = \gamma H_0$, $\omega_1 = \gamma h$. The constant C may be determined from the condition $M_x^2 + M_y^2 + M_z^2 = M_0^2$ which

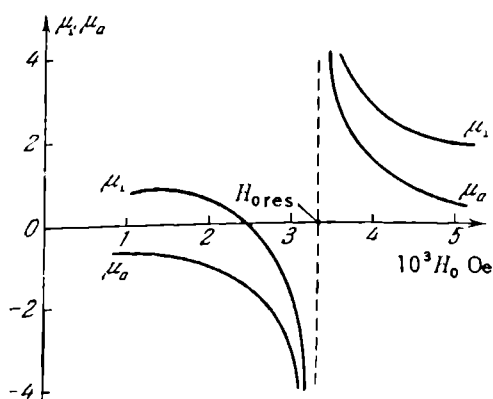


Fig. 54.

follows from the Landau-Lifshits equation:

$$C = \frac{|\Delta\omega|}{\Omega} M_0,$$

where $\Omega = (\Delta\omega^2 + \omega_1^2)^{\frac{1}{2}}$. The expression for C includes the modulus $|\Delta\omega|$ since $M_z > 0$. The components of $\mathbf{M}$ are given by

$$\left. \begin{aligned} M_x &= \pm \frac{\omega_1}{\Omega} M_0 \cos \omega t = \chi h_x, \\ M_y &= \pm \frac{\omega_1}{\Omega} M_0 \sin \omega t = \chi h_y, \quad M_z = \frac{|\Delta\omega|}{\Omega} M_0. \end{aligned} \right\} \quad (1)$$

The sign $\pm$ corresponds to the sign of $\Delta\omega$. It follows from these equations that the relation between $\mathbf{M}$ and $\mathbf{h}$ is not linear, and the coefficient of proportionality χ is a function of h :

$$\chi = \pm \frac{\gamma M_0}{\sqrt{\Delta\omega^2 + \omega_1^2}}.$$

The precession angle ϑ (the angle between $\mathbf{M}$ and $\mathbf{H}_0$) is given by

$$\sin \vartheta = \frac{M_{\perp}}{M_0} = \frac{\omega_1}{\Omega},$$

where $M_{\perp} = (M_x^2 + M_y^2)^{\frac{1}{2}}$. At the ferromagnetic resonance $\Delta\omega = 0$ and equations (1) yield

$$M_x = \pm M_0 \cos \omega t, \quad M_y = \pm M_0 \sin \omega t, \quad M_z = 0.$$

The vector $\mathbf{M}$ will then rotate with a frequency ω in the plane perpendicular to $\mathbf{H}_0$, and its components will remain finite.

331. $\mathbf{M} = \mathbf{M}_0 + \mathbf{m} \exp(-i\omega t)$ where $\mathbf{M}_0$ lies in the direction of $\mathbf{H}_0$ and the components of $\mathbf{m}$ are given by

$$m_x = \chi_0 \frac{\Omega^2 - i\omega\omega_r}{\Omega^2 - \omega^2 - 2i\omega\omega_r} h_x - i\chi_0 \frac{\omega\omega_0}{\Omega^2 - \omega^2 - 2i\omega\omega_r} h_y,$$

$$m_y = i\chi_0 \frac{\omega\omega_0}{\Omega^2 - \omega^2 - 2i\omega\omega_r} h_x + \chi_0 \frac{\Omega^2 - i\omega\omega_r}{\Omega^2 - \omega^2 - 2i\omega\omega_r} h_y,$$

$$m_z = \chi_0 \frac{\omega_r}{\omega_r - i\omega} h_z,$$

$$\Omega = \sqrt{\omega_0^2 + \omega_r^2}, \quad \omega_0 = \gamma H_0.$$

As can be seen from these formulae, the presence of losses

($\omega_r \neq 0$) will ensure that the amplitude of $\mathbf{m}$ will remain finite at resonance.

332.

$$\mu_{lh} = \begin{pmatrix} \mu_{\perp} & -i\mu_a & 0 \\ i\mu_a & \mu_{\perp} & 0 \\ 0 & 0 & \mu_{\parallel} \end{pmatrix}, \quad \begin{aligned} \mu_{\perp} &= \mu'_{\perp} + i\mu''_{\perp}, \\ \mu_a &= \mu'_a + i\mu''_a, \end{aligned}$$

$$\mu'_{\perp} = 1 + 4\pi\chi_0 \frac{\Omega^2(\Omega^2 - \omega^2) + 2\omega^2\omega_r^2}{(\Omega^2 - \omega^2)^2 + 4\omega^2\omega_r^2},$$

$$\mu''_{\perp} = 4\pi\chi_0 \frac{\omega\omega_r(\Omega^2 + \omega^2)}{(\Omega^2 - \omega^2)^2 + 4\omega^2\omega_r^2},$$

$$\mu'_a = 4\pi\chi_0 \frac{\omega\omega_0(\Omega^2 - \omega^2)}{(\Omega^2 - \omega^2)^2 + 4\omega^2\omega_r^2},$$

$$\mu''_a = \frac{\omega^2\omega_0\omega_r}{(\Omega^2 - \omega^2)^2 + 4\omega^2\omega_r^2},$$

where

$$\Omega = \sqrt{\omega_0^2 + \omega_r^2}, \quad \omega_0 = \gamma H_0,$$

$$\mu_{\parallel} = 1 + 4\pi\chi_0 \frac{\omega_r}{\omega_r - i\omega},$$

$$H_{\text{ores}} \simeq 3400 \text{ Oe.}$$

Fig. 55 shows the dependence of $\mu'_{\perp}$ and $\mu''_{\perp}$ on the constant field H_0 . The dependence of μ'_a and μ''_a on H_0 is similar. The imaginary parts of $\mu'_{\perp}$ and μ''_a have a maximum at $H_0 = H_{0 \text{ res}} \simeq \omega/\gamma$, while the real parts of $\mu'_{\perp}$, μ'_a reach extremal values at $H_0 \simeq (\omega \pm \omega_r)/\gamma$.

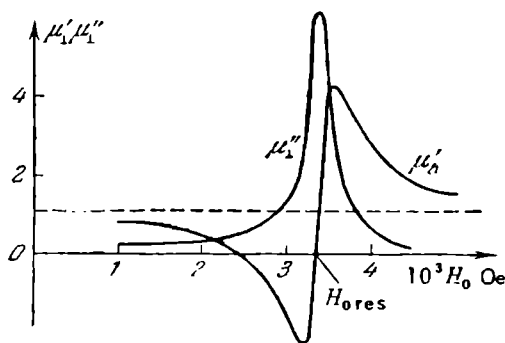


Fig. 55.

The curves shown in Fig. 55 are similar to the dispersion curves for $\varepsilon(\omega)$ (cf. Fig. 16).

The imaginary parts of the components $\mu''_{\perp}$, μ''_a , and $\mu''_{\parallel}$ determine the dissipation of electromagnetic energy. They are zero at $\omega_r = 0$.

$$333. \quad \Delta H_0 = \omega_r/\gamma.$$

334. Let the coordinate axes lie along the principal axes of the ellipsoid, with the z -axis lying along the field $\mathbf{H}_0$. The tensor N_{ikh} will then be diagonal, and hence the components of

the Landau-Lifshits equation along the coordinate axes will be

$$\begin{aligned}\dot{M}_x &= -\gamma [H_0 + 4\pi(N^{(y)} - N^{(z)}) M_z] M_y, \\ \dot{M}_y &= \gamma [H_0 + 4\pi(N^{(x)} - N^{(z)}) M_z] M_x, \\ \dot{M}_z &= -4\pi\gamma (N^{(x)} - N^{(y)}) M_x M_y.\end{aligned}\quad (1)$$

The equations are therefore non-linear. Assuming that the departure of the vector $\mathbf{M}$ from the equilibrium position (the direction of the z -axis) is small, the solution will be sought in the form

$$\mathbf{M} = \mathbf{M}_0 + m e^{-i\omega t}, \quad (2)$$

where the vector $\mathbf{M}_0$ is parallel to the z -axis. The system of equations given by (1) may be linearised by substituting (2) into it and then neglecting terms containing m^2 . Equating the determinant of the system to zero, we have

$$\omega^2 \equiv \omega_k^2 = \gamma^2 [H_0 + 4\pi(N^{(x)} - N^{(z)}) M_0] [H_0 + 4\pi(N^{(y)} - N^{(z)}) M_0].$$

$$335. \quad \omega = \omega_k + i\omega_r \left[1 + \frac{\chi_0(N^{(x)} + N^{(y)})}{2} \right], \quad \chi_0 = \frac{M_0}{H_0 - N^{(z)} M_0}.$$

The value of ω_k is given in the solution of the preceding problem.

$$336. \quad \chi_{ik} = \begin{pmatrix} \chi_1 & -i\chi_a & 0 \\ i\chi_a & \chi_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the z -axis lies along $\mathbf{H}_0$, and

$$\begin{aligned}\chi_1 &= \frac{1}{\Delta} \{ \gamma^2 M_0 [H_0 + (N^{(y)} - N^{(z)}) M_0] - i\chi_0 \omega \omega_r \}, \\ \chi_2 &= \frac{1}{\Delta} \{ \gamma^2 M_0 [H_0 + (N^{(x)} - N^{(z)}) M_0] - i\chi_0 \omega \omega_r \},\end{aligned}$$

where

$$\begin{aligned}\Delta &= (\omega_k^2 - \omega^2) - i\omega \omega_r [2 + \chi_0(N^{(x)} + N^{(y)})], \\ \chi_0 &= \frac{M_0}{H_0 - N^{(z)} M_0}, \quad \chi_a = -\frac{1}{\Delta} \gamma \omega M_0.\end{aligned}$$

Since the demagnetising factors enter into the expressions for the components of the tensor χ_{ik} , the position of the resonance and the width of the resonance line will depend on the shape of the body.

337. The equations of motion for the magnetisation vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ are of the form

$$\frac{d\mathbf{M}_1}{dt} = -\gamma \mathbf{M}_1 \times (\mathbf{H}_0 - \lambda \mathbf{M}_2), \quad \frac{d\mathbf{M}_2}{dt} = -\gamma \mathbf{M}_2 \times (\mathbf{H}_0 - \lambda \mathbf{M}_1). \quad (1)$$

The solution will be sought in the form $\mathbf{M}_1 = \mathbf{M}_{10} + \mathbf{m}_1 \exp(-i\omega t)$, $\mathbf{M}_2 = \mathbf{M}_{20} + \mathbf{m}_2 \exp(-i\omega t)$, where M_{10} and M_{20} are the equilibrium values of M_1 and M_2 respectively.

It is convenient to use the cyclic components

$$m_{j\pm} = m_{jx} \pm im_{jy} \quad (j = 1, 2).$$

The natural precession frequencies are given by

$$\omega_{01} = \gamma H_0, \quad \omega_{02} = \gamma \lambda |M_{10} - M_{20}|. \quad (2)$$

These formulae will hold provided $\lambda |M_{10} - M_{20}| \gg H_0$. The frequency ω_{01} is the same as in the case of a ferromagnetic without sub-lattices. The frequency ω_{02} depends on the molecular field and is usually much greater than ω_{01} .

Chapter 7

QUASI-STATIONARY ELECTROMAGNETIC FIELDS

1. Quasi-stationary phenomena in linear conductors

$$338. \quad \mathcal{J}(t) = \frac{\pi a^2 \omega H_0}{c \sqrt{R^2 + \left(\frac{\omega L}{c^2}\right)^2}} \sin(\omega t - \varphi),$$

where $\tan \varphi = \omega L / c^2 R$,

$$N(t) = - \frac{\omega (\pi a^2 H_0)^2}{c^2 \sqrt{R^2 + \left(\frac{\omega L}{c^2}\right)^2}} \sin \omega t \sin(\omega t - \varphi),$$

$$\bar{P} = \frac{\omega^2}{2c^2} \cdot \frac{(\pi a^2 H_0)^2 R}{\left[R^2 + \left(\frac{\omega L}{c^2}\right)^2\right]} = \frac{1}{2} \mathcal{J}_0^2 R.$$

In these expressions L is the inductance of the ring (*cf.* Problem 272), R is its resistance, and $\mathcal{J}_0$ is the amplitude of the current in the ring. The origin of time is chosen so that at $t = 0$ the plane of the loop is perpendicular to H_0 .

$$339. \quad \bar{N} = \frac{\omega}{2c^2} \cdot \frac{(SH_0)^2 R}{R^2 + \left(\frac{\omega L}{c^2} - \frac{1}{\omega C}\right)^2}.$$

340. The average generalised force which tends to increase the generalised coordinate q_i is equal to

$$- \frac{\mathcal{J}_0^2}{2c^6} \cdot \frac{\omega^2 L L_{12}}{R^2 + \left(\frac{\omega L}{c^2}\right)^2} \cdot \frac{\partial L_{12}}{\partial q_i},$$

where R and L are the resistance and inductance of the second circuit respectively, and L_{12} is the coefficient of mutual inductance.

$$341. \quad \bar{F} = - \frac{\omega^2 L L_{12} |\mathcal{E}_0|^2}{2c^6 \left\{ \left[R^2 + \frac{\omega^2 (L_{12}^2 - L^2)}{c^4} \right]^2 + \frac{4\omega^2 L^2 R^2}{c^4} \right\}} \frac{\partial L_{12}}{\partial q_i}.$$

342.

$$\omega_{1,2}^2 = \frac{c^2 [(L_1 + L_2) C_1 + L_1 C_1 - L_2 C_2] \pm c^2 \{ [L_1 (C_1 + C_2) - L_2 (C_1 - C_2)]^2 + 4 L_1 L_2 C_2^2 \}^{\frac{1}{2}}}{2 L_1 L_2 (C_1 C_2 + C C_1 + C C_2)}.$$

When there is no coupling between the circuits, *i.e.* when $C = 0$, the frequencies ω_1 and ω_2 become equal to $c/(L_1 C_1)^{\frac{1}{2}}$ and $c/(L_2 C_2)^{\frac{1}{2}}$, which corresponds to independent oscillations in each of the circuits. For very tight coupling ($C \gg C_1, C_2$) there is only a single frequency $\omega = c/(L' C')^{\frac{1}{2}}$, where $L' = L_1 L_2 / (L_1 + L_2)$, $C' = C_1 + C_2$. This corresponds to oscillations in a single circuit in which the capacitances C_1, C_2 and inductances L_1, L_2 are connected in parallel.

$$343. \quad \omega_{1,2}^2 = \frac{c^2}{2} \left(\frac{1}{LC_1} + \frac{1}{LC_2} + \frac{1}{L_1 C_1} + \frac{1}{L_2 C_2} \right) \pm \frac{c^2}{2} \left\{ \left[\frac{1}{C_1} \left(\frac{1}{L} + \frac{1}{L_1} \right) - \frac{1}{C_2} \left(\frac{1}{L} + \frac{1}{L_2} \right) \right]^2 + \frac{4}{L^2 C_1 C_2} \right\}^{\frac{1}{2}}.$$

$$344. \quad \omega_{1,2}^2 = c^2 \frac{L_1 C_1 + L_2 C_2 \pm [(L_1 C_1 - L_2 C_2)^2 + 4 C_1 C_2 L_{12}^2]^{\frac{1}{2}}}{2 C_1 C_2 (L_1 L_2 - L_{12}^2)}.$$

345. When the system of equations for the currents is set up and the determinant of the system is equated to zero, we have after some rearrangement the following fourth-order equation

$$\omega^4 + i\omega^3 \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} \right) - \omega^2 (\omega_1^2 + \omega_2^2) - i\omega \left(\frac{\omega_1^2}{\tau_2} + \frac{\omega_2^2}{\tau_1} \right) + \omega_1^2 \omega_2^2 = 0, \quad (1)$$

where

$$\omega_1 = \frac{c}{\sqrt{L_1 C_1}}, \quad \omega_2 = \frac{c}{\sqrt{L_2 C_2}}, \quad \tau_1 = RC_1, \quad \tau_2 = RC_2.$$

The coefficients of this equation are complex and hence the frequency ω will also be complex, so that $\omega = \omega' + i\omega''$. On the zero-order approximation the terms including τ_1 and τ_2 in Eqn. (1) may be neglected, so that the equation will assume the simpler form

$$\omega^4 - \omega^2 (\omega_1^2 + \omega_2^2) + \omega_1^2 \omega_2^2 = 0. \quad (2)$$

This equation has the following solutions: $\omega_1^{(0)} = \omega_1$ and $\omega_2^{(0)} = \omega_2$. Thus, on this approximation, $\omega'' = 0$ and there is no dissipation of energy (since it was assumed that R is infinite). The oscillations in each circuit take place independently. On the next approximation $\omega = \omega^{(0)} + \Delta\omega' + i\omega''$, where ω'' , $\Delta\omega'$ are of the order of τ^{-1} or higher. Accordingly, we shall neglect higher-order terms. If ω is substituted into (1), and (2) used, we have, since the real and imaginary parts are separately zero,

$$\Delta\omega' = 0, \quad \omega_1'' = -\frac{1}{2\tau_2}, \quad \omega_2'' = -\frac{1}{2\tau_1}. \quad (3)$$

The correction to ω' which contains R will only appear on the next approximation.

$$346. \quad \mathcal{J}_1 = \frac{\mathcal{E}}{Z_1 \left(1 + \frac{\omega^2 L_{12}^2}{c^2 Z_1 Z_2} \right)}, \quad \mathcal{J}_2 = \frac{i\omega L_{12}}{c^2 Z_2} \mathcal{J}_1;$$

$$Z_1 = R_1 + i \left(\frac{1}{\omega C_1} - \frac{\omega L_1}{c^2} \right), \quad Z_2 = R_2 + i \left(\frac{1}{\omega C_2} - \frac{\omega L_2}{c^2} \right);$$

$$\mathcal{J}_{1\max} = \frac{\mathcal{E}}{R} \quad \text{when} \quad \omega = \frac{c}{\sqrt{L_1 C_1 \left(1 - \frac{L_{12}^2}{L_1 L_2} \right)}}.$$

$$347. \quad Z = \frac{R - \frac{i\omega L}{c^2}}{1 - \frac{\omega^2}{\omega_1^2} - i\omega RC},$$

where $\omega_1 = c(LC)^{-\frac{1}{2}}$ is the natural frequency of oscillations in the circuit. The impedance Z becomes infinite when $R = 0$ and $\omega = \omega_1$. This property of the two-terminal network is used in electronics (rejector circuit).

$$348. \quad C = C_0, \quad L = L_0, \quad R = \frac{\gamma L_0}{c^2},$$

where

$$L_0 = \frac{c^2}{\omega_p^2 C_0}.$$

$$349. \quad Q = \frac{1}{2} \operatorname{Re} (U \mathcal{J}^*) = \frac{1}{2} |\mathcal{J}|^2 \operatorname{Re} \left(\frac{1}{Z} \right) = \frac{1}{2} \cdot \frac{\gamma \omega_p^2}{\omega^2 + \gamma^2} C_0 |U_0|^2,$$

$$\bar{W} = \frac{1}{4} \left(1 + \frac{\omega_p^2}{\omega^2 + \gamma^2} \right) C_0 |U_0|^2.$$

$$350. \quad C = C_0, \quad L = \frac{c^2}{\omega_p^2 C_0}, \quad C_1 = \frac{\omega_p^2}{\omega_0^2} C_0, \quad R = \frac{\gamma L}{c^2} = \frac{\gamma}{\omega_p^2 C_0}.$$

$$351. \quad Q = \frac{1}{2} \cdot \frac{\omega^2 \omega_p^2 \gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2 \gamma^2} \cdot C_0 |U_0|^2,$$

$$\bar{W} = \frac{1}{4} \left[1 + \frac{\omega_p^2 (\omega^2 + \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + \omega^2 \gamma^2} \right] C_0 |U_0|^2.$$

352. Let the currents flowing through the inductance, capacitance, and the battery be denoted by $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$. According to Kirchhoff's laws

$$\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 = 0, \quad \frac{L}{c^2} \mathcal{J}_1 = \frac{\dot{q}(t)}{c} = \mathcal{E}(t) + \mathcal{J}_3 R, \quad (1)$$

where $q(t)$ is the charge on the capacitor plate, which is related to $\mathcal{I}_2$ by the formula $\mathcal{I}_2 = \dot{q}$, and

$$\mathcal{E}(t) = \begin{cases} 0 & \text{when } t < 0, \\ \mathcal{E} & \text{when } t > 0. \end{cases}$$

From (1) we obtain a second-order equation for the current $\mathcal{I}_1$. The corresponding characteristic equation has the following roots

$$x = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \omega_0^2}, \quad \omega_0^2 = \frac{c^2}{LC}.$$

The following three cases are possible, depending on the relation between R , L , and C :

(a) $\omega_0 > (2RC)^{-1}$. Using the method of variation of arbitrary constants due to Lagrange, we have

$$\mathcal{I}_1(t) = \frac{\mathcal{E}}{R} \left[1 - e^{-\frac{t}{2RC}} \left(\frac{\sin \omega t}{2\omega RC} + \cos \omega t \right) \right],$$

where $\omega = [\omega_0^2 - (2RC)^{-2}]^{\frac{1}{2}}$.

(b) $\omega_0 < (2RC)^{-1}$. Here

$$\mathcal{I}_1(t) = \frac{\mathcal{E}}{R} \left[1 - e^{-\frac{t}{2RC}} \left(\frac{\sinh \Omega t}{2\Omega RC} + \cosh \Omega t \right) \right],$$

where $\Omega = [(4R^2C^2)^{-1} - \omega_0^2]^{\frac{1}{2}}$.

(c) $\omega_0 = (2RC)^{-1}$:

$$\mathcal{I}_1(t) = \frac{\mathcal{E}}{R} \left[1 - \left(1 + \frac{t}{2RC} \right) e^{-\frac{t}{2RC}} \right].$$

Under the conditions (b) and (c) the transients are completely aperiodic and there are no oscillations.

$$353. \quad U_2(t) = \begin{cases} 0 & \text{when } t < 0, \\ U_0 e^{-\frac{t}{RC}} & \text{when } 0 < t < T, \\ U_0 \left(e^{-\frac{t}{RC}} - e^{-\frac{t-T}{RC}} \right) & \text{when } t > T. \end{cases}$$

$$354. \quad U_2(t) = \begin{cases} 0 & \text{when } t < 0, \\ U_0 e^{-\frac{Rc^2 t}{L}} & \text{when } 0 < t < T, \\ U_0 \left(e^{-\frac{Rc^2 t}{L}} - e^{-\frac{Rc^2 (t-T)}{L}} \right) & \text{when } t > T. \end{cases}$$

355. The form of the pulse should be

$$U_1(t) = \begin{cases} 0 & \text{when } t < -T, \\ hE_0 \left(1 + \frac{t}{T} + \frac{RC}{T}\right) & \text{when } -T < t < 0, \\ hE_0 \left(1 - \frac{t}{T}\right) & \text{when } 0 < t < T, \\ 0 & \text{when } t > T. \end{cases}$$

The origin of time is chosen so that the field between the plates of the capacitor is a maximum at $t = 0$.

$$356. \quad I(t) = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\frac{\omega L}{c^2}\right)^2}} \left[\cos(\omega t + \varphi_0 - \varphi) - e^{-\frac{RC^2 t}{L}} \cos(\varphi_0 - \varphi) \right],$$

where $\tan \varphi = \omega L / c^2 R$. The transient is absent when $\tan \varphi_0 = -RC^2 / \omega L$. This condition has a simple interpretation: the stationary value of the current should be zero at $t = 0$.

357. For a simple harmonic dependence of the currents on time, the Kirchhoff equation for the n -th section is of the form

$$-\frac{\omega L}{c^2} \mathcal{J}_n + \frac{1}{\omega C} (2\mathcal{J}_n - \mathcal{J}_{n-1} - \mathcal{J}_{n+1}) = 0. \quad (1)$$

This latter equation is a linear difference equation with the independent variable n assuming integral values. It has two linearly independent solutions, namely $\sin \kappa n$ and $\cos \kappa n$ (cf. Problem 223); the frequencies of natural oscillations can be expressed in terms of the parameter κ :

$$\omega^2 = 4\omega_0^2 \sin^2 \frac{\kappa}{2}, \quad \omega_0 = \frac{c}{\sqrt{LC}}. \quad (2)$$

Using the boundary conditions $\mathcal{J}_0 = \mathcal{J}_N = 0$, we have

$$\mathcal{J}_n = A \sin \kappa n, \quad \kappa = \frac{\pi r}{N}, \quad (3)$$

where $r = 1, 2, \dots$. When $r = 0$, the current in the circuit is zero. However, owing to the periodicity of the factor $\sin(\kappa/2)$ which enters into (2), the number of natural frequencies of the system will be finite. To obtain the frequency spectrum it is sufficient to vary r within the range $1 \leq r \leq N$. The parameter κ will then vary in the range $0 < \kappa \leq \pi$, and to each κ there will be one natural frequency. The total number of such frequencies will be N , as should be the case for a system of N coupled circuits. They will lie in the range $0 < \omega \leq 2\omega_0$.

In order to determine the significance of κ introduce the coordinate $y_n = an$, where n refers to the n -th section and a is the "length" of a section. Eqn. (3) can then be written in the form

$$\mathcal{J}_n(t) = \mathcal{J}_0 \sin ky_n e^{-i\omega_k t}, \quad (4)$$

where the time factor has been included and $k = \kappa/a$.

Eqn. (4) represents a superposition of two waves travelling in opposite directions. The quantity k plays the role of a "wave vector". The phase and group velocities of these waves are given by the formulae

$$v_\varphi = \frac{\omega}{k}, \quad v_g = \frac{d\omega}{dk}. \quad (5)$$

Since ω is a non-linear function of k the phase and group velocities are different, *i.e.* dispersion takes place. From (2) we have

$$\begin{aligned} v_\varphi &= \frac{2\omega_0}{k} \sin \frac{ka}{2}, \\ v_g &= \omega_0 a \cos \frac{ka}{2}. \end{aligned} \quad (6)$$

The quantity $2\pi/k$ may be interpreted as the "wavelength". For long waves ($\lambda \gg a$) we have $ka \ll 1$ and hence $v_\varphi = v_g = \omega_0 a$, *i.e.* the phase and group velocities are independent of k and there is no dispersion. Fig. 56 shows plots of v_g and ω as functions of k .

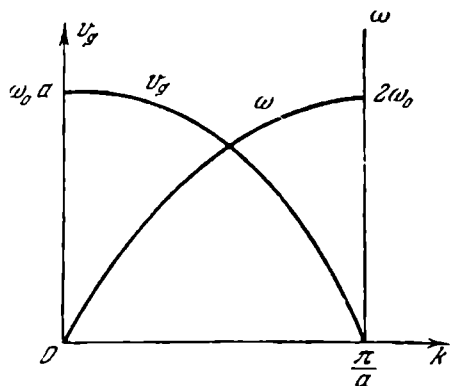


Fig. 56.

The electrical oscillations of the system considered in this problem are analogous to the mechanical oscillations of a linear atomic chain, which may be used as a one-dimensional model of a crystal. The inductance

L is then analogous to the mass of an atom, while $1/C$ is analogous to the elastic constant.

$$358. \quad \Delta r = \frac{2N}{\pi} \cdot \frac{\Delta \omega}{\sqrt{4\omega_0^2 - \omega^2}}.$$

359. Let the currents through the section with self-inductances L_1 and L_2 be $\mathcal{J}$ and $\mathcal{J}'$. The Kirchhoff equation will then be of the

form

$$\left. \begin{aligned} -\frac{\omega L_1}{c^2} \mathcal{J}_n + \frac{1}{\omega C} (2\mathcal{J}_n - \mathcal{J}'_n - \mathcal{J}'_{n-1}) &= 0, \\ -\frac{\omega L_2}{c^2} \mathcal{J}'_n + \frac{1}{\omega C} (2\mathcal{J}'_n - \mathcal{J}_n - \mathcal{J}_{n+1}) &= 0. \end{aligned} \right\} \quad (1)$$

Substituting $\omega_1 = c(L_1 C)^{-\frac{1}{2}}$ and $\omega_2 = c(L_2 C)^{-\frac{1}{2}}$ we have

$$\left. \begin{aligned} (2\omega_1^2 - \omega^2) \mathcal{J}_n &= \omega_1^2 (\mathcal{J}'_n + \mathcal{J}'_{n-1}), \\ (2\omega_2^2 - \omega^2) \mathcal{J}'_n &= \omega_2^2 (\mathcal{J}_n + \mathcal{J}_{n+1}). \end{aligned} \right\} \quad (2)$$

The solution of this system of equations will be sought in the form

$$\mathcal{J}_n = A e^{i\kappa n}, \quad \mathcal{J}'_n = B e^{i\kappa n}, \quad (3)$$

where A , B , and κ are constants. Substitution into (2) yields

$$A(2\omega_1^2 - \omega^2) = B\omega_1^2(1 + e^{-i\kappa}), \quad B(2\omega_2^2 - \omega^2) = A\omega_2^2(1 + e^{i\kappa}). \quad (4)$$

Equating the determinant of this system to zero, we have the following relation between the frequency ω and κ :

$$\omega^2 = \omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2\omega_2^2 \sin^2 \frac{\kappa}{2}}. \quad (5)$$

To obtain the complete spectrum of oscillations, the quantity κ must be varied between 0 and π . As in Problem 357, the values of κ will be found from the boundary conditions.

The most important difference between the present arrangement and that involving identical sections is the fact that there

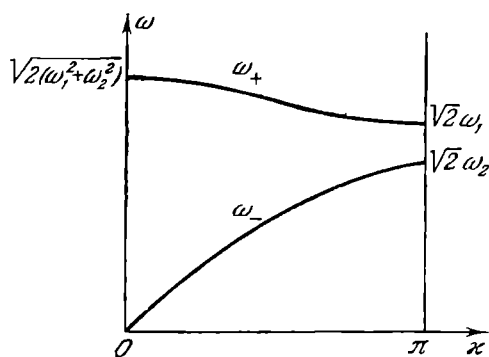


Fig. 57.

are now two frequencies to each value of κ , which can be seen from (5). There are, therefore, two branches of oscillations.

Let us denote the frequencies of these oscillations by ω_+ and ω_- where the subscripts $+$ and $-$ correspond to the same signs in (5). Fig. 57 shows the frequencies as functions of κ . The oscillations corresponding to ω_- are

analogous to those in the case of identical sections. In particular, for small κ (long waves) we have

$$\omega_- = \frac{\omega_1 \omega_2}{\sqrt{2(\omega_1^2 + \omega_2^2)}} \kappa,$$

i.e. there is no dispersion. For the ω_+ branch, the dispersion relation for small κ is

$$\omega_+ = a + b\kappa^2.$$

When $\kappa \rightarrow 0$ the phase velocity becomes infinite while the group velocity tends to zero.

To investigate the character of the oscillations in both branches, consider the ratio of the current amplitudes in adjacent sections for very long ($\kappa \ll 1$) and very short ($\kappa \sim \pi$) waves. Using (4), we have for $\kappa \ll 1$

$$\left(\frac{A}{B}\right)_- \simeq 1,$$

and

$$\left(\frac{A}{B}\right)_+ \simeq -\frac{\omega_1^2}{\omega_2^2} = -\frac{L_2}{L_1}.$$

For the oscillations corresponding to the ω_- branch the amplitudes are equal and the oscillations are in phase. In the other case, the oscillations in adjacent sections are in antiphase while the amplitudes are inversely proportional to the inductances.

When $\kappa = \pi$,

$$\omega_+ = \sqrt{2} \omega_1, \quad \omega_- = \sqrt{2} \omega_2.$$

On passing to the limit $\kappa \rightarrow \pi$ in (4), we have

$$\left(\frac{B}{A}\right)_+ \rightarrow 0, \quad \left(\frac{A}{B}\right)_- \rightarrow 0.$$

Thus in the limit $\kappa = \pi$, oscillations with frequency $\omega_+ = c(2/L_1 C)^{\frac{1}{2}}$ will take place only in the sections with inductances L_1 , while oscillations with the frequency $\omega_- = c(2/L_2 C)^{\frac{1}{2}}$ will take place in sections with inductances L_2 .

The oscillations considered in this problem are an analogue of acoustic and optical oscillations in a linear atomic chain consisting of two types of atoms with different masses.

$$360. \quad \mathcal{J}_n = Aq_1^n + Bq_2^n, \quad (1)$$

where q_1 and q_2 are the roots of the equation

$$q^2 - \left(2 + \frac{Z_1}{Z_2}\right)q + 1 = 0. \quad (2)$$

The constants A and B may be determined from the boundary

conditions $\mathcal{J}_N = 0$, $(\mathcal{J}_0 - \mathcal{J}_1)Z_2 = U_1$. The second condition means that a voltage U_1 is applied across the points $a'b'$ (cf. Fig. 23). Since $q_1 q_2 = 1$, which follows from (2), we have

$$U_2 = \mathcal{J}_{N-1} Z_2 = U_1 \frac{q_2 - q_1}{(1 - q_1) q_2^N - (1 - q_2) q_1^N}.$$

361. The transmission coefficient K can be determined from the results obtained in the preceding solution:

$$K = \frac{q_1 - q_2}{(1 - q_2) q_1^N - (1 - q_1) q_2^N}.$$

The denominator of this expression includes q_1^N and q_2^N . Since $q_1 \cdot q_2 = 1$, there are two possible cases, namely $|q_1| = |q_2| = 1$ and $|q_1| > 1$, $|q_2| < 1$. In the first case the moduli of q_1^N and q_2^N will be equal to unity and K will also be of the order of unity. In the second case $|q_1^N| \gg 1$, $|q_2^N| \ll 1$ when $N \gg 1$ and hence

$$K = \frac{q_1 - q_2}{(1 - q_2) q_1^N} \ll 1.$$

The frequency ranges may be determined from Eqn. (2) of Problem 360, from which it follows that

$$q_{1,2} = 1 + \frac{Z_1}{2Z_2} \pm \sqrt{\left(1 + \frac{Z_1}{2Z_2}\right)^2 - 1}.$$

If the expression under the square root is negative, then q_1 and q_2 are conjugate roots whose moduli are equal to unity, and this corresponds to $|q_1| = |q_2| = 1$. When this expression is positive, q_1 and q_2 are real and different and this corresponds to $|q_1| > 1$, $|q_2| < 1$. The range of Z_1 and Z_2 for the first of the two cases can be found by equating to zero the expression under the square root, and the result is

$$-4 \leq \frac{Z_1}{Z_2} \leq 0.$$

This corresponds to values of ω^2 lying between $c^2/L_1 C_1$ and $c^2(4C_1 + C_2)/C_1 C_2(4L_2 + L_1)$.

362. Consider the n -th closed section of the long artificial transmission line (Fig. 58). It may be regarded as the equivalent circuit for a section of a line with distributed parameters having a length a , inductance ΔL , and capacitance ΔC . For an arbitrary dependence of the current in the line on time, the Kirchhoff

equation for this section will be of the form

$$-\frac{1}{c^2} \Delta L \frac{\partial \mathcal{J}_n}{\partial t} + \frac{q_{n-1,n}}{\Delta C} - \frac{q_{n+1,n}}{\Delta C} = 0, \quad (1)$$

where $q_{n-1,n}$ and $q_{n+1,n}$ are the charges on the upper plates of the two capacitors. From the derivative of (1) with respect to time and relationships $\dot{q}_{n-1,n} = -\mathcal{J}_n + \mathcal{J}_{n-1}$ and $\dot{q}_{n,n+1} = \mathcal{J}_n - \mathcal{J}_{n+1}$, we have

$$\frac{1}{c^2} \Delta L \frac{\partial^2 \mathcal{J}_n}{\partial t^2} + \frac{1}{\Delta C} (2\mathcal{J}_n - \mathcal{J}_{n-1} - \mathcal{J}_{n+1}) = 0. \quad (2)$$

It is now necessary to transform from the variable n to z , which is the distance along the line with distributed parameters. We shall, therefore, substitute

$$\begin{aligned} \mathcal{J}_n(t) &= \mathcal{J}(z, t), \quad \mathcal{J}_{n-1}(t) = \mathcal{J}(z - a, t), \\ \mathcal{J}_{n+1}(t) &= \mathcal{J}(z + a, t) \end{aligned}$$

and evaluate the differences

$$\begin{aligned} \mathcal{J}_n - \mathcal{J}_{n-1} &\simeq \frac{\partial \mathcal{J}}{\partial z} a - \frac{1}{2} \frac{\partial^2 \mathcal{J}}{\partial z^2} a^2, \\ \mathcal{J}_n - \mathcal{J}_{n+1} &\simeq -\frac{\partial \mathcal{J}}{\partial z} a - \frac{1}{2} \frac{\partial^2 \mathcal{J}}{\partial z^2} a^2. \end{aligned}$$

If these differences are substituted into (2), together with $L = \Delta L/a$ and $C = \Delta C/a$, which are the inductance and capacitance per unit length, we have

$$\frac{L}{c^2} \frac{\partial^2 \mathcal{J}}{\partial t^2} = \frac{1}{C} \frac{\partial^2 \mathcal{J}}{\partial z^2}. \quad (3)$$

This is the differential equation for a long lossless line. A real long line will always exhibit losses associated with the finite resistance of the leads and imperfect insulation.

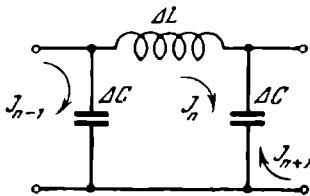


Fig. 58.

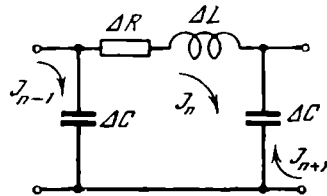


Fig. 59.

Fig. 59 shows the equivalent circuit when the insulation is not assumed to be perfect. The equation for a long line of this kind

(the telegraph equation) can be obtained in a similar way and is

$$\frac{L}{c^2} \frac{\partial^2 \mathcal{J}}{\partial t^2} + R \frac{\partial \mathcal{J}}{\partial t} = \frac{1}{C} \frac{\partial^2 \mathcal{J}}{\partial z^2}, \quad (4)$$

where R is the active resistance of the leads per unit length.

363. From the solution of Eqn. (3) obtained in the preceding solution, we have $\omega = vk$, where $v = c(LC)^{-\frac{1}{2}}$ is the velocity of propagation of waves in the long line, $k = \pi r/l$, $r = 1, 2, 3, \dots$, and L and C are respectively the inductance and capacitance per unit length. In distinction to the line with lumped parameters, here the number of natural oscillations is infinite. This is due to the fact that the long line has an infinite number of degrees of freedom while with lumped parameters the number of degrees of freedom, N , was finite. The absence of dispersion is characteristic of a perfect long line.

364. We shall use Ohm's law in the differential form $\mathbf{j} = \sigma(\mathbf{E} + \mathbf{E}_e)$, where $\mathbf{E}_e$ is the external field strength. The field strengths can be expressed in terms of the potential:

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{E}_e = \frac{\mathbf{j}}{\sigma} + \nabla\varphi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$

Assuming that the conductor is thin, we can integrate both parts of the latter equation along the circuit:

$$\oint \mathbf{E}_e \cdot d\mathbf{l} = \oint \frac{\mathbf{j}}{\sigma} \cdot d\mathbf{l} + \oint \nabla\varphi \cdot d\mathbf{l} + \frac{1}{c} \oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l}. \quad (1)$$

The integral on the left-hand side of (1) is equal to the external e.m.f. $\mathcal{E}_e$ which is applied to the circuit. The integral $\oint (\mathbf{j}/\sigma) \cdot d\mathbf{l} = \mathcal{J}R$ represents the Joule losses per unit time, while $\oint \nabla\varphi \cdot d\mathbf{l} = \oint d\varphi = 0$. The last integral may be transformed as follows. If

$$\mathbf{A} = \frac{1}{c} \oint \frac{\mathcal{J} \left(t - \frac{r}{c} \right) d\mathbf{l}'}{r},$$

$$\mathcal{J} \left(t - \frac{r}{c} \right) = \mathcal{J}_0 e^{-i\omega \left(t - \frac{r}{c} \right)}, \quad \frac{\partial \mathbf{A}}{\partial t} = -i\omega \mathbf{A},$$

is substituted into (1), and the real and imaginary parts are separated, we have

$$\mathcal{E}_e(t) = \mathcal{J}(t) \left[\left(R + \frac{\omega}{c^2} \oint \oint \frac{\sin \frac{\omega r}{c} d\mathbf{l} \cdot d\mathbf{l}'}{r} \right) - \frac{i\omega}{c^2} \oint \oint \frac{\cos \frac{\omega r}{c} d\mathbf{l} \cdot d\mathbf{l}'}{r} \right].$$

The expression in square brackets represents the complex impedance of the circuit. The effective resistance is equal to $R + R_r(\omega)$ where

$$R_r(\omega) = \frac{\omega}{c^2} \oint \oint \frac{\sin \frac{\omega r}{c}}{r} dl \cdot dl'.$$

The resistance R is associated with losses giving rise to the heating of the conductor, while $R_r(\omega)$ represents the energy lost by radiation and is referred to as the radiation resistance (cf. next problem).

The reactance is equal to $-i\omega L(\omega)/c^2$ where

$$L(\omega) = \oint \oint \frac{\cos \omega r/c}{r} dl \cdot dl'$$

is the inductance and is a function of frequency.

Consider the limit where $c/\omega = \lambda/2\pi \gg l$, where the quantity l is a linear dimension of the circuit. In the region of integration $\omega r/c \ll 1$. When the cosine is expanded into a series, and third- and higher-order terms are neglected, we have

$$L(\omega) \approx \oint \oint \frac{dl \cdot dl'}{r} - \frac{1}{2} \left(\frac{\omega}{c} \right)^2 \oint \oint r dl \cdot dl'.$$

The first term in this equation is independent of frequency and represents the ordinary inductance, while the second term gives a correction which is significant at high frequencies. In practice, the self-inductance should be calculated from (V.18), since the integral $\oint \oint dl \cdot dl'/r$ is divergent. This divergence is due to the fact that the conductor is assumed to be infinitely thin (linear).

In the expansion for the sine in the expression for $R_r(\omega)$, the third-order term must also be included since the integral of the first (linear) term is zero. The radiation resistance is then given by

$$R_r(\omega) = -\frac{\omega^4}{6c^5} \oint \oint r^2 dl \cdot dl'.$$

$$365. \quad \begin{aligned} L(\omega) &= L + \frac{64\pi^4}{3} \cdot \frac{a^4}{\lambda^3}, \\ R_r(\omega) &= \frac{2\pi^2}{3c} \left(\frac{2\pi a}{\lambda} \right)^4. \end{aligned}$$

The current-carrying ring acts as a magnetic dipole. The energy

emitted per unit time is equal to $2\ddot{\mathbf{m}}^2/3c^3$ where $\mathbf{m}$ is the magnetic dipole moment. The coefficient of proportionality between the emitted energy and the mean square current is equal to $2\pi^2 a^2 \omega^4/3c^5$ and is identical with $R_r(\omega)$.

2. Eddy currents and skin effect

$$366. \quad H(x) = H_0 \left(\frac{\sinh^2 x/b + \cos^2 x/b}{\sinh^2 h/b + \cos^2 h/b} \right)^{\frac{1}{2}}, \quad H_0 = \frac{4\pi}{c} \mathcal{J}_0 n; \quad \delta = \frac{c}{\sqrt{2\pi\mu\sigma\omega}}.$$

When $\delta \ll h$, the magnetic field is $H(x) = H_0 \exp[-(h - |x|)/\delta]$. When $\delta \gg h$, the field is $H(x) = H_0$ (cf. Problem 247).

367. Since the system is symmetric with respect to the axis of the cylinder and the primary magnetic field H_0 is uniform, it is clear that the currents induced in the cylinder will follow circular paths which lie in planes perpendicular to the axis. These currents will produce the same magnetic field as that due to a large number of separate coaxial solenoids. However, the field due to the solenoid is zero outside the system; inside it will be parallel to the axis of the solenoid. Thus, the total magnetic field outside the cylinder is equal to H_0 while inside the cylinder it is given by the first equation in (VII.12), which in view of the axial symmetry is of the form

$$\frac{d^2 H}{dr^2} + \frac{1}{r} \frac{dH}{dr} + k^2 H = 0,$$

where

$$k^2 = \frac{1+i}{\delta}, \quad H = H_z(r), \quad H_a = H_r = 0.$$

The boundary condition is $H(a) = H_0$.

The solution which is finite at $r = 0$ and which satisfies this boundary condition can be expressed in terms of the zero-order Bessel function:

$$H = H_0 \frac{J_0(kr)}{J_0(ka)}.$$

Outside the cylinder $H = H_0$ when $a \leq r \leq b$, and $H = 0$ when $r > b$. The current density and the electric field inside the cylinder may be calculated from (VII.11):

$$j = j_a = \sigma E_a = \frac{kc}{4\pi} \cdot \frac{J_1(kr)}{J_0(ka)} H_0, \quad E_r = E_z = 0.$$

To determine the electric field outside the cylinder we shall use the Maxwell equation for curl $\mathbf{E}$ in the integral form

$$\oint E_t dl = \frac{i\omega}{c} \int B_n dS.$$

Inside the cylinder there is only a single component of the electric field (E_α). It follows from the boundary condition on the surface of the cylinder and from the symmetry of the system that outside the cylinder the field $\mathbf{E}$ will also have the single component E_α which is a function of r only. If the contour of integration is taken to be circular, the line integral turns out to be $2\pi r E_\alpha$. The surface integral may be evaluated with the aid of Eqn. (12) of Appendix III. The final result is

$$\begin{aligned} E_\alpha &= \frac{kcH_0}{4\pi a} \cdot \frac{J_1(ka)}{J_0(ka)} \cdot \frac{a}{r} + \frac{H_0}{2r} (r^2 - a^2), & a \leq r \leq b, \\ E_\alpha &= \frac{kcH_0}{4\pi a} \cdot \frac{J_1(ka)}{J_0(ka)} \cdot \frac{a}{r} + \frac{H_0}{2r} (b^2 - a^2), & r > b. \end{aligned}$$

In the absence of the cylinder, *i.e.* when $a = 0$, the field is given by

$$E_\alpha = \frac{1}{2} H_0 r \quad (r < b), \quad E_\alpha = \frac{H_0 b^2}{2r} \quad (r > b).$$

Thus, the additional magnetic field due to the presence of the cylinder is equal to zero for $r > a$, even though the additional electric field is finite. This is due to the fact that the exact equation $\text{curl } \mathbf{H} = (\partial \mathbf{D} / \partial t) / c$ which holds outside the conductor was replaced by the approximate equation $\text{curl } \mathbf{H} = 0$ (the displacement current is neglected on the quasi-stationary approximation). In the rigorous solution the additional magnetic field outside the conductor will also be finite (*cf.* Problem 418 where the diffraction of a plane wave by a conducting cylinder is considered).

368. At low frequencies ($|ka| \ll 1$ or $\delta \gg a$)

$$j = i \frac{cH_0}{4\pi} \cdot \frac{r}{\delta^2} = \frac{i\omega H_0}{2c} r,$$

and hence the current density is a linear function of r and is proportional to the frequency.

At high frequencies ($|ka| \gg 1$ or $\delta \ll a$) it is possible to use the asymptotic expression for the Bessel function and the result is

$$j = (i - 1) \frac{cH_0}{4\pi\delta} \sqrt{\frac{a}{r}} e^{-(1+i)\frac{a-r}{\delta}}.$$

When $a - r \gg \delta$, the current density becomes infinitesimal. Thus at high frequencies the current is largely concentrated in a thin surface layer.

369.

$$Q = -\frac{\pi a^2 n^2 \mathcal{J}_0^2}{\sigma} \operatorname{Re} \left[\frac{k J_1(ka)}{J_0(ka)} \right], \quad k^2 = i \frac{4\pi\mu\sigma\omega}{c^2}.$$

When $|ka| \ll 1$ (low frequencies)

$$Q = \frac{\pi n^2 \mathcal{J}_0^2}{16\sigma} \left(\frac{a}{\delta} \right)^4 = \pi^3 \sigma \left(\frac{a^2 \mu n \omega \mathcal{J}_0}{c^2} \right)^2.$$

When $|ka| \gg 1$ (high frequencies)

$$Q = \frac{\pi n^2 \mathcal{J}_0^2}{\sigma} \left(\frac{a}{\delta} \right) = \frac{a n^2 \mathcal{J}_0^2}{c} \sqrt{\frac{2\pi^3 \mu \omega}{\sigma}}.$$

The energy dissipation at low and high frequencies is proportional to ω^2 and $\omega^{\frac{1}{2}}$ respectively.

370.

$$\beta = \beta' + i\beta'' = -\frac{a^2}{4} \left[1 - \frac{2}{ka} \cdot \frac{J_1(ka)}{J_0(ka)} \right], \quad k^2 = i \frac{4\pi\sigma\omega}{c^2}.$$

When $|ka| \gg 1$ (high frequencies)

$$\beta' = -\frac{a^2}{4} \left(1 - \frac{c}{a \sqrt{2\pi\sigma\omega}} \right), \quad \beta'' = \frac{ca}{4 \sqrt{2\pi\sigma\omega}}.$$

It follows that at high frequencies $\beta'' \rightarrow 0$, *i.e.* the losses are reduced as a result of the absence of the field in the conductor.

When $|ka| \ll 1$ (low frequencies)

$$\beta' = -\frac{\pi^2 a^6 \sigma^2 \omega^2}{12c^4}, \quad \beta'' = \frac{\pi a^4 \sigma \omega}{8c^2}.$$

Thus $\beta \rightarrow 0$ when $\omega \rightarrow 0$; this is related to the fact that $\mu = 1$, *i.e.* the static magnetic polarisability is zero.

371. The magnetic moment due to the induced currents will, in view of the symmetry of the system, be parallel to the external magnetic field. Hence, the total magnetic field $\mathbf{H}_2$ outside the cylinder will be of the form

$$\mathbf{H}_2(\mathbf{r}) = \frac{4\mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^4} - \frac{2\mathbf{m}}{r^2} + \mathbf{H}_0, \quad (1)$$

where $\mathbf{m}$ is the unknown magnetic moment per unit length of the cylinder, which is parallel to $\mathbf{H}_0$, and $\mathbf{r}$ is the position vector in the plane perpendicular to the axis of the cylinder. The vector

potential corresponding to $\mathbf{H}_2$ is $\mathbf{A}_2 = 2(\mathbf{m} \times \mathbf{r})/r^2 + (\mathbf{H}_0 \times \mathbf{r})$. The components of $\mathbf{A}_2$ are

$$A_{2z} \equiv A_2 = \left(\frac{2m}{r} + H_0 r\right) \sin \alpha, \quad A_{2r} = A_{2\alpha} = 0. \quad (2)$$

The angle α is measured from the direction of $\mathbf{H}_0$.

Thus, the vector potential in the external space has only a longitudinal component which is proportional to $\sin \alpha$. The continuity conditions for the field components at the boundary may be satisfied if the vector potential in the internal region is sought in an analogous form:

$$A_{1z} \equiv A_1 = F(r) \sin \alpha, \quad A_{1r} = A_{1\alpha} = 0. \quad (3)$$

The electric field $\mathbf{E}$ will in general depend on both $\mathbf{A}$ and φ . As usual, we shall impose the additional condition

$$\operatorname{div} \mathbf{A} + \frac{\epsilon}{c} \frac{\partial \varphi}{\partial t} = 0.$$

Since $\operatorname{div} \mathbf{A} = 0$, which follows from (2) and (3), we have $\partial \varphi / \partial t = -i\omega\varphi = 0$ so that $\mathbf{E} = -(\partial \mathbf{A} / \partial t) / c = i\omega \mathbf{A} / c$. It follows that $\mathbf{A}$ will satisfy the same equation as the electric field [cf. (VII.12)]. The solution of this equation which is bounded at $r = 0$ is the Bessel function

$$F(r) = C J_1(kr), \quad A_1 = C J_1(kr) \sin \alpha. \quad (4)$$

The constants C and m in (4) and (2) may be determined from the condition that the internal ($\mathbf{H}_1$) and external ($\mathbf{H}_2$) fields must be equal at $r = a$ (surface of the cylinder). Using Eqn. (9) of Appendix III we have

$$C = \frac{2H_0}{kJ_0(ka)}, \quad m = -\frac{a^2 H_0}{2} \left(1 - \frac{2}{ka} \cdot \frac{J_1(ka)}{J_0(ka)}\right). \quad (5)$$

It follows from the expression for m that the transverse magnetic polarisability of the cylinder

$$\beta = -\frac{a^2}{2} \left[1 - \frac{2}{ka} \cdot \frac{J_1(ka)}{J_0(ka)}\right] \quad (6)$$

is larger by a factor of two as compared with its longitudinal polarisability (cf. Problem 370). The components of the magnetic field inside the cylinder can be determined from (4) and (5)

and are given by

$$\begin{aligned} H_{1r} &= \frac{1}{r} \frac{\partial A_1}{\partial \alpha} = 2H_0 \frac{J_1(kr)}{krJ_0(ka)} \cos \alpha, \quad H_{1z} = 0, \\ H_{1\alpha} &= -\frac{\partial A_1}{\partial r} = -2H_0 \frac{J'_1(kr)}{J_0(ka)} \sin \alpha. \end{aligned} \quad (7)$$

Finally, the current density is given by $\mathbf{j} = (c/4\pi) \text{curl } \mathbf{H}$, and hence

$$j_z = -\frac{cH_0}{2\pi} \cdot \frac{J_1(kr)}{J_0(ka)} \sin \alpha, \quad j_\alpha = j_r = 0. \quad (8)$$

It is clear from this formula that the currents in the two halves of the cylinder, $0 \leq \alpha \leq \pi$ and $\pi \leq \alpha \leq 2\pi$, will always flow in opposite directions. The total current through the cross-section of the cylinder is zero. The radial dependence of the current is the same as for a cylinder placed in a longitudinal field (*cf.* Problem 368). It should be noted, however, that for the longitudinal field the currents flow over circles in planes perpendicular to the axis of the cylinder, while for the transverse field the currents flow along the axis of the cylinder.

372. The average energy dissipation per unit length of the cylinder is most simply calculated with the aid of (VII.17) by considering the flux of energy through the lateral surface of the cylinder. From the results of Problem 371, we have

$$Q = -\frac{ac^2H_0^2}{8\pi\sigma} \operatorname{Re} \left(\frac{kJ_1(ka)}{J_0(ka)} \right).$$

The same result can be obtained with the aid of (VII.16) where the integration of the product of Bessel functions can be carried out with the aid of Eqn. (13) of Appendix III.

373. The electric and magnetic fields inside the cylinder may be found as in Problem 371 for a linearly polarised external field:

$$\left. \begin{aligned} H_r &= -\frac{2H_0 J_1(kr)}{krJ_0(ka)} e^{i\alpha}, \quad H_\alpha = -2iH_0 \frac{J'_1(kr)}{J_0(ka)} e^{i\alpha}, \\ j_z &= \frac{ickH_0}{2\pi} \cdot \frac{J_1(kr)}{J_0(ka)} e^{i\alpha}. \end{aligned} \right\} \quad (1)$$

The force per unit volume of the cylinder is given by

$$\mathbf{f} = \frac{1}{c} (\mathbf{j} \times \mathbf{H}), \quad (2)$$

where it is assumed that $\mu = 1$ inside the cylinder. The radial

component of this force gives rise to a radial pressure, while the azimuthal component is responsible for a couple. Since $\mathbf{j}$ and $\mathbf{H}$ are complex quantities, the average value of the azimuthal component of the force is

$$\bar{f}_\alpha = \frac{1}{2c} \operatorname{Re} (j_z H_r^*). \quad (3)$$

The couple per unit length of the cylinder may be obtained by multiplying the average force given by (3) by r and integrating over the cross-section of the cylinder. The integral may be evaluated with the aid of Eqn. (13) of Appendix III. The result is

$$\bar{N} = -\frac{aH_0^2}{|k|^2} \operatorname{Re} \left(k \frac{J_1(ka)}{J_0(ka)} \right). \quad (4)$$

The same result may be obtained in another way. The moment of forces may be expressed in terms of the magnetic moment of the system

$$\mathbf{N}(t) = \mathbf{m}(t) \times \mathbf{H}_0(t). \quad (5)$$

Writing $\bar{N}_z = \bar{N}$ in terms of the complex amplitudes of $\mathbf{H}_0$ and $\mathbf{m}$, and $\mathbf{m}$ in terms of the transverse magnetic polarisability of the cylinder (*cf.* Problem 371), we arrive again at Eqn. (4).

At low frequencies, Eqn. (4) yields

$$\bar{N} = \frac{1}{4} \cdot \frac{a^4 H_0^2}{\delta^2} = \frac{\pi \sigma \omega}{4c^2} H_0^2 a^4, \quad (6)$$

while at high frequencies,

$$\bar{N} = \frac{1}{2} a \delta H_0^2 = \frac{ca}{\sqrt{8\pi\sigma\omega}} H_0^2. \quad (7)$$

It is clear from these formulae that the couple is zero both at very low and very high frequencies. If the field is linearly polarised, the average couple is zero (formally this follows from the fact that the integral with respect to α in the formula for N is zero; *cf.* Problem 371 in which $\mathbf{j}$ and $\mathbf{H}$ are found for this situation). The couple is produced by a "rotating" field. The induction motor is based on the phenomenon discussed in this problem.

374. Consider a system of coordinates ξ, η, z which rotates together with the cylinder and whose z -axis lies along the axis of the cylinder. In this system the external magnetic field may be

written in the form

$$\mathbf{H}_0(t) = (\mathbf{H}_{01} - i\mathbf{H}_{02}) e^{-i\omega t},$$

where $\mathbf{H}_{01}$ and $\mathbf{H}_{02}$ are constant vectors of equal length H_0 and are parallel to the ξ and η axes. A field of this kind was discussed in Problem 373. The couple, which will tend to retard the cylinder, is given by

$$\bar{N} = -\frac{aH_0^2}{|k|^2} \operatorname{Re} \left(k \frac{J_1(ka)}{J_0(ka)} \right).$$

375. It was shown in the solution of Problem 367 that the currents induced in the cylinder as a result of changes in the external longitudinal field do not give rise to an additional magnetic field outside the cylinder. Inside the cylinder, on the other hand, the field due to these currents is longitudinal and is a function of r only. This field satisfies the equation

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} - \frac{4\pi\mu\sigma}{c^2} \frac{\partial H}{\partial t} = 0. \quad (1)$$

It is clear that the magnetic field inside the cylinder will eventually be damped out. Hence, the special solution of (1) will be sought in the form $F(r) \exp(-\gamma t)$ where $\gamma > 0$ is a constant. The function $F(r)$ satisfies the Bessel equation

$$F''(r) + \frac{1}{r} F'(r) + k^2 F(r) = 0, \quad (2)$$

where $k^2 = 4\pi\mu\sigma\gamma/c^2$.

The solution of (2) which is finite at $r = 0$ is of the form $F(r) = CJ_0(kr)$. Since the external field H_0 is switched off, and the additional field due to the induced currents outside the cylinder is zero, the condition at the boundary is $H|_{r=a} = 0$, *i.e.*

$$J_0(ka) = 0. \quad (3)$$

It follows that $k_m a = \beta_m$, $m = 1, 2, \dots$, where the quantities β_m are the zeros of J_0 . The possible values of γ are

$$\gamma_m = \frac{c^2 \beta_m^2}{4\pi\mu\sigma a^2}. \quad (4)$$

The general solution of (1) which corresponds to the boundary value problem under consideration is

$$H(r, t) = \sum_m C_m J_0(k_m r) e^{-\gamma_m t}, \quad (5)$$

where C_m may be determined from the initial condition

$$H(r, 0) = \sum_m C_m J_0(k_m r). \quad (6)$$

Using the orthogonality condition

$$\int_0^1 x J_0(k_m x) J_0(k_n x) dx = \frac{1}{2} [J'_0(k_m)]^2 \delta_{mn}, \quad (7)$$

we have

$$C_m = \frac{2}{a^2 [J'_0(k_m a)]^2} \int_0^a H(r, 0) J_0(k_m r) r dr. \quad (8)$$

Initially, the field $H(r, 0)$ is equal to the external field H_0 , since the constant magnetic field is unaffected by the presence of an infinite cylinder whose axis is parallel to it. Using Eqns. (12) and (9) of Appendix III we have

$$C_m = \frac{2H_0}{(k_m a) J_1(k_m a)}. \quad (9)$$

The rate of attenuation of the field will be determined by the smallest of the γ_m , *i.e.* by γ_1 . The latter quantity may be obtained by substituting the smallest root of the Bessel function ($\beta_1 \simeq 2.4$) into (4).

376. On the zero-order approximation (in frequency) the magnetic field inside the sphere is given by

$$\mathbf{H} = \frac{3}{\mu + 2} \mathbf{H}_0, \quad (1)$$

which was obtained in Problem 281. The electric field inside the sphere will be zero on this approximation [*cf.* (VII.11)], since a constant magnetic field does not give rise to an electric field. In order to determine the electric field on the next approximation (linear in frequency) we shall use Eqn. (VII.10) in an integral form.

It is clear from the symmetry of the system that the currents in the sphere will flow in circular paths in planes perpendicular to $\mathbf{H}_0$ and the electric field will be in the same direction. In the spherical system of coordinates with the z -axis parallel to $\mathbf{H}_0$ we have

$$E = \frac{i\omega H}{2c} r \sin \vartheta, \quad j = \sigma E, \quad (2)$$

where H is given by (1). The heat Q liberated in the sphere is

obtained by integrating $q = \frac{1}{2}\sigma |E|^2$ over the volume of the sphere. The result is

$$Q = \frac{3\pi a^5 \omega^2 H_0}{5c^2 (\mu + 2)^2}. \quad (3)$$

377. The magnetic field outside the sphere is

$$\mathbf{H} = \mathbf{H}_0 + \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{m}}{r^3},$$

where $\mathbf{m} = -a^3 \mathbf{H}_0 / 2$ and $\beta = -a^3 / 2$ is the magnetic polarisability of the sphere for a strong skin effect. Inside the sphere

$$H_\vartheta = -\frac{3}{2} H_0 e^{-(1-l)\frac{z}{a}} \sin \vartheta, \quad H_r = H_a = 0,$$

where z is measured from the surface along the inward normal to the conductor, and the polar axis of the spherical system is parallel to $\mathbf{H}_0$;

$$Q = \frac{3a^2 c}{8} \sqrt{\frac{\mu \omega}{2\pi \sigma}} H_0^2.$$

378. For a strong skin effect the field inside the ellipsoid is zero, while the field outside the ellipsoid satisfies the equations $\text{curl } \mathbf{H} = 0$, $\text{div } \mathbf{H} = 0$ and the boundary conditions $H_n|_S = 0$, $\mathbf{H}|_{r \rightarrow \infty} \rightarrow \mathbf{H}_0$ where $\mathbf{H}_0$ is the external field and S represents the surface of the ellipsoid.

Compare now this problem with the case of a dielectric ellipsoid with $\varepsilon = 0$ placed in a uniform electric field. The electric field outside the ellipsoid will satisfy the equation

$$\text{curl } \mathbf{E} = 0, \quad \text{div } \mathbf{E} = 0, \quad (1)$$

and the boundary conditions

$$E_n|_S = \varepsilon E_{n \text{ int}}|_S = 0, \quad \mathbf{E}|_{r \rightarrow \infty} \rightarrow \mathbf{E}_0. \quad (2)$$

The conditions for the tangential components of $\mathbf{E}$ need not be considered since (1) and (2) uniquely define $\mathbf{E}$ in the external region.

We see that the problem of a conducting ellipsoid for a strong skin effect is identical with the problem of a dielectric ellipsoid for which $\varepsilon = 0$. The magnetic polarisabilities in the direction of the principal axes of the ellipsoid are obtained by substituting $\varepsilon_1 = 0$ into the formulae given in the solution

of Problem 200:

$$\beta^{(i)} = - \frac{V}{4\pi(1-n^{(i)})}, \quad (3)$$

where $n^{(i)}$ is the depolarisation coefficient and V is the volume of the ellipsoid.

For a very prolate ellipsoid of revolution with semi-axis a and $b \gg a$ (a rod), we have (cf. Problem 198)

$$\beta_{\perp} = -\frac{2}{3}a^2b, \quad \beta_{\parallel} = -\frac{1}{3}a^2b.$$

For a very oblate ellipsoid ($b \ll a$; a disc)

$$\beta_{\perp} = -\frac{2a^3}{3\pi}, \quad \beta_{\parallel} = -\frac{1}{3}a^2b \rightarrow 0 \quad \text{when } b \rightarrow 0.$$

379. In view of the axial symmetry of the system, both the distribution of eddy currents in the sphere and the electric field will exhibit axial symmetry. Hence, the electric field will have only a single component, which cannot be a function of α , so that $E_{\alpha} = f(r, \vartheta)$.

The solution of Eqn. (VII.12) for the resultant electric field $\mathbf{E}$ will be sought in the form

$$E_{\alpha} = F(r) \sin \vartheta, \quad E_r = E_{\vartheta} = 0.$$

Using the expression for the Laplacian of a vector in spherical coordinates (cf. Problem 47), we obtain the equation for $F(r)$, and this can be reduced to the Bessel equation by substituting $F(r) = r^{-\frac{1}{2}}\chi(r)$. The solution of this equation which is bounded at $r = 0$ is

$$\chi(r) = AJ_{\frac{3}{2}}(kr).$$

The magnetic field inside the sphere may be determined from Eqn. (VII.10). The magnetic field in the external region is the sum of the external field $\mathbf{H}_0$ and the magnetic field due to the dipole $\mathbf{m}$ whose direction is parallel to that of $\mathbf{H}_0$. Thus,

$$\mathbf{H}_2 = \mathbf{H}_0 + \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{m}}{r^3}.$$

The constants A and m can be determined from the boundary conditions for $\mathbf{H}$ at the surface of the sphere. If the Bessel functions of half-integral order are expressed in terms of trigonometric

functions, then

$$m = -\frac{a^3}{2} \left(1 - \frac{3}{k^2 a^2} + \frac{3}{ka} \cotan ka \right) H_0, \quad A = \frac{3la\omega}{c \sin ka} \sqrt{\frac{\pi}{8k}} H_0.$$

380.

$$Q = -\frac{3a\delta^2 H_0^2}{8} \left(1 - \frac{a}{\delta} \cdot \frac{\sinh 2a/\delta + \sin 2a/\delta}{\cosh 2a/\delta - \cos 2a/\delta} \right).$$

381.

$$R = \frac{l}{ca} \sqrt{\frac{\omega}{\pi\sigma}} \operatorname{Re} \left[(1 + i) \frac{J_0(ka)}{J_1(ka)} \right],$$

where $kc = (1 + i)(2\pi\sigma\omega)^{\frac{1}{2}}$.

When $|ka| \ll 1$ (low frequencies)

$$R = R_0 \left[1 + \frac{1}{12} \left(\frac{\pi\sigma\omega a^2}{c^2} \right)^2 \right],$$

where $R_0 = l/\pi\sigma a^2$ is the d.c. resistance.

When $|ka| \gg 1$ (high frequencies)

$$R = \frac{1}{\sigma} \cdot \frac{l}{2\pi a\delta} = \frac{l}{ca} \sqrt{\frac{\omega}{2\pi\sigma}}.$$

It follows from the latter formula that the effective cross-section of a conductor for a strong skin effect is equal to $2\pi a\delta$.

382.

$$R = \frac{\omega\delta_2}{2ac^2} \cdot \frac{(\delta_1^2 - \delta_2^2) \sin 2h/\delta_2 - (\delta_1^2 + \delta_2^2) \sinh 2h/\delta_2 + 2\delta_1\delta_2 \cos 2h/\delta_2}{(\delta_1 \sin h/\delta_2 + \delta_2 \cos h/\delta_2)^2 + (\delta_1^2 + \delta_2^2) \sinh^2 h/\delta_2},$$

where

$$\delta_1 = \frac{c}{\sqrt{2\pi\sigma_1\omega}}, \quad \delta_2 = \frac{c}{\sqrt{2\pi\sigma_2\omega}}.$$

383.

$$H' = \frac{2H_0}{ku \sinh kh + 2 \cosh kh},$$

where $k = (1 + i)/\delta$.

When $|kh| \ll 1$ (low frequencies), $H' = H_0$ and hence the presence of the cylindrical shell has no effect on the magnitude of the field. When $|kh| \gg 1$ (high frequencies), we have

$$\sinh kh \simeq \cosh kh \simeq \frac{1}{2} e^{(1+l)\frac{h}{\delta}}.$$

Since $a \gg \delta$, it follows that

$$H' = (1-l)\frac{\delta}{2a} e^{-(1+l)\frac{h}{\delta}} H_0, \quad H_0 \gg |H'|.$$

The strong reduction in the field is due to the fact that the currents, induced in the shell give rise to an additional field in the cavity which acts in the opposite direction.

384.

$$j = \frac{2i\mathcal{G}_0\mu\sigma\omega}{c^2ka} \cdot \frac{\sinh k(h-x)}{\cosh kh},$$

where x is measured from the surface along the inward radial direction;

$$R = \frac{1}{2\pi a \delta \sigma} \cdot \frac{\sinh 2h/\delta - \sin 2h/\delta}{2(\sinh^2 h/\delta + \cos^2 h/\delta)}.$$

When $\delta \ll h$, the hollow and solid conductors have the same resistance.

385. Consider the cylindrical system of coordinates illustrated in Fig. 60. For a weak skin effect, the component of the

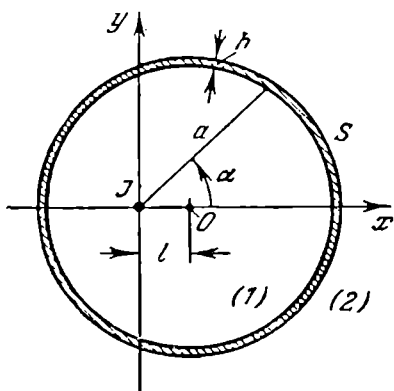


Fig. 60.

magnetic field which is tangential to the wall of the tube at the surface S of this wall must satisfy the condition

$$H_{2\tau} - H_{1\tau} = \frac{4\pi}{c} i, \quad (1)$$

where $i = \sigma h E = \zeta E$ is the surface current and ζ is the surface conductance.

The electric field will clearly have a single component only (z component) and should be continuous across S :

$$E_1 = E_2 = E. \quad (2)$$

The remaining steps of the solution are very similar to those in the solution of Problem 159. To within terms of the order of l/a ,

the equation of the boundary may be written in the form

$$r = a + l \cos \alpha. \quad (3)$$

The vector potential, whose direction is parallel to that of the current, will be sought in the form

$$\begin{aligned} A_1 &= -\frac{2\mathcal{J}}{c} \ln \frac{r}{a} + C_1 r \cos \alpha + C, \\ A_2 &= -\frac{2\mathcal{J}'}{c} \ln \frac{r}{a} + \frac{B_1}{r} \cos \alpha, \end{aligned} \quad (4)$$

where C_1 and B_1 are functions of time and are of the first order of small quantities with respect to l/a , and $\mathcal{J}'$ is of zero order with respect to l/a .

For a weak skin effect ($h \ll \delta$), the vector potential satisfies the condition

$$A_1 = A_2 \quad \text{for} \quad r = a + l \cos \alpha. \quad (5)$$

Hence when terms of the order of $(l/a)^2$ are neglected

$$B_1 = a^2 C_1 + \frac{2(\mathcal{J}' - \mathcal{J})l}{c}, \quad C = 0. \quad (6)$$

The quantity H_r in the boundary condition (1) may be replaced by H_α . It can be easily verified that this will give rise to an error of the order of $(l/a)^2$. Since

$$H_\alpha = -\frac{\partial A}{\partial r}, \quad l = \zeta E = -\frac{\zeta}{c} \frac{\partial A}{\partial t},$$

we have over S

$$\frac{\partial A_1}{\partial r} - \frac{\partial A_2}{\partial r} = -\frac{4\pi\zeta}{c^2} \frac{\partial A}{\partial t},$$

or, to within terms of the order of l/a ,

$$\frac{2(\mathcal{J}' - \mathcal{J})}{ca} + 2C_1 \cos \alpha = \frac{4\pi\zeta}{c^2} \left[\frac{2}{ca} \frac{d(\mathcal{J}l)}{dt} + a \frac{dC_1}{dt} \right] \cos \alpha.$$

It follows at once that $\mathcal{J} = \mathcal{J}'$, which is due to the fact that the skin effect was assumed to be weak. The differential equation for C_1 is

$$\frac{dC_1}{dt} + \rho C_1 = \frac{2}{a^2 c} \frac{d(\mathcal{J}l)}{dt}. \quad (7)$$

The parameter $\rho = c^2/2\pi a \zeta$ is equal to the resistance per unit length of the tube in electromagnetic units.

The solution of Eqn. (7) may easily be obtained by the method of variation of arbitrary constants. It is of the form

$$C_1 = \frac{2}{ca^2} \int_{-\infty}^t e^{\rho(\tau-t)} \frac{d}{d\tau} [\mathcal{J}(\tau) l(\tau)] d\tau,$$

where it has been assumed that the current is zero for $t \rightarrow -\infty$.

The force f per unit length of the current $\mathcal{J}$ may be calculated from the formula

$$f_x = -\frac{1}{c} \mathcal{J} H'_y,$$

where H'_y is the magnetic field along the straight line parallel to the current $\mathcal{J}$ due to the current flowing in the tube. The vector potential for this field is

$$A' = C_1 r \cos \alpha = C_1 y,$$

and hence

$$H'_y = -\frac{\partial A'}{\partial y} = -C_1.$$

Finally,

$$f_x = \frac{2\mathcal{J}(t)}{c^2 a^2} \int_{-\infty}^t e^{\rho(\tau-t)} \frac{d}{d\tau} [\mathcal{J}(\tau) l(\tau)] d\tau.$$

Consider some special cases. When $\mathcal{J} = \text{const}$,

$$f_x = \frac{2\mathcal{J}^2}{c^2 a^2} \int_{-\infty}^t e^{\rho(\tau-t)} \dot{l}(\tau) d\tau.$$

When the current departs from the axis of the cylinder ($\dot{l} > 0$) there will be a force tending to prevent this departure. For a slow motion ($\ddot{l} \ll \rho \dot{l}$), integration by parts will yield

$$f_x = \frac{2\mathcal{J}^2}{c^2 a^2} \left(\frac{\dot{l}}{\rho} - \frac{\ddot{l}}{\rho^2} + \dots \right).$$

In particular, for a uniform displacement $l = vt$ the retarding force will be

$$f_x = \frac{2\mathcal{J}^2 v}{c^2 a^2 \rho}.$$

386.

$$f_x = \frac{2\mathcal{J}^2(t) l(t)}{c^2 a^2}.$$

Chapter

8

PROPAGATION OF ELECTROMAGNETIC WAVES

1. Plane waves in a homogeneous medium.

Reflection and refraction.

Wave packets

387. The amplitude of the first wave is $\mathbf{E}_1 = a\mathbf{e}_x$, the amplitude of the second wave is $\mathbf{E}_2 = b \exp(i\chi)\mathbf{e}_y$; a and b are real. The resultant amplitude is

$$\mathbf{E}_0 = \mathbf{E}_1 + \mathbf{E}_2 = a\mathbf{e}_x + be^{i\chi}\mathbf{e}_y.$$

In order to discuss the polarisation it is convenient to shift the origin from which the phases are measured so that the vibrations

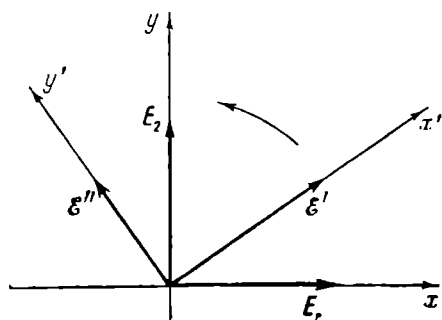


Fig. 61.

taking place along the mutually perpendicular directions differ in phase by $\pi/2$. Consider a new amplitude given by $\mathbf{E}'_0 = \mathbf{E}_0 \exp(i\alpha) = \mathcal{E}' - i\mathcal{E}''$, where the vectors $\mathcal{E}'$ and $\mathcal{E}''$ are real and $\mathcal{E}' \cdot \mathcal{E}'' = 0$ (Fig. 61). Hence

$$\begin{aligned} \mathcal{E}' &= a \cos \alpha \cdot \mathbf{e}_x + b \cos(\alpha - \chi) \cdot \mathbf{e}_y \\ \mathcal{E}'' &= a \sin \alpha \cdot \mathbf{e}_x + b \sin(\alpha - \chi) \cdot \mathbf{e}_y. \end{aligned} \quad (1)$$

The phase difference α may be obtained from the condition $\mathcal{E}' \cdot \mathcal{E}'' = 0$:

$$a^2 \cos \alpha \sin \alpha + b^2 \sin(\alpha - \chi) \cos(\alpha - \chi) = 0,$$

so that

$$\tan 2\alpha = \frac{b^2 \sin 2\chi}{a^2 + b^2 \cos 2\chi}. \quad (2)$$

Having determined α from Eqn. (2), we can substitute it into Eqn. (1) and determine $\mathcal{E}'$ and $\mathcal{E}''$. Let us now introduce new axes $x' \parallel \mathcal{E}'$ and $y' \parallel \mathcal{E}''$ in the xy plane. The components along

these axes are then given by

$$E_{x'} = \mathcal{E}' \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \alpha),$$

$$E_{y'} = \mathcal{E}'' \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \alpha).$$

It is clear that $E_{x'}^2/\mathcal{E}'^2 + E_{y'}^2/\mathcal{E}''^2 = 1$, *i.e.* the end point of the vector $\mathbf{E}$ describes an ellipse.

In general $\mathcal{E}', \mathcal{E}'' \neq 0$. The vibrations along the x' -axis lead the vibrations along the y' -axis by $\pi/2$. If the x', y', z system forms a right-handed set (Fig. 61), then for an observer looking along the negative direction of the z -axis, the vector $\mathbf{E}$ will rotate in the anti-clockwise direction. The polarisation is then said to be elliptical and laevorotatory. If the x', y', z -axes form a left-handed system, then $\mathbf{E}$ will rotate in a clockwise direction and the elliptical polarisation will be dextrorotatory. Circular and linear polarisation corresponds to $\mathcal{E}' = \mathcal{E}''$ and $\mathcal{E}' = 0$ or $\mathcal{E}'' = 0$, respectively.

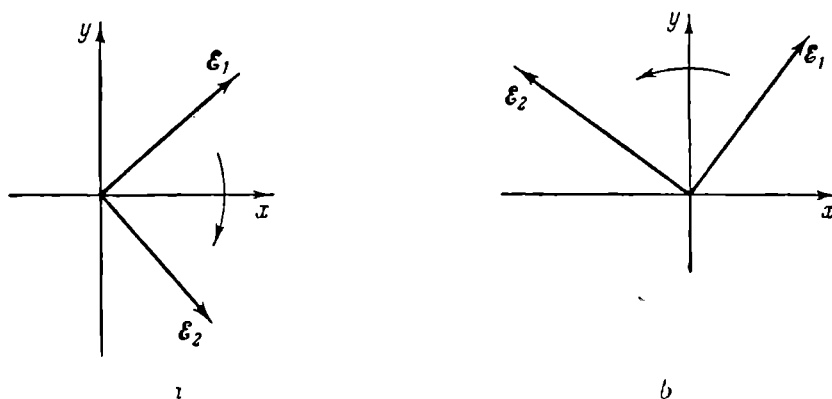


Fig. 62.

388. When $\chi = 0$, the polarisation is linear and the plane of polarisation contains the bisector of the angle between the x - and y -axes. When $\chi = \pi$, the polarisation is also linear and the plane of polarisation passes through the bisector of the angle between the x and $-y$ axes. When $\chi = \pi/2$, the polarisation is circular and dextrorotatory (Fig. 62a). When $\chi = -\pi/2$, the polarisation is circular and laevorotatory (Fig. 62b). In the remaining cases the polarisation is elliptical and dextrorotatory when $0 < \chi < \pi$ [$\cos(\chi/2) > 0$, $\sin(\chi/2) > 0$], with axes as shown in Fig. 62a and laevorotatory when $-\pi < \chi < 0$ (Fig. 62b).

389. When $a = b$ the polarisation is linear. When $a > b$ the polarisation is elliptical and dextrorotatory. When $a < b$ the

polarisation is elliptical and laevorotatory. Circular polarisation occurs only when $b = 0$ (dextrorotatory) or $a = 0$ (laevorotatory).

390. The amplitude of the resultant wave is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = E(\mathbf{e}^{(1)} + \mathbf{e}^{(2)} e^{i\alpha}),$$

where α is a randomly varying phase difference and $|\mathbf{E}|^2 = I$. By definition, the components of the polarisation tensor are given by [cf. Eqn. (VIII.14)]

$$I_{ik} = \overline{E_i E_k^*} = \overline{I(\mathbf{e}^{(1)} + \mathbf{e}^{(2)} e^{i\alpha})_i (\mathbf{e}^{(1)} + \mathbf{e}^{(2)} e^{-i\alpha})_k}.$$

Averaging with respect to time yields $\overline{\exp(\pm i\alpha)} = 0$ and hence the polarisation tensor becomes

$$I_{ik} = I \begin{pmatrix} 1 + \cos^2 \vartheta & \sin \vartheta \cos \vartheta \\ \sin \vartheta \cos \vartheta & 1 - \cos^2 \vartheta \end{pmatrix}.$$

In order to determine the degree of depolarisation, the tensor I_{ik} should be reduced to the form

$$I_{ik} = I_1 n_i^{(1)} n_k^{(1)*} + I_2 n_i^{(2)} n_k^{(2)*},$$

where I_i and $\mathbf{n}^{(i)}$ are determined from

$$I_{ik} n_k = I n_i.$$

On equating to zero the determinant of the system, it is found that

$$I_1 = 1 + \cos \vartheta, \quad I_2 = 1 - \cos \vartheta,$$

and hence the degree of polarisation is given by

$$\rho = \frac{1 - \cos \vartheta}{1 + \cos \vartheta} \quad \left(0 \leq \vartheta \leq \frac{\pi}{2}\right).$$

In the above case the vectors $\mathbf{n}^{(i)}$ are real and their components are $\mathbf{n}^{(1)}(\cos \vartheta/2, \sin \vartheta/2)$ and $\mathbf{n}^{(2)}(-\sin \vartheta/2, \cos \vartheta/2)$ (Fig. 63).

The wave is completely polarised (although not monochromatic)

when $\vartheta = 0$ ($\rho = 0$). When $\vartheta = \pi/2$ there is complete depolarisation ($\rho = 1$). The polarisation tensor is then given by $I_{ik} = I \delta_{ik}$.

391. The polarisation tensor is given by

$$I_{ik} = \begin{pmatrix} I_1 + \frac{1}{2} I_2 & \frac{1}{2} I_2 \\ \frac{1}{2} I_2 & \frac{1}{2} I_2 \end{pmatrix},$$

where the x_1 -axis lies along the

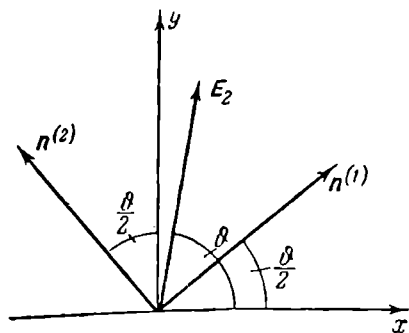


Fig. 63.

direction of polarisation of the first wave. The degree of polarisation is

$$\rho = \frac{I_1 + I_2 - \sqrt{I_1^2 + I_2^2}}{I_1 + I_2 + \sqrt{I_1^2 + I_2^2}}.$$

392. $\rho = (1 - \xi)/(1 + \xi)$. When $\xi = 0$ the wave is unpolarised; when $\xi = 1$ it is completely polarised. The quantity ξ is therefore called the degree of polarisation. Substituting $\xi_i = \xi \eta_i$ where $\eta_1^2 + \eta_2^2 + \eta_3^2 = 1$, we have

$$I_{ik} = \frac{I}{2}(1 - \xi)\delta_{ik} + \frac{I\xi}{2}\left(1 + \sum_{l=1}^3 \eta_l \tau_{ik}^{(l)}\right).$$

The first term in this expression corresponds to a completely unpolarised state, and the second term corresponds to a completely polarised state. In case (a), $\eta_3 = 1$, $\eta_1 = \eta_2 = 0$.

On comparing

$$I''_{ik} = I\xi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with $I_{ik} = I n_i n_k^*$, we see that in this case $n_1 = 1$, $n_2 = 0$, *i.e.* the tensor I''_{ik} describes a wave which is linearly polarised along the direction of the x -axis (the waves propagating along the z -axis).

Similarly, it is easy to show that in case (b) $\eta_1 = 1$, $\eta_2 = \eta_3 = 0$ and the wave is linearly polarised along the direction at 45° to the x -axis, while in case (c) $\eta_2 = 1$, $\eta_1 = \eta_3 = 0$ and the wave is circularly polarised.

393. Since the vector $\mathbf{E}$ is linearly polarised, the amplitude $\mathbf{E}_0$ may be assumed to be real. Since $\text{div } \mathbf{E} = 0$, we have $\mathbf{k}' \cdot \mathbf{E}_0 = 0 = \mathbf{k}'' \cdot \mathbf{E}_0$, *i.e.* $\mathbf{E}_0$ is perpendicular to the $(\mathbf{k}', \mathbf{k}'')$ plane. Since $\text{curl } \mathbf{E} = -(\mu/c)\partial\mathbf{H}/\partial t$, it follows that

$$\frac{\mu\omega}{c} \mathfrak{H}_1 = \mathbf{k}' \times \mathbf{E}_0,$$

$$\frac{\mu\omega}{c} \mathfrak{H}_2 = \mathbf{k}'' \times \mathbf{E}_0,$$

i.e. $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are perpendicular to $\mathbf{E}_0$ and $\mathfrak{H}_1 \perp \mathbf{k}'$, $\mathfrak{H}_2 \perp \mathbf{k}''$. The

end point of the vector $\mathbf{H}$ describes an ellipse in the $(\mathbf{k}', \mathbf{k}'')$ plane (Fig. 64).

394. Both waves will be elliptically polarised. One of the principal axes of the polarisation ellipse lies in the plane of

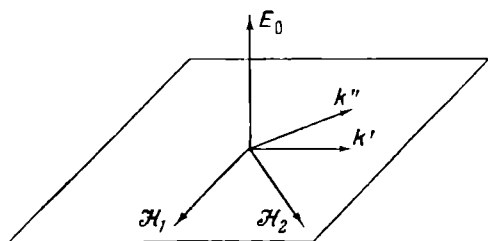


Fig. 64.

incidence while the other is perpendicular to it. The semi-axes are given by the following formulae:

(1) reflected wave:

$$E_{\parallel} = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} E_0, \quad E_{\perp} = \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)} E_0;$$

(2) refracted wave:

$$E_{\parallel} = \frac{2 \cos \theta_0 \sin \theta_2}{\sin(\theta_0 + \theta_2) \cos(\theta_0 - \theta_2)} E_0, \quad E_{\perp} = \frac{2 \cos \theta_0 \sin \theta_2}{\sin(\theta_0 + \theta_2)} E_0,$$

where θ_0 is the angle of incidence, θ_2 is the angle of refraction, and E_0 is the absolute magnitude of the amplitude of the incident wave. When $\theta_0 = \pi/2 - \theta_2$ (Brewster's angle), the reflected wave is linearly polarised.

395. Unpolarised (natural) light may be looked upon as an incoherent superposition of two waves of "complementary" polarisation and equal intensity. The incident beam can therefore be looked upon as consisting of two incoherent components one of which ($E_{\parallel}$) is polarised in the plane of incidence and the other ($E_{\perp}$) is polarised in the perpendicular plane. The intensities of these waves are equal, so that $I_{\parallel} = I_{\perp} = I$.

After reflection the two components will remain incoherent. Fresnel's formulae then give

$$I_{ik}^{(1)} = I \frac{\sin^2(\theta_0 - \theta_2)}{\sin^2(\theta_0 + \theta_2)} \left(e_i^{\perp} e_k^{\perp} + \frac{\cos^2(\theta_0 + \theta_2)}{\cos^2(\theta_0 - \theta_2)} e_i^{\parallel} e_k^{\parallel} \right),$$

$$\rho_1 = \frac{\cos^2(\theta_0 + \theta_2)}{\cos^2(\theta_0 - \theta_2)} < 1,$$

where $e^{\perp}$ and $e^{\parallel}$ are unit vectors in the direction of the polarisation of the transverse and longitudinal components. These vectors lie in the plane perpendicular to the direction of propagation of the reflected light. The degree of depolarisation of the incident light is 1. The light becomes polarised after reflection.

Similarly, for refracted light we have

$$I_{ik}^{(2)} = \frac{4I \cos^2 \theta_0 \sin^2 \theta_2}{\sin^2(\theta_0 + \theta_2)} \left(e_i^{\perp} e_k^{\perp} + \frac{e_i^{\parallel} e_k^{\parallel}}{\cos^2(\theta_0 - \theta_2)} \right),$$

$$\rho_2 = \cos^2(\theta_0 - \theta_2) < 1.$$

396.

$$R = \frac{(\epsilon_1 - \epsilon_2)^2}{2(\epsilon_1 + \epsilon_2)}, \quad \rho_1 = 0, \quad \rho_2 = \frac{4\epsilon_1\epsilon_2}{(\epsilon_1 + \epsilon_2)^2}.$$

where ε_1 and ε_2 are the permittivities of the two dielectrics.
397.

$$E_{\perp 1} = (-1 + 2\zeta \cos \theta_0) E_{\perp 0}, \quad E_{\parallel 1} = \left(1 - \frac{2\zeta}{\cos \theta_0}\right) E_{\parallel 0},$$

$$E_{\perp 2} = 2\zeta \cos \theta_0 E_{\perp 0}, \quad E_{\parallel 2} = 2\zeta E_{\parallel 0}.$$

The expressions for $E_{\parallel 1}$ and $E_{\parallel 2}$ will hold only if the glancing angle is such that $\varphi_0 = \pi/2 - \theta_0 \gg |\zeta|$. The following formulae will hold for $\varphi_0 \ll 1$:

$$E_{\parallel 1} = \frac{\varphi_0 - \zeta}{\varphi_0 + \zeta} E_{\parallel 0}, \quad E_{\parallel 2} = \frac{\varphi_0 \zeta}{\varphi_0 + \zeta} E_{\parallel 0},$$

where $|\zeta|$ and φ_0 are arbitrary.

398.

$$R_{\perp} = 1 - 4\zeta' \cos \theta_0.$$

$R_{\perp}$ is very nearly equal to unity for all angles of incidence and reaches a minimum at $\theta_0 = 0$ (normal incidence). Moreover,

$$R_{\parallel} = 1 - \frac{4\zeta'}{\cos \theta_0} \quad \text{when} \quad \varphi_0 = \frac{\pi}{2} - \theta_0 \gg 4\zeta',$$

$$R_{\parallel} = \frac{(\varphi_0 - \zeta')^2 + \zeta''^2}{(\varphi_0 + \zeta')^2 + \zeta''^2} \quad \text{when} \quad \varphi_0 \ll 1.$$

The angle φ_0 at which $R_{\parallel}$ is a minimum may be found from the condition $\partial R_{\parallel} / \partial \varphi_0 = 0$ and the result is

$$\varphi_0 = \Phi_0 = |\zeta|, \quad R_{\parallel} = \frac{|\zeta| - \zeta'}{|\zeta| + \zeta'}.$$

The angle Φ_0 is an analogue of the Brewster angle since the magnitude of $R_{\parallel}$ is a minimum at $\varphi_0 = \Phi_0$ (when the wave is incident on the boundary of the dielectric at the Brewster angle, the coefficient $R_{\parallel}$ is also a minimum and is in fact equal to zero).

399. The polarisation of the reflected wave depends on the phase difference between the longitudinal and the transverse components. Using the results of the two preceding problems, we find that

$$E_{\perp 1} \simeq -E_{\perp 0} = e^{i\delta_{\perp}} E_{\perp 0}, \quad \delta_{\perp} = \pi;$$

$$E_{\parallel 1} = \left[\frac{|\zeta| - \zeta'}{|\zeta| + \zeta'} \right]^{\frac{1}{2}} e^{i\delta_{\parallel}} E_{\parallel 0}, \quad \tan \delta_{\parallel} = -\frac{2\Phi_0 \zeta''}{\Phi_0^2 - |\zeta|^2} \rightarrow \infty,$$

i.e. $\delta_{\parallel} = \pi/2$.

Thus the phase difference is $\delta = \delta_{\perp} - \delta_{\parallel} = \pi/2$. In general, the reflected wave is elliptically polarised and one of the axes of the ellipse lies in the plane of incidence. When $|E_{\parallel 1}| = |E_{\perp 1}|$ the polarisation is circular. When $E_{\parallel 0} = 0$ or $E_{\perp 0} = 0$ the polarisation is linear.

400. Fresnel's formulae give

$$n' = \frac{\sin \theta_0 \tan \theta_0 \cos 2\rho}{1 + \sin^2 2\rho \cos \delta} \quad n'' = \frac{\sin \theta_0 \tan \theta_0 \sin 2\rho \sin \delta}{1 + \sin 2\rho \cos \delta}$$

401.

$$R = \frac{(\sqrt{\varepsilon} - \sqrt{\varepsilon'})^2}{(\sqrt{\varepsilon} + \sqrt{\varepsilon'})^2} + \frac{4}{(\sqrt{\varepsilon} + \sqrt{\varepsilon'})^4} \cdot \sqrt{\frac{\varepsilon'}{\varepsilon}} \left(\frac{2\pi \sigma}{\omega} \right)^2,$$

where ε is the permittivity of the medium from which the light emerges, and ε' is the real part of the permittivity of the conducting medium.

402. The phase difference between $E_{\perp 1}$, E_0 and $E_{\parallel 1}$, E_0 may be determined from the Fresnel formulae:

$$\tan \frac{\delta_{\perp}}{2} = \frac{\sqrt{\sin^2 \theta_0 - n^2}}{\cos \theta_0}, \quad \tan \frac{\delta_{\parallel}}{2} = \frac{\sqrt{\sin^2 \theta_0 - n^2}}{n^2 \cos \theta_0}. \quad (1)$$

Since $\delta_{\perp} \neq \delta_{\parallel}$, the wave is elliptically polarised.

The polarisation will become circular when the following conditions are satisfied

$$\delta = \delta_{\parallel} - \delta_{\perp} = \frac{\pi}{2}; \quad E_{\parallel 0} = E_{\perp 0}.$$

The second of these conditions requires that the incident wave should be polarised in the plane making an angle of $\pi/4$ with the plane of incidence. Let us consider whether the first of the above two conditions can be satisfied. From Eqn. (1) we have

$$\tan \frac{\delta}{2} = \frac{\cos \theta_0 \sqrt{\sin^2 \theta_0 - n^2}}{\sin^2 \theta_0}. \quad (2)$$

It follows that $\delta = 0$ when $\theta_0 = \sin^{-1} n$ and $\theta_0 = \pi/2$. The maximum value of δ occurs between these two points. It is easy to show that $\tan(\delta_{\max}/2) = (1 - n^2)/2n$. In order that $\tan(\delta/2)$ should be equal to unity ($\delta = \pi/2$), the following inequalities must be satisfied: $1 - n^2 \geq 2n$, $n \leq 0.414$.

403. When the vector $\mathbf{E}_0$ is normal to the plane of incidence, the transverse and longitudinal components of Poynting's vector are given by

$$\left. \begin{aligned} \gamma_{\perp} &= \frac{c^2 k''}{8\pi\omega} E_0^2 e^{-2k''z} \sin 2(k'x - \omega t), \\ \gamma_{\parallel} &= \frac{c^2 k''}{8\pi\omega} E_0^2 e^{-2k''z} [1 - \cos 2(k'x - \omega t)], \end{aligned} \right\} \quad (1)$$

where the z -axis is normal to the boundary of the media and the x -axis is the line of intersection of the plane of incidence and the

separation boundary,

$$k' = k_2 \sin \theta_0, \quad k'' = k_2 \sqrt{\sin^2 \theta_0 - n^2},$$

where $k_2 = \omega n_2 / c$ is the wave vector in the second medium and θ_0 is the angle of incidence.

It is clear from Eqn. (1) that the frequency of the vibrations in the direction perpendicular to the boundary is 2ω . The time average of the energy flux entering the second medium is zero. The average magnitude of $\gamma_{||}$ is not zero, so that there is a finite flow of energy along the separation boundary.

The equation for the Poynting vector lines in the second medium is

$$z = \frac{1}{k''} \ln \frac{|\sin k' x|}{C}, \quad (2)$$

where C is a constant of integration. Fig. 65 illustrates the form of these lines. In the first medium the γ lines have a more complicated form.

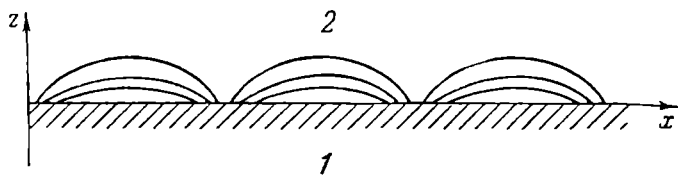


Fig. 65.

404. In this case the law of refraction is

$$k_1 \sin \theta_0 = k_2 \sin \theta_2, \quad k_1 = \frac{\omega}{c} \sqrt{\epsilon_1}, \quad k_2 = \frac{\omega}{c} \sqrt{\epsilon'_2 + i \frac{4\pi \sigma_2}{\omega}} = k'_2 + i k''_2,$$

where $\sin \theta_2$ and $\cos \theta_2$ are complex quantities. Let $\cos \theta_2 = \rho \exp(i\alpha)$ where ρ and α are real constants which are functions of θ_0 and of the electrical constants of the medium. The parameters ρ , α can be determined from

$$\begin{aligned} \rho^2 \cos 2\alpha &= 1 - \frac{k_1^2}{|k_2|^2} \sin^2 \theta_0, \\ \rho^2 \sin 2\alpha &= \frac{2k_1^2 k'_2 k''_2}{|k_2|^4} \sin^2 \theta_0. \end{aligned}$$

The wave in the second conducting medium 2 is

$$\mathbf{E}_2(\mathbf{r}, t) = \mathbf{E}_2 e^{i(k_2 \mathbf{e}_2 \cdot \mathbf{r} - \omega t)}.$$

Separating the real and imaginary parts of the product $k_2 \mathbf{e}_2 \cdot \mathbf{r}$ we have

$$k_2 \mathbf{e}_2 \cdot \mathbf{r} = (k'_2 + ik''_2)(x \sin \theta_2 + z \cos \theta_2) = izp(\theta_0) + xk_1 \sin \theta_0 + zq(\theta_0),$$

where

$$p(\theta_0) = \rho(k'_2 \sin \alpha + k''_2 \cos \alpha), \quad q(\theta_0) = \rho(k'_2 \cos \alpha - k''_2 \sin \alpha).$$

Thus,

$$\mathbf{E}_2(\mathbf{r}, t) = \mathbf{E}_2 e^{-p z} e^{i(xk_1 \sin \theta_0 + zq - \omega t)}.$$

It is clear that the directions of propagation and damping are not the same. The planes of constant amplitude $z = \text{const}$ are parallel to the surface of the conductor. The planes of constant phase are given by

$$xk_1 \sin \theta_0 + zq(\theta_0) = \text{const},$$

from which it follows that the vector $\mathbf{k}'_2$ which defines the direction of propagation of the wave makes an angle $\psi = \tan^{-1}[k_1 \sin \theta_0 / q(\theta_0)]$ with the z -axis (Fig. 66). The phase velocity in the conducting

medium is a function of the angle of incidence and is given by

$$v_\varphi = \frac{\omega}{\sqrt{q^2(\theta_0) + k_1^2 \sin^2 \theta_0}}.$$

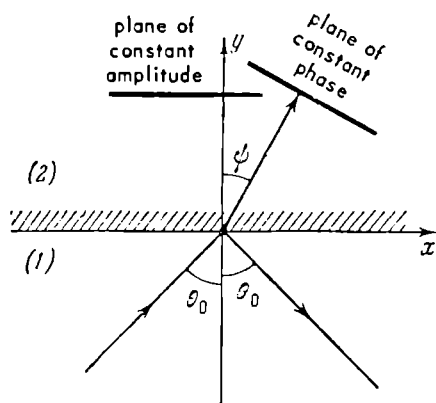


Fig. 66.

405. In order to determine the reflection coefficient for the plane layer it is necessary to find the relation between the amplitudes of the reflected and incident waves. This relation may be obtained by two methods.

In the first method it is determined from the boundary conditions. Since the tangential components of the vectors $\mathbf{E}$ and $\mathbf{H}$ are continuous across the boundaries $z = 0$ and $z = a$, and since in front of the layer there are waves propagating in both directions while behind the layer there is only the transmitted wave propagating in the positive direction of the z -axis, we have from the boundary conditions

$$E_1 = \frac{\alpha_{12} + \alpha_{23} e^{-2ik_2 a}}{1 + \alpha_{12} \alpha_{23} e^{-2ik_2 a}} E_0, \quad (1)$$

where E_1 and E_0 are the amplitudes of the reflected and incident waves respectively, and

$$\alpha_{12} = \frac{1 - n_{12}}{1 + n_{12}}, \quad \alpha_{23} = \frac{1 - n_{23}}{1 + n_{23}}, \quad n_{lk} = \sqrt{\frac{\epsilon_k}{\epsilon_l}}, \quad k_2 = \frac{\omega}{c} \sqrt{\epsilon_2}.$$

The second method of solution is based on a consideration of multiple reflections from the separation boundaries. Using the Fresnel formulae at normal incidence, we find that the amplitude of the wave which has been reflected from the boundary $z = 0$ only once is $\mathcal{E}_0 = \alpha_{12} E_0$. The amplitude of the wave which has entered the layer is $\mathcal{E}_\alpha = \beta_{12} E_0$ where

$$\beta_{12} = \frac{2}{1 + n_{12}}.$$

The amplitude of the wave passing from the layer into the region $z < 0$ after single reflection at the $z = a$ plane is

$$\mathcal{E}_1 = \beta_{21} \alpha_{23} \beta_{12} E_0 e^{-2ik_2 a}.$$

The amplitude of the wave entering the region $z < 0$ after s reflections at $z = a$ is

$$\mathcal{E}_s = \beta_{21} \beta_{12} \alpha_{23} e^{-2ik_2 a} (\alpha_{21} \alpha_{23} e^{-2ik_2 a})^{s-1}.$$

The total amplitude E_1 of the wave reflected from the plane layer is equal to the sum of all the $\mathcal{E}_s$:

$$E_1 = \sum_{s=0}^{\infty} \mathcal{E}_s = \alpha_{12} E_0 + \beta_{21} \beta_{12} \alpha_{23} e^{-2ik_2 a} \sum_{s=1}^{\infty} (\alpha_{21} \alpha_{23} e^{-2ik_2 a})^{s-1}.$$

Using the formula for the sum of an infinite geometric progression, this expression can be shown to be identical with Eqn. (1).

The reflection coefficient is defined by $R = |E_1|^2 / |E_0|^2$. The minimum of R may be found in the usual way. The reflection coefficient is a minimum if the thickness of the layer satisfies the condition

$$a = a_n = n \frac{\lambda_2}{4}, \quad n = 1, 2, 3, \dots, \quad (2)$$

where λ_2 is the wavelength inside the layer.

Consider the minimum thickness of the layer $a = \lambda_2/4$. Equating R to zero, we find that the condition for the absence of reflection is

$$\epsilon_2 = \sqrt{\epsilon_1 \epsilon_3}.$$

406. The equation satisfied by the electric field is [cf. Eqn. (VIII.12)]

$$\frac{d^2 E}{dz^2} + \frac{\omega^2}{c^2} \left(\epsilon - \frac{\Delta \epsilon}{\epsilon \frac{z}{a} + 1} \right) E = 0. \quad (1)$$

It is required to find the solution of this problem which is bounded for all z and satisfies the appropriate conditions at $z \rightarrow \pm\infty$. When $z \rightarrow -\infty$, the solution should be

$$E(z) \rightarrow A e^{i k_0 z} + B e^{-i k_0 z}, \quad (2)$$

where $k_0 = \omega/c$. This is a superposition of two waves, namely, the incident and the reflected waves. When $z \rightarrow +\infty$, there should only be the transmitted wave, so that

$$E(z) \rightarrow C e^{i k z}, \quad (3)$$

where $k = \omega \epsilon^{\frac{1}{2}}/c$.

Let us substitute $\xi = -\exp(-z/a)$ into Eqn. (1). The new variable is such that $-\infty \leq \xi \leq 0$ when z varies from $-\infty$ to $+\infty$. Using the substitution $E(\xi) = \xi^{-i k a} \psi(\xi)$ we obtain the following equation for the new unknown function $\psi(\xi)$:

$$\xi(1-\xi)\psi'' + (1-2ika)(1-\xi)\psi' + \kappa^2 a^2 \psi = 0, \quad (4)$$

where $\kappa^2 = \omega^2 \Delta \epsilon / c^2$. This is the hypergeometric equation.

It follows from Eqn. (3) that the function $\psi(\xi)$ should tend to a constant limit when $\xi \rightarrow 0$. The hypergeometric function which is a solution of Eqn. (4) and satisfies the above conditions is given by

$$F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1) \cdot 2!} z^2 + \dots$$

The solution of Eqn. (4) may therefore be written in the form

$$\psi = C F \left[-l(k+k_0)a, -l(k-k_0)a, 1-2ika, -e^{-\frac{z}{a}} \right]. \quad (5)$$

In order to find the form of the function ψ when $\xi \rightarrow -\infty$, we shall use the asymptotic expression

$$F(\alpha, \beta, \gamma, \xi) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-\xi)^{-\alpha} + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} (-\xi)^{-\beta}. \quad (6)$$

An examination of this equation will show that the condition given by Eqn. (2) is satisfied.

The reflection coefficient is therefore given by

$$R = \left| \frac{B}{A} \right|^2 = \left| \frac{\Gamma(2ik_0a) \Gamma[1-i(k+k_0)a] \Gamma[-i(k+k_0)a]}{\Gamma(-2ik_0a) \Gamma[1-i(k-k_0)a] \Gamma[-i(k-k_0)a]} \right|^2 \quad (7)$$

This equation may be simplified if we recall that

$$\left| \frac{\Gamma(2ik_0a)}{\Gamma(-2ik_0a)} \right| = \left| \frac{\Gamma(2ik_0a)}{\Gamma^*(2ik_0a)} \right| = 1 \quad \text{and} \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

The final expression for R is therefore

$$R = \frac{\sinh^2 \pi a (k - k_0)}{\sinh^2 \pi a (k + k_0)} \quad (8)$$

For small a ($ka \ll 1$), the expression for the reflection coefficient becomes identical with the well-known expression which holds for a step change in ε :

$$R = \frac{(k - k_0)^2}{(k + k_0)^2}.$$

R is a monotonically decreasing function of a . For large ka , the reduction is exponential:

$$R = e^{-4\pi k_0 a}, \quad ka \gg 1.$$

407. For normal incidence on a non-homogeneous layer, the electric field is a function of z only and satisfies the equation

$$\frac{d^2 E}{dz^2} + \frac{\omega^2}{c^2} \varepsilon(\omega, z) E = 0. \quad (1)$$

Let $z_1 = m\omega^2/4\pi e^2 N_0$ so that $\varepsilon = 1 - z/z_1$. Substituting $\xi = (\omega^2/c^2 z_1)^{1/2} (z_1 - z)$ into Eqn. (1) we have†

$$\frac{d^2 E}{d\xi^2} + \xi E = 0. \quad (2)$$

The simplest way to obtain the solution of Eqn. (2) is to expand $E(\xi)$ into a Fourier integral:

$$E(\xi) = \int_{-\infty}^{\infty} E(u) e^{i\xi u} du, \quad E(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\xi) e^{-i\xi u} d\xi.$$

† A similar equation is used in quantum mechanics to describe the motion of a particle in a uniform field.

Substitution into Eqn. (2) yields the first-order differential equation

$$\frac{dE(u)}{du} + iu^2E(u) = 0. \quad (3)$$

The solution of this equation is

$$E(u) = A'e^{-\frac{iu^3}{3}},$$

hence

$$E(\xi) = A' \int_{-\infty}^{\infty} e^{-i\left(\frac{u^3}{3} - \xi u\right)} du.$$

Rewriting $\exp -i(u^3/3 - \xi u)$ in its trigonometric form and noting that the integral of $\sin(u^3/3 - \xi u)$ is zero, we have

$$E(\xi) = \frac{A}{\sqrt{\pi}} \int_0^{\infty} \cos\left(\frac{u^3}{3} - \xi u\right) du. \quad (4)$$

The function

$$\Phi(\xi) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \cos\left(\frac{u^3}{3} + \xi u\right) du$$

is known as Airy's function. Airy's function may be expressed in terms of Bessel functions of order $\frac{1}{3}$. Hence, finally

$$E(\xi) = A\Phi(-\xi).$$

The constant A should be determined from the boundary conditions.

Consider now the behaviour of $E(\xi)$ at large $|\xi|$. Using the asymptotic form of $\Phi(\xi)$ we have for large positive ξ

$$E(\xi) = \frac{A}{\xi^{1/4}} \sin\left(\frac{2}{3}\xi^{3/2} + \frac{\pi}{4}\right),$$

so that the field is oscillatory in character.

For large negative ξ ,

$$E(\xi) = \frac{A}{2|\xi|^{1/4}} e^{-\frac{2}{3}|\xi|^{3/2}},$$

so that the field falls off exponentially. The reason for this is that negative ξ corresponds to negative values of the permittivity ϵ . When $\epsilon < 0$, the wave vector $k = \omega\epsilon^{1/2}/c$ is purely imaginary, *i.e.* absorption takes place. However, this absorption is not a process in which the electromagnetic energy is converted

into heat. In fact, the absorption gives rise to a reflection of the wave from a layer with negative ϵ . This is so because the permittivity is real, and therefore there are no losses.

408.

$$\Psi(x, 0) = A(x, 0) e^{ik_0 x},$$

where

$$A(x, 0) = a_0 \sqrt{\pi \Delta k} e^{-\frac{x^2 \Delta k^2}{4}}$$

The amplitude $A(x, 0)$ of the wave packet is gaussian. It becomes negligibly small when $|x \Delta k| \gg 1$. It follows that the width of the packet in ordinary space is related to its "width" in k -space by $\Delta k \cdot \Delta x \simeq 1$. This is a universal relation which holds for electromagnetic or any other wave packets. It is of particular importance for probability waves in quantum mechanics, since it leads to the uncertainty relation for the space and momentum coordinates of fundamental particles.

409.

$$\Psi(0, t) = A(0, t) e^{-i\omega_0 t},$$

where

$$A(0, t) = a_0 \sqrt{\pi \Delta \omega} e^{-\frac{t^2 \Delta \omega^2}{4}},$$

$$\Delta t \cdot \Delta \omega \simeq 1.$$

410. $\Delta x_{\min} = \lambda / (2\pi \sin \theta)$, where θ is the angle of the cone formed by the rays drawn from the objective of the microscope to the object under consideration.

411. The wave pulse sent out by a radar set has a width Δx which is related to the transverse spread $k_{\perp}$ of the wave vectors by $\Delta x \cdot k_{\perp} \geq 1$. On the other hand, it is clear that $\Delta x / l \simeq k_{\perp} / k$. Hence the uncertainty in the position of the object is

$$\Delta x \geq \sqrt{l \lambda}.$$

412. The wave packet is described by

$$\Psi(\mathbf{r}, t) = 4\pi a_0 \sqrt{\frac{\pi q^3}{2\rho^3}} J_{\frac{3}{2}}(\rho q) e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)},$$

where $J_{\frac{3}{2}}(x) = (2/\pi x)^{\frac{1}{2}}(\sin x/x - \cos x)$ is the Bessel function and $\rho = |\mathbf{r} - \mathbf{v}_g t|$. The group velocity $\mathbf{v}_g = \partial \omega / \partial \mathbf{k}$ is a vector with components $\partial \omega / \partial k_x$, $\partial \omega / \partial k_y$, $\partial \omega / \partial k_z$. The amplitude of the wave packet is now appreciably different from zero only in the (spherically symmetric) region $\rho q \leq 1$.

It is clear from the expression for $\Psi(\mathbf{r}, t)$ that the shape of the wave packet is independent of time. This is due to the linear

dispersion law which is strictly true only for waves travelling in a vacuum. When higher terms in the expansion of ω as a function of k are taken into account, the packet will spread out, and its form will no longer be time independent. The packet as a whole moves with the group velocity $\mathbf{v}_g$.

413. Writing the function $\omega(k)$ in the form

$$\omega = \omega_0 + v_g(k - k_0) + \beta(k - k_0)^2,$$

we have

$$\Psi(x, t) = a_0 \sqrt{\frac{\pi}{\alpha + i\beta t}} e^{-\frac{(x - v_g t)^2}{4(\alpha + i\beta t)} + i(k_0 x - \omega_0 t)}.$$

It is convenient to consider the square of the modulus of this expression:

$$|A(x, t)|^2 = \frac{\pi a_0^2}{\sqrt{\alpha^2 + (\beta t)^2}} e^{-\frac{\alpha(x - v_g t)^2}{2(\alpha^2 + \beta^2 t^2)}},$$

which determines the intensity of the wave. It is clear from this expression that the intensity of the wave as a function of x at given t is a gaussian curve whose width l is given by

$$l = \sqrt{\frac{2(\alpha^2 + \beta^2 t^2)}{\alpha}}.$$

Thus the width increases with time, but the height of the packet decreases, owing to the presence of the factor $(\alpha^2 + \beta^2 t^2)^{-\frac{1}{2}}$.

The wave packet spreads out symmetrically both in the $t = +\infty$ and $t = -\infty$ directions. This spread is not associated with the absorption of energy because k is real. The absence of dissipation is clear from the fact that the integral $\int_{-\infty}^{+\infty} |A(x, t)|^2 dx = (\pi/2\alpha)^{\frac{1}{2}} a_0^2$ is independent of time, *i. e.* the "total intensity" is conserved. The reason for the spread of the wave packet is that the phase velocities $v_\varphi = \omega/k$ of the various plane waves entering into the series which represents the wave are different: owing to dispersion, the ratio ω/k is a function of k .

414. When $\omega \ll \omega_0$

$$v_\varphi = \frac{c}{\sqrt{\epsilon_0}} \left(1 - \frac{1}{2} \frac{\omega_p^2 \omega^2}{\epsilon_0 \omega_0^4} \right) < c,$$

$$v_g = \frac{c}{\sqrt{\epsilon_0}} \left(1 - \frac{3}{2} \frac{\omega_p^2 \omega^2}{\epsilon_0 \omega_0^4} \right) < c,$$

where $\varepsilon_0 = \varepsilon(0)$. When $\omega \gg \omega_0$

$$v_\varphi = c \left(1 + \frac{\omega_p^2}{2\omega^2} \right) > c,$$

$$v_g = c \left(1 - \frac{\omega_p^2}{2\omega^2} \right) < c.$$

In the latter case $v_\varphi \cdot v_g \simeq c^2$. Near the resonance frequency ($\omega \simeq \omega_0$), the concept of group velocity loses its significance.

415. It follows from the solution of Problems 412 and 413 that the function describing the wave packet is

$$E(x, t) = E_0(x, t) e^{i(k_0 x - \omega_0 t)}.$$

The amplitude $E_0(x, t)$ varies much more slowly than the function $\exp[i(k_0 x - \omega_0 t)]$ (the periods of these functions are in the ratio of $\Delta k/k_0$). Neglecting the change in E_0 as compared with $\exp[i(k_0 x - \omega_0 t)]$, we find from Maxwell's equations that

$$H(x, t) = \frac{k_0 c}{\omega_0 \mu} E(x, t) = \sqrt{\frac{\varepsilon}{\mu}} E_0(x, t) e^{i(k_0 x - \omega_0 t)}.$$

The energy flux averaged over the period $2\pi/\omega_0$ of the high frequency component is given by

$$\bar{\gamma}(x, t) = \frac{c}{8\pi} |\operatorname{Re}(\mathbf{E} \times \mathbf{H}^*)| = \frac{c}{8\pi} \sqrt{\frac{\varepsilon}{\mu}} E_0(x, t) E_0^*(x, t).$$

Since $\bar{\gamma} = v\bar{w}$, the rate of energy transport is

$$v = \frac{c}{\frac{d}{d\omega}(\omega \sqrt{\varepsilon \mu})} = \frac{d\omega}{dk} = v_g$$

2. Scattering of electromagnetic waves by macroscopic bodies. Diffraction

416. It is convenient to introduce cylindrical coordinates with the z -axis lying along the axis of the cylinder, and the angle α measured from the direction of the wave vector $\mathbf{k}$ of the incident wave. In view of the symmetry of the problem, the field vectors are independent of z , and the only finite components are E_z , H_r , and H_α . In what follows we shall omit the time factor $\exp(-i\omega t)$ and will use the wave equation (VIII.6) for $\mathbf{E}$ and the Maxwell equation given by (VIII.1). The first of these equations will give

E_z , and the second will give H_r and H_α in terms of E_z :

$$H_r = \frac{1}{ikr} \frac{\partial E_z}{\partial \alpha}, \quad H_\alpha = -\frac{1}{ik} \frac{\partial E_z}{\partial r}. \quad (1)$$

The secondary field $E' = E - E_0$, which is due to the presence of the cylinder, satisfies the equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E'}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E'}{\partial \alpha^2} + k^2 E' = 0. \quad (2)$$

Let $E' = R(r)\Phi(\alpha)$ and separate the variables in Eqn. (2):

$$R''_m + \frac{1}{r} R'_m + \left(k^2 - \frac{m^2}{r^2} \right) R_m = 0, \quad (3)$$

$$\Phi''_m + m^2 \Phi_m = 0, \quad (4)$$

where m^2 is the separation constant. The general solution of Eqn. (2) is then the sum of the special solutions over all the allowed values of m :

$$E'(r, \alpha) = \sum_m \Phi_m(\alpha) R_m(r). \quad (5)$$

In order to obtain the solution of the Bessel equation [Eqn. (3)] in a form convenient for our purposes, consider the boundary condition for $r \rightarrow \infty$. Since E' represents the secondary field, which is due to the currents induced in the cylinder, it follows that when $r \rightarrow \infty$ this field will take the form of cylindrical waves, travelling in the outward direction. This means that E' should, in this region, be of the form

$$E' = \mathcal{E}_0 f(\alpha) \frac{e^{ikr}}{\sqrt{r}}. \quad (6)$$

The latter condition will be satisfied if the Hankel function $H_m^{(1)}(kr)$ is taken as the solution of Eqn. (3). For large r this function is of the form (*cf.* Appendix III)

$$H_m^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}} e^{i\left(kr - \frac{m\pi}{2} - \frac{\pi}{4}\right)} \quad (kr \gg 1).$$

The second linearly independent solution will contain a term of the form $\text{const } r^{-\frac{1}{2}} \exp(-ikr)$ which represents a cylindrical wave flowing in the inward direction. This wave cannot appear under the conditions of the present problem. Hence, the solution of Eqn. (3) may be written in the form $R_m(r) = H_m^{(1)}(kr)$. Eqn. (4)

has the following solution:

$$\Phi_m(\alpha) = A_m e^{im\alpha} + B_m e^{-im\alpha}.$$

Since the field remains unaltered when α changes by 2π , the number m should be an integer. If it is considered that m may assume both positive and negative values, then the final solution will be of the form

$$E'(r, \alpha) = \mathcal{E}_0 \sum_{m=-\infty}^{\infty} A_m H_m^{(1)}(kr) e^{im\alpha} \quad (7)$$

At large distances (7) will become identical with (6) and

$$f(\alpha) = \sqrt{\frac{2}{\pi k}} \sum_m A_m e^{i\left(m\alpha - \frac{m\pi}{2} - \frac{\pi}{4}\right)}.$$

The coefficients A_m in the series given by (7) may be determined from the boundary condition for the surface of the cylinder. Since the cylinder is a perfect conductor, it follows that

$$E' + E_0 = 0 \quad \text{when } r = a \quad (8)$$

or

$$e^{ika \cos \alpha} + \sum_{m=-\infty}^{\infty} A_m H_m^{(1)}(ka) e^{im\alpha} = 0. \quad (9)$$

Since the functions $\exp(im\alpha)$ are orthogonal, we have

$$\int_0^{2\pi} e^{i(ka \cos \alpha - m'\alpha)} d\alpha + 2\pi A_{m'} H_{m'}^{(1)}(ka) = 0,$$

and hence, using Eqn. (11) of Appendix III, we have

$$A_m = \frac{i^m J_m(ka)}{H_m^{(1)}(ka)}. \quad (10)$$

The resultant electric field is therefore given by

$$E(r, \alpha) = \mathcal{E}_0 e^{ikr \cos \alpha} - \mathcal{E}_0 \sum_m \frac{i^m J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(kr) e^{im\alpha}. \quad (11)$$

The components of the magnetic field can be determined with the aid of Eqn. (1) and are given by

$$H_r = -\mathcal{E}_0 \sin \alpha e^{ikr \cos \alpha} - \mathcal{E}_0 \sum_m \frac{i^m m J_m(ka)}{H_m^{(1)}(ka)} \cdot \frac{H_m^{(1)}(kr)}{kr} e^{im\alpha}, \quad (12)$$

$$H_\alpha = -\mathcal{E}_0 \cos \alpha e^{ikr \cos \alpha} + \mathcal{E}_0 \sum_m \frac{i^{m-1} J_m(ka)}{H_m^{(1)}(ka)} \cdot \frac{dH_m^{(1)}(kr)}{d(kr)} e^{im\alpha}. \quad (13)$$

The secondary electric field is transverse in all space; the secondary magnetic field becomes transverse at large distances

from the cylinder $kr \gg 1$ (wave zone), when the longitudinal component H_r vanishes owing to the presence of the factor kr in the denominator.

The surface current density can be determined from the boundary condition for the tangential component of $\mathbf{H}$:

$$l(\alpha) = l_z(\alpha) = \frac{c}{4\pi} H_\alpha(a, \alpha).$$

Finally, the total current is given by

$$\mathcal{J} = -\frac{i}{2} ca\mathcal{E}_0 \left[J_1(ka) - \frac{J_0(ka) H_1^{(1)}(ka)}{H_0^{(1)}(ka)} \right]$$

417. Since the field is two-dimensional, the quantity dI in the general formula $d\sigma_s = dI/\bar{\gamma}_0$ (VIII.22) must be looked upon as the intensity of secondary waves per unit length within an angle $d\alpha$, so that $dI = \bar{\gamma} r d\alpha$. The effective differential scattering cross-section will then have the dimensions of length. Using the results of Problem 416, we have

$$d\sigma_s = |f(\alpha)|^2 d\alpha,$$

where

$$f(\alpha) = \sqrt{\frac{2}{\pi k}} \sum_m i^m \frac{J_m(ka)}{H_m^{(1)}(ka)} e^{i\left(m\alpha - \frac{m\pi}{2} - \frac{\pi}{4}\right)} \quad (1)$$

In general, the latter result is very complicated. However, it becomes much simpler when $ka \ll 1$. It is then sufficient to retain only the term with $m = 0$, which yields an isotropic distribution of the secondary radiation:

$$d\sigma_s = \frac{\pi dz}{2k \ln^2(ka)} = \frac{\lambda}{4 \ln^2(ka)} dz. \quad (2)$$

The total cross-section is obtained by integrating (1) with respect to α . Using the orthogonality of the functions $\exp(im\alpha)$, we have

$$\sigma_s = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J_m(ka)}{H_m^{(1)}(ka)} \right|^2. \quad (3)$$

When $ka \ll 1$, Eqn. (3) assumes the simple form

$$\sigma_s = \frac{\pi\lambda}{2 \ln^2(ka)}.$$

418.

$$\begin{aligned}
 H_z &= \mathcal{H}_0 \left[e^{ikr \cos \alpha} - \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H_m^{(1)'}(ka)} H_m^{(1)}(kr) e^{im\alpha} \right], \\
 E_r &= \mathcal{H}_0 \left[\sin \alpha e^{ikr \cos \alpha} + \frac{1}{kr} \sum_{m=-\infty}^{\infty} i^m \frac{m J'_m(ka)}{H_m^{(1)'}(ka)} H_m^{(1)}(kr) e^{im\alpha} \right], \\
 E_\alpha &= \mathcal{H}_0 \left[\cos \alpha e^{ikr \cos \alpha} + \sum_{m=-\infty}^{\infty} i^{m+1} \frac{J'_m(ka)}{H_m^{(1)'}(ka)} H_m^{(1)}(kr) e^{im\alpha} \right],
 \end{aligned}$$

where α is measured from the direction of $\mathbf{k}$, and the axis of the cylindrical system of coordinates lies along the axis of the cylinder.

$$\begin{aligned}
 d\sigma_s(\alpha) &= \frac{\pi (ka)^3}{8} a (1 - 2 \cos \alpha)^2 d\alpha, \\
 \sigma_s &= \frac{3}{4} \pi^2 k^3 a^4.
 \end{aligned}$$

419.

$$\begin{aligned}
 d\sigma'_s &= \cos^2 \varphi d\sigma_{\parallel} + \sin^2 \varphi d\sigma_{\perp}, \\
 d\sigma''_s &= \frac{1}{2} (d\sigma_{\parallel} + d\sigma_{\perp}).
 \end{aligned}$$

420. The unpolarised wave can be looked upon as a superposition of two incoherent components of equal intensity. The electric vector $\mathbf{E}$ of one of the two components can be taken to lie along the axis of the cylinder, while the electric vector of the other component is then perpendicular to the axis. The scattering cross-sections for the first and second components were obtained in Problems 417 and 418. The degree of depolarisation ρ is given by the ratio of the intensities of the scattered waves:

$$\rho = \frac{d\sigma_{\perp}}{d\sigma_{\parallel}} = \frac{1}{4} (ka)^4 \ln^2(ka) (1 - 2 \cos \alpha)^2.$$

Since $ka \ll 1$, it follows that ρ is very small, *i.e.* the scattered waves are almost completely polarised for all scattering angles. When $\cos \alpha = 0.5$, *i.e.* when $\alpha = 60^\circ$, the depolarisation coefficient is zero.

421.

$$H_z = \mathcal{H}_0 \left[e^{ikr \cos \alpha} + \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka) - i\zeta J_m(ka)}{i\zeta H_m^{(1)}(ka) - H_m^{(1)'}(ka)} H_m^{(1)}(kr) e^{im\alpha} \right],$$

where ζ is the surface impedance of the metal,

$$H_a = H_r = 0, \quad \mathbf{E} = \frac{i}{k} \operatorname{curl} \mathbf{H}.$$

422.

$$Q = \frac{ac\zeta' \mathcal{E}_0^2}{4} \sum_m \left| \frac{J'_m N_m - J_m N'_m}{i\zeta H_m^{(1)} - H_m^{(1)'}} \right|^2,$$

where ζ' is the real part of the surface impedance. The cylindrical functions J_m , N_m , and $H_m^{(1)}$ (cf. Appendix III) and their derivatives are taken at the point ka . The absorption cross-section is given by

$$\sigma_a = \frac{Q}{\gamma_0} = 2\pi a \zeta' \sum_m \left| \frac{J'_m N_m - J_m N'_m}{i\zeta H_m^{(1)} - H_m^{(1)'}} \right|^2.$$

When $ka \ll 1$, i.e. when $\lambda \gg a$, the field in the neighbourhood of the cylinder is quasi-stationary (conducting cylinder in a longitudinal quasi-stationary magnetic field; cf. Problem 367).

Hence, if we express ζ' in terms of the conductivity σ with the aid of (VIII.9) and (VIII.11), we have the following expression for Q :

$$Q = \frac{ac\mathcal{E}_0^2}{8} \sqrt{\frac{\mu\omega}{2\pi\sigma}},$$

which is the same as that found in the solution of Problem 369 for a strong skin effect.

423. When $r > a$

$$E_z = \mathcal{E}_0 \left[e^{ikr \cos \alpha} + \sum_{m=-\infty}^{\infty} i^m \frac{\zeta J'_m(ka) J_m(k'a) - J_m(ka) J'_m(k'a)}{H_m^{(1)}(ka) J'_m(k'a) - \zeta H_m^{(1)'}(ka) J_m(k'a)} \times \right. \\ \left. \times H_m^{(1)}(kr) e^{im\alpha} \right],$$

and when $r < a$

$$E_z = \mathcal{E}_0 \zeta \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(k'a) H_m^{(1)}(ka) - J'_m(k'a) H_m^{(1)'}(ka)}{J'_m(k'a) H_m^{(1)}(ka) - \zeta J_m(k'a) H_m^{(1)}(ka)} J_m(k'r) e^{im\alpha},$$

where $\mathcal{E}_0$ is the amplitude of the incident wave, $\zeta = (\mu/\varepsilon)^{\frac{1}{2}}$, $k = \omega/c$, $k' = \omega(\varepsilon\mu)^{\frac{1}{2}}/c$, and the remaining components of $\mathbf{E}$ are zero. The field $\mathbf{H}$ may be calculated from the formula

$$\mathbf{H} = \frac{c}{i\omega\mu} \operatorname{curl} \mathbf{E}.$$

424. The dipole moments of the sphere may be written in the form

$$\mathbf{p} = \beta_e \mathbf{E}_0 e^{-i\omega t}, \quad \mathbf{m} = \beta_m \mathbf{H}_0 e^{-i\omega t},$$

where β_e and β_m are the electric and magnetic polarisabilities of the sphere, which in general are complex.

The components of the vectors $\mathbf{E}$ and $\mathbf{H}$ of the scattered wave may be obtained from (XII.17) and (XII.20):

$$H_z = E_\theta = \frac{\omega^2 E_0}{c^2 r} (\beta_e \cos \theta + \beta_m) \cos \alpha,$$

$$H_\theta = -E_z = \frac{\omega^2 E_0}{c^2 r} (\beta_e + \beta_m) \sin \alpha.$$

The angles θ and α define the direction of scattering as shown in Fig. 67.

The differential scattering cross-section can be determined from Eqn. (VIII.22) and is given by

$$\frac{d\sigma_s(\theta, \alpha)}{d\Omega} = \frac{\omega^4}{c^4} \left[|\beta_e|^2 (\cos^2 \theta \cos^2 \alpha + \sin^2 \alpha) + |\beta_m|^2 (\cos^2 \theta \sin^2 \alpha + \cos^2 \alpha) + (\beta_e \beta_m^* + \beta_e^* \beta_m) \cos \theta \right].$$

425.

$$\begin{aligned} d\sigma_s(\theta) &= \frac{1}{2} \left[d\sigma_s(\theta, \alpha) + d\sigma_s\left(\theta, \alpha + \frac{\pi}{2}\right) \right] = \\ &= \frac{\omega^4}{2c^4} \left[(|\beta_e|^2 + |\beta_m|^2) (1 + \cos^2 \theta) + 2(\beta_e \beta_m^* + \beta_e^* \beta_m) \cos \theta \right] d\Omega, \\ \sigma_s &= \frac{8\pi\omega^4}{3c^4} (|\beta_e|^2 + |\beta_m|^2). \end{aligned}$$

In order to determine the degree of depolarisation of the scattered radiation, it is necessary to find the principal directions of the polarisation tensor. This can easily be done in the present example because of the symmetry of the problem. For given $\mathbf{k}$ and $\mathbf{n}$ (cf. Fig. 66), the special directions for $\mathbf{E}_0$ will be the direction of the normal to the plane of scattering and the direction lying in the plane of scattering and perpendicular to $\mathbf{k}$.

The differential cross-sections for these two directions were obtained in the solution of the preceding problem. The degree of depolarisation ρ is given by the ratio of the smaller to the larger of these two quantities. When $|\beta_m| < |\beta_e|$,

$$\rho = \frac{d\sigma_s(\theta, 0)}{d\sigma_s\left(\theta, \frac{\pi}{2}\right)} = \left| \frac{\beta_m + \beta_e \cos \theta}{\beta_m \cos \theta + \beta_e} \right|^2.$$

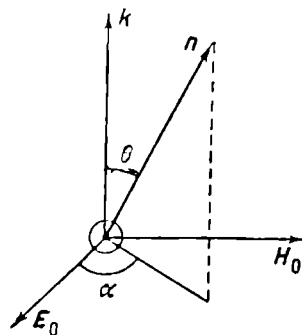


Fig. 67.

426. For a dielectric sphere

$$d\sigma_{s\,d} = \frac{\omega^4 a^6}{2c^4} \left(\frac{\epsilon - 1}{\epsilon + 2} \right)^2 (1 + \cos^2 \theta) d\Omega,$$

$$\sigma_{s\,d} = \frac{8\pi\omega^4 a^6}{3c^4} \left(\frac{\epsilon - 1}{\epsilon + 2} \right)^2; \quad \rho_d = \cos^2 \theta.$$

For a perfectly conducting sphere

$$d\sigma_{s\,cond} = \frac{\omega^4 a^6}{8c^4} [5(1 + \cos^2 \theta) - 8 \cos \theta] d\Omega,$$

$$\sigma_{s\,cond} = \frac{10\pi\omega^4 a^6}{3c^4}, \quad \rho_{cond} = \left(\frac{1 - 2 \cos \theta}{2 - \cos \theta} \right)^2.$$

It is clear from the formula for $d\sigma_{s\,d}$ that the scattering cross-section of a dielectric sphere is symmetric with respect to the forward ($\theta = 0$) and the backward ($\theta = \pi$) directions. Moreover, $d\sigma_{s\,d}(0)/d\sigma_{s\,d}(\pi) = 1$. The scattering cross-section for a conducting sphere is much more anisotropic and asymmetric: $d\sigma_{s\,cond}(0)/d\sigma_{s\,cond}(\pi) = 1/9$. Light scattered by a dielectric sphere through an angle $\theta = \pi/2$ will be completely polarised. Scattering by a perfectly conducting sphere results in complete polarisation when $\cos \theta = 0.5$ ($\theta = 60^\circ$).

The above formulae for the dielectric sphere are rigorous provided it is possible to neglect effects associated with the finite

velocity of propagation of the electromagnetic wave inside the sphere, *i.e.* when the wavelength inside the sphere is large compared with its radius. For a perfectly conducting sphere, the wave will not propagate inside the sphere, and it is sufficient for the condition $a \ll \lambda$ to be satisfied, where λ is the wavelength in the medium surrounding the sphere.

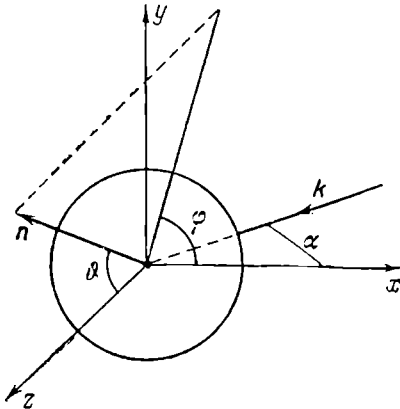


Fig. 68.

427. Just as in Problem 424,

it is necessary to consider the emission of radiation by the induced electric and magnetic dipoles $\mathbf{p}$ and $\mathbf{m}$. Consider the system of coordinates illustrated in Fig. 68. The vector $\mathbf{k}$ of the primary wave lies in the xz plane. Consider two conditions of polarisation of the incident wave, namely, (a) $\mathbf{E}_0$ lying in the plane of incidence, *i.e.* the xz plane,

and (b) $\mathbf{E}_0$ normal to the plane of incidence. For (a) the component of the electric field which is parallel to the plane of the disc is given by $E_{0\parallel} = -E_{0x} = E_0 \cos \alpha$, while the perpendicular component is given by $E_{0\perp} = -E_{0z} = E_0 \sin \alpha$. On this approximation ($a \ll \lambda$), the electric moment $\mathbf{p}$ may be calculated on the assumption that it is the same as the static moment of a conducting disc in a uniform electric field.

According to the solutions of Problems 197 and 199, the longitudinal polarisability of the disc is $\beta_{e\parallel} = 4a^3/3\pi$, while the transverse polarisability is $\beta_{e\perp} = 0$. Hence,

$$p_x = \beta_{e\parallel} E_{0x} = -\frac{4a^3}{3\pi} E_0 \cos \alpha, \quad p_y = p_z = 0.$$

The magnetic field will only have a longitudinal component. However, the longitudinal magnetic polarisability of the disc is zero (cf. Problem 378), and hence $\mathbf{m} = 0$.

The differential scattering cross-section is given by

$$d\sigma_s = \frac{16a^6\omega^4}{9\pi^2c^4} \cos^2 \alpha (1 - \sin^2 \theta \cos^2 \varphi) d\Omega, \quad (1)$$

and the total scattering cross-section is given by

$$\sigma_s = \frac{128a^6\omega^4}{27\pi c^4} \cos^2 \alpha. \quad (2)$$

For condition (b) we have

$$\begin{aligned} p_y &= \frac{4a^3}{3\pi} E_0, \quad p_x = p_z = 0, \quad m_z = \frac{2a^3}{3\pi} E_0 \sin \alpha, \quad m_x = m_y = 0, \\ d\sigma_s &= \frac{16a^6\omega^4}{9\pi^2c^4} \left[1 + \sin^2 \theta \left(\frac{1}{4} \sin^2 \alpha - \sin^2 \varphi \right) + \sin \theta \sin \alpha \cos \varphi \right] d\Omega, \quad (3) \\ \sigma_s &= \frac{128a^6\omega^4}{27\pi c^4} \left(1 + \frac{1}{4} \sin^2 \alpha \right). \end{aligned}$$

For an unpolarised wave, we have from Eqns. (1), (2), and (3)

$$\begin{aligned} d\sigma_s &= \frac{8a^6\omega^4}{9\pi^2c^4} \left[1 + \sin^2 \theta \left(1 - \frac{1}{4} \sin^2 \alpha - \sin^2 \alpha \cos^2 \varphi \right) + \right. \\ &\quad \left. + \cos^2 \alpha + \sin \theta \sin \alpha \cos \varphi \right] d\Omega, \quad (4) \\ \sigma_s &= \frac{128a^6\omega^4}{27\pi c^4} \left(1 - \frac{3}{8} \sin^2 \alpha \right). \end{aligned}$$

428.

$$d\sigma_s = \frac{a^4 h^2 \omega^4 (\epsilon - 1)^2}{18c^4 \epsilon^2} (1 + \cos^2 \theta) d\Omega,$$

where θ is the scattering angle, and hence

$$\sigma_s = \frac{8\pi a^4 h^2 \omega^4 (\epsilon - 1)^2}{27c^4 \epsilon^2}.$$

429. Consider the coordinate system shown in Fig. 69. The vector $\mathbf{k}$ of the primary wave lies in the xz plane. The cylinder may be approximately represented by an elongated ellipsoid of

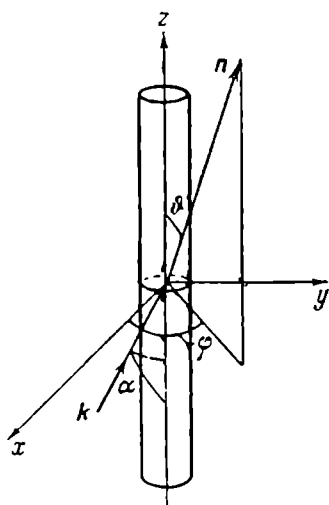


Fig. 69.

revolution with semi-axes a and h . It follows from the solutions of Problems 197, 198, and 378 that the longitudinal component of the electrical polarisability of a very elongated ellipsoid of revolution is larger than its transverse electric and magnetic polarisability by a factor of roughly h/a . Hence, this scattering cross-section will be very dependent on whether there is a finite longitudinal component of the electric field in the incident wave. If this component is appreciable, then the secondary radiation will be

due to the z -component of the electric dipole moment. The remaining components of the electric moment, and also the magnetic moment, may be neglected. Assuming that $\mathbf{E}_0$ lies in the xz plane, we have

$$d\sigma_s = \frac{\omega^4 h^6}{9c^4 |\mathbf{n}|^2 (h/a)} \sin^2 \alpha \sin^2 \theta d\Omega,$$

$$\sigma_s = \frac{8\pi \omega^4 h^6}{27c^4 |\mathbf{n}|^2 (h/a)} \sin^2 \alpha.$$

If the longitudinal component of $\mathbf{E}_0$ is zero, then the scattering is due to the transverse components of the electric moment and the magnetic moment, which have the same order of magnitude. Then,

$$d\sigma_s = \frac{a^4 h^2 \omega^4}{9c^4} \left[(1 + 2n_x \sin \alpha)^2 + 3 \cos^2 \alpha + n_z^2 (4 - \sin^2 \alpha) + 8n_z \cos \alpha + 2n_x n_z \sin 2\alpha \right] d\Omega,$$

$$\sigma_s = \frac{40\pi a^4 h^2 \omega^4}{27c^4} \left(1 + \frac{3}{5} \cos^2 \alpha \right),$$

where n_i ($i = x, y, z$) are the components of the unit vector in the direction of scattering. The scattering cross-section for an unpolarised wave is given by

$$\frac{d\sigma_s}{d\Omega} = \frac{\omega^4 h^6}{18c^4 \ln^2(h/a)} \sin^2 \alpha \sin^2 \vartheta,$$

$$\sigma_s = \frac{4\pi\omega^4 h^6}{27c^4 \ln^2(h/a)} \sin^2 \alpha.$$

430. When the vector $\mathbf{E}_0$ is polarised in the xz plane (Fig. 69),

$$d\sigma_{s\parallel} = \frac{4\omega^4 a^4 h^2}{9c^4} \cdot \left(\frac{\epsilon-1}{\epsilon+1}\right)^2 \left[(1-n_x^2) \cos^2 \alpha + \frac{1}{4} (\epsilon+1)^2 (1-n_z^2) \times \right. \\ \left. \times \sin^2 \alpha - \frac{1}{2} (\epsilon+1) n_x n_z \sin 2\alpha \right] d\Omega.$$

When the vector $\mathbf{E}_0$ is polarised at right angles to the xz plane

$$d\sigma_{s\perp} = \frac{4\omega^4 a^4 h^2}{9c^4} \cdot \left(\frac{\epsilon-1}{\epsilon+1}\right)^2 \cdot (1 - \sin^2 \vartheta \sin^2 \varphi) d\Omega.$$

431. The total intensity of the electric field at a general point in space may be written in the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}, t) + \mathbf{E}'(\mathbf{r}, t), \quad (1)$$

where $\mathbf{E}_0(\mathbf{r}, t) = \mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ represents the incident wave, and $\mathbf{E}'(\mathbf{r}, t)$ represents the scattered (secondary) radiation. At each point inside the body (which may be inhomogeneous) the polarisation vector $\mathbf{P}(\mathbf{r}, t)$ is proportional to $\mathbf{E}$:

$$\mathbf{P}(\mathbf{r}, t) = \frac{\epsilon(\mathbf{r}) - 1}{4\pi} \mathbf{E}(\mathbf{r}, t) = \alpha(\mathbf{r}) \mathbf{E}(\mathbf{r}, t). \quad (2)$$

The secondary field $\mathbf{E}'$ is due to the electric dipole polarisation which is distributed throughout the body with the density given by (2). In order to calculate $\mathbf{E}'$ consider the polarisation potential (Hertz vector) $\mathbf{Z}(\mathbf{r}, t)$ which is due to the polarisation $\mathbf{P}(\mathbf{r}, t)$ [cf. Chapter XII, Eqn. (XII.13)]:

$$\mathbf{Z}(\mathbf{r}, t) = \int \frac{\mathbf{P}(\mathbf{r}') e^{i(kR - \omega t)}}{R} dV'. \quad (3)$$

Substituting

$$\mathbf{E}' = \text{curl curl } \mathbf{Z} - 4\pi \mathbf{P} \quad (4)$$

into (1), and omitting the common time factor $\exp(-i\omega t)$, we

obtain the following integral equation

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r})} + \text{curl curl} \int \frac{\alpha(\mathbf{r}') \mathbf{E}(\mathbf{r}') e^{ikR}}{R} dV' - 4\pi\alpha(\mathbf{r}) \mathbf{E}(\mathbf{r}). \quad (5)$$

This integral equation may be useful in the solution of problems associated with the scattering of electromagnetic waves by bodies whose permittivities approach unity. In such cases, the integral term in (5) is small because $\alpha = (\epsilon - 1)/4\pi$ is small, and the solution can then be found by the method of successive approximations in the form of a power series in α .

432. Inside the sphere ($r < a$), the integral equation (5) of the preceding problem will be of the following form

$$\mathbf{E}_1(\mathbf{r}) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r})} + \alpha \text{curl curl} \int \mathbf{E}_1(\mathbf{r}') \frac{e^{ikR}}{R} dV' - 4\pi\alpha \mathbf{E}_1(\mathbf{r}). \quad (1)$$

Outside the sphere ($r > a$),

$$\mathbf{E}_2(\mathbf{r}) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r})} + \alpha \text{curl curl} \int \mathbf{E}_1(\mathbf{r}') \frac{e^{ikR}}{R} dV', \quad (2)$$

where $\alpha = (\epsilon - 1)/4\pi \ll 1$ is a constant, and the integration is carried out over the volume of the sphere.

The solutions inside and outside the sphere may be sought in the form of a power series in α :

$$\mathbf{E}_1 = \mathbf{E}_1^{(0)} + \alpha \mathbf{E}_1^{(1)} + \dots, \quad \mathbf{E}_2 = \mathbf{E}_2^{(0)} + \alpha \mathbf{E}_2^{(1)} + \dots$$

On substituting these series into (1) and (2) and equating the coefficients of equal powers of α , we have the following zero-order approximation:

$$\mathbf{E}_1^{(0)} = \mathbf{E}_2^{(0)} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r})},$$

which is the undisturbed incident wave. The next approximation yields

$$\begin{aligned} \mathbf{E}_1^{(1)} &= \text{curl curl} \int \mathbf{E}_0 \frac{e^{i(\mathbf{k} \cdot \mathbf{r}' + kR)}}{R} dV' - 4\pi \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r})}, \\ \mathbf{E}_2^{(1)} &= \text{curl curl} \int \mathbf{E}_0 \frac{e^{i(\mathbf{k} \cdot \mathbf{r}' + kR)}}{R} dV'. \end{aligned} \quad (3)$$

These formulae will hold provided the correction term $\alpha \mathbf{E}^{(1)}$ is small compared with the main term $\mathbf{E}_0$, *i.e.*

$$|\alpha \mathbf{E}^{(1)}| \ll \mathbf{E}_0.$$

This inequality should be satisfied for all r . By estimating the integrals in (3) it is possible to derive the condition which must be satisfied by ω , ϵ , and a .

To calculate the cross-section, it is sufficient to determine the field at large distances from the sphere. R in the denominator of the integrand can then be replaced by r , while in the argument of the exponential it can be replaced by $r - \mathbf{n} \cdot \mathbf{r}'$ where $\mathbf{n} = \mathbf{r}/r$. The result is (omitting subscripts):

$$\mathbf{E}^{(1)} = \text{curl curl } \mathcal{E}_0 \frac{e^{ikr}}{r} \int e^{i(\mathbf{k} - k\mathbf{n}) \cdot \mathbf{r}'} dV'. \quad (4)$$

The difference $\mathbf{k} - k\mathbf{n}$ represents the change in the wave vector on scattering. Let this change be denoted by $\mathbf{q}$ where $q = 2k \sin \theta/2$ and θ is the angle of scattering. In order to evaluate the integral, take the polar axis in the direction of $\mathbf{q}$, when

$$\int e^{i(\mathbf{q} \cdot \mathbf{r}') r'^2} d\mathbf{r}' d\Omega = 4\pi \frac{\sin qa - qa \cos qa}{q^3}.$$

In evaluating the double curl in (4) retain only terms proportional to $1/r$, so that

$$\text{curl curl } \mathcal{E}_0 \frac{e^{ikr}}{r} = k^2 [\mathbf{n} \times (\mathcal{E}_0 \times \mathbf{n})] \frac{e^{ikr}}{r}.$$

Finally, the expression for the secondary field $\mathbf{E}' = \alpha \mathbf{E}^{(1)}$ is

$$\mathbf{E}' = \frac{\omega^2 a^3}{c^2} \cdot \frac{\epsilon - 1}{3} [\mathbf{n} \times (\mathcal{E}_0 \times \mathbf{n})] \varphi(qa) \frac{e^{ikr}}{r}, \quad (5)$$

where

$$\varphi(qa) = \frac{3(\sin qa - qa \cos qa)}{(qa)^3} = 3 \sqrt{\frac{\pi}{(qa)^3}} J_{3/2}(qa).$$

Compare the expression given by (5) with the expression which holds for small a (cf. Problem 426). Taking (5) to the limit $qa \ll 1$,

$$\mathbf{E}' = \frac{\omega^2 a^3}{c^2} \cdot \frac{\epsilon - 1}{3} [\mathbf{n} \times (\mathcal{E}_0 \times \mathbf{n})] \frac{e^{ikr}}{r}, \quad (6)$$

since $\varphi(qa) \simeq 1$ when $qa \ll 1$. On the other hand, if $\mathbf{E}'$ is evaluated from the formula

$$\mathbf{E}' = \mathbf{n} \times (\ddot{\mathbf{p}} \times \mathbf{n}) \frac{e^{ikr}}{c^2 r},$$

where

$$\mathbf{p} = \frac{\epsilon - 1}{\epsilon + 2} a^3 \mathbf{E}_0$$

is the static dipole moment of the sphere, we have

$$\mathbf{E}' = \frac{\omega^2 a^3}{c^2} \cdot \frac{\epsilon - 1}{\epsilon + 2} \mathbf{n} \times (\mathbf{E}_0 \times \mathbf{n}) \frac{e^{ikr}}{r}. \quad (6')$$

It is clear that the factor $1/3$ has been replaced by $(\epsilon + 2)^{-1}$. However, this difference between (6) and (6') is not really significant because (6) is valid only to within a factor $(\epsilon - 1)$.

The differential scattering cross-section is

$$\frac{da_s(\theta, \alpha)}{d\Omega} = \frac{\omega^4 a^6 (\epsilon - 1)^2}{9c^4} \varphi^2(qa) (\sin^2 \alpha + \cos^2 \alpha \cos^2 \theta) \quad (7)$$

where the angles θ and α are defined in Fig. 67.

This cross-section differs from the scattering cross-section for a small dielectric sphere (*cf.* solution to Problem 426) by the presence of the new factor $\varphi^2(qa)$ and the fact that the term $(\epsilon + 2)^2$ in the denominator has been replaced by 9. The new factor, $\varphi^2(qa)$, represents the interference between secondary waves from different elements of the sphere. The degree of depolarisation of the scattered light will therefore be the same as for a small dielectric sphere:

$$\rho = \cos^2 \theta. \quad (8)$$

Averaging over the polarisations yields

$$\frac{da_s(\theta)}{d\Omega} = \frac{\omega^4 a^6 (\epsilon - 1)^2}{18c^4} \varphi^2(qa) (1 + \cos^2 \theta).$$

Finally, consider a very large sphere ($ka \gg 1$). If the angles are such that $qa \gg 1$, then $\varphi(qa) \rightarrow 0$, and the cross-section for this range of angles is very small. It follows from the explicit expression for q that $qa \gg 1$ is equivalent to the condition $\theta \gg (ka)^{-1}$. Thus, if the sphere is large, the scattering is mainly in the forward direction, in the angular range $\theta \leq (ka)^{-1}$.

We note that the method involving expansions in powers of a small parameter, which is used in this and in the preceding problem, is analogous to Born's method in quantum mechanics. The latter method is widely used in the theory of scattering of particles by quantum mechanical systems.

433. When $ka \gg 1$, the function $\varphi^2(qa)$, which enters into the expression for the differential cross-section (*cf.* preceding problem), is appreciably different from zero only within the angular range $\theta \leq (ka)^{-1}$. In this angular range, the factor $(1 + \cos^2 \theta)$ may be regarded as constant and equal to 2. Hence,

$$\sigma_s = \frac{2\pi\omega^4 a^6 (\varepsilon - 1)^2}{9c^4} \int_0^\pi \varphi^2(qa) \sin \theta d\theta.$$

Substitute $y = qa = 2ka \sin \theta/2$. In the limit $ka \gg 1$

$$\sigma_s = \frac{\pi\omega^4 a^4 (\varepsilon - 1)^2}{18c^2}.$$

For a small sphere ($ka \ll 1$) we can replace $\varepsilon + 2$ by 3 (*cf.* solution of Problem 426), so that

$$\sigma_s = \frac{8\pi\omega^4 a^6 (\varepsilon - 1)^2}{27c^4}.$$

As can be seen from these results, the dependence on the frequency and on the radius of the sphere is different in the two extremes. Thus, the frequency dependence is of the form $\sim \omega^4$ and $\sim \omega^2$, while the dependence on the radius is of the form $\sim a^6$ and $\sim a^4$.

434. Consider the result

$$\sigma_a = -\frac{1}{E_0^2} \operatorname{Re} \int (\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{n} r^2 d\Omega, \quad (1)$$

where $\mathbf{n} = \mathbf{r}/r$ and σ_a is the absorption cross-section. The integration is carried out over the surface of a large sphere surrounding the scatterer. Eqn. (1) expresses the fact that the scattering cross-section is proportional to the energy flux through the surface of the sphere in the inward direction. Substituting the expression for $\mathbf{E}$ (as given in the problem) into (1) and the following expression for $\mathbf{H}$

$$\mathbf{H} = E_0 \left\{ (\mathbf{n}_0 \times \mathbf{e}) e^{ikz} + [\mathbf{n} \times \mathbf{F}(\mathbf{n})] \frac{e^{ikr}}{r} \right\}$$

and remembering that $\mathbf{n} \cdot \mathbf{F}(\mathbf{n}) = 0$, we have

$$\begin{aligned} \frac{1}{E_0^2} \operatorname{Re} (\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{n} &= (\mathbf{n}_0 \cdot \mathbf{n}) + \frac{|\mathbf{F}|^2}{r^2} + \\ &+ \frac{1}{2} [(\mathbf{e} \cdot \mathbf{F}) + (\mathbf{n}_0 \cdot \mathbf{n})(\mathbf{e} \cdot \mathbf{F}) - (\mathbf{e} \cdot \mathbf{n})(\mathbf{n}_0 \cdot \mathbf{F})] \frac{e^{ik(r-z)}}{r} + \\ &+ \frac{1}{2} [(\mathbf{e}^* \cdot \mathbf{F}^*) + (\mathbf{n}_0 \cdot \mathbf{n})(\mathbf{e}^* \cdot \mathbf{F}^*) - (\mathbf{e}^* \cdot \mathbf{n})(\mathbf{n}_0 \cdot \mathbf{F}^*)] \frac{e^{-ik(r-z)}}{r}. \end{aligned} \quad (2)$$

The integral of the first term with respect to the angles is zero, while the integral of the second term is equal to the total scattering cross-section σ_s . The integrals of the remaining components may be transformed by integrating by parts:

$$\begin{aligned} \frac{1}{r} \int (\mathbf{n}_0 \cdot \mathbf{n})(\mathbf{e} \cdot \mathbf{F}) e^{ik(r-z)} r^2 d\Omega = \\ = \frac{1}{ik} \int_0^{2\pi} d\varphi \left\{ [(\mathbf{n}_0 \cdot \mathbf{n})(\mathbf{e} \cdot \mathbf{F}) e^{ikr(1-\cos\vartheta)}] \Big|_{\vartheta=0}^{\vartheta=\pi} - \right. \\ \left. - \int_0^{\pi} e^{ikr(1-\cos\vartheta)} \frac{\partial}{\partial \cos\vartheta} (\mathbf{n}_0 \cdot \mathbf{n})(\mathbf{e} \cdot \mathbf{F}) d \cos\vartheta \right\}. \end{aligned}$$

Repeated integration by parts in the last integral will give terms proportional to $1/r$, and may therefore be neglected. Moreover, the term with the oscillating factor $\exp(2ikr)$ can also be neglected, since it will not contribute to the total energy flux. In order to show this, we recall that strictly monochromatic waves do not, in fact, exist. In reality, any "monochromatic" wave is really a superposition of harmonics whose frequencies lie within a more or less narrow interval $\Delta\omega$. Since r is large, the result of averaging the factor $\exp(2ikr)$ over any such interval will be zero. It follows that

$$\frac{1}{r} \int (\mathbf{n}_0 \cdot \mathbf{n})(\mathbf{e} \cdot \mathbf{F}) e^{ik(r-z)} r^2 d\Omega = \frac{2\pi i}{k} [\mathbf{e} \cdot \mathbf{F}(\mathbf{n}_0)].$$

The integrals of the remaining components may be evaluated in a similar way. Terms containing the factors $(\mathbf{e} \cdot \mathbf{n})$ and $(\mathbf{e}^* \cdot \mathbf{n})$ will not contribute to the final result in view of the fact that $(\mathbf{e} \cdot \mathbf{n}_0) = 0$. Substituting the evaluated integrals into (1), we have the following final expression

$$\sigma_t = \frac{4\pi}{k} \operatorname{Im} [\mathbf{e} \cdot \mathbf{F}(\mathbf{n}_0)]. \quad (3)$$

The latter result admits of a simple physical interpretation. The total cross-section is a measure of the attenuation of the primary wave. This attenuation is due to the interference between the incident wave and that part of the scattered wave which has the same polarisation and direction of propagation as the incident wave. It follows that the total cross-section depends on the forward scattering amplitude.

435. The scattered wave is due to the electric and magnetic dipole moments which are induced by the incident wave.

The scattering amplitude $\mathbf{F}(\mathbf{n})$ (cf. preceding problem) can be determined from (XII.17) and (XII.20).

The final result is

$$\sigma_a = \frac{4\pi\omega}{c} (\beta_e'' + \beta_m'').$$

436. $\sigma_a = 6\pi b^2 \zeta.$

437. The force is in the direction of the wave vector of the incident wave, and is of the form

$$\bar{F} = \frac{\bar{\gamma}_0}{c} \left[\sigma_a + \int (1 - \cos \vartheta) \left(\frac{d\sigma_s(\vartheta, \varphi)}{d\Omega} \right) d\Omega \right],$$

where $\bar{\gamma}_0$ is the average energy flux density in the incident wave, and the integration is carried out over the entire solid angle.

438. For a perfectly conducting sphere

$$\bar{F} = \frac{43a^6\omega^4}{96c^4} E_0^2.$$

For a dielectric sphere

$$\bar{F} = \frac{a^6\omega^4}{3c^4} \left(\frac{\epsilon - 1}{\epsilon + 2} \right)^2 E_0^2.$$

439. The diffraction formula (VIII.25) may be used. The plane in which the screen is located can be taken as the plane of integration. It follows that on the surface of integration

$$u = A \frac{e^{ikR_1}}{R_1}, \quad dS_n = 2\pi r dr \cos(R_1, z) = 2\pi \frac{z_1 r}{R_1} dr,$$

where $A = \text{const.}$ Substituting these expressions into (VIII.25), and transforming to the new variable $\rho = R + R_1$, we have

$$u_p(z) = -ikAz_1 \int_a^\infty \frac{e^{ik(R+R_1)}}{RR_1^2} r dr = -ikAz_1 \int_{\rho_0}^\infty \frac{e^{ik\rho}}{\rho R_1(\rho)} d\rho. \quad (1)$$

where

$$\rho_0 = \sqrt{a^2 + z^2} + \sqrt{a^2 + z_1^2}.$$

Integration by parts may be used to rewrite (1) in the form of a series with increasing negative powers of $k\rho$. Since $\lambda \ll a$, the higher-order terms are negligible, and only the first term need be retained. Hence,

$$u_p(z) = u_0 \frac{z_1 e^{ik\sqrt{a^2 + z^2}}}{\rho_0},$$

where $u_0 = A(a^2 + z_1^2)^{-1/2} \exp[ik(a^2 + z_1^2)^{1/2}]$ is the amplitude of the incident wave at the surface of the screen. Since the intensity

is proportional to the square of the amplitude, we have

$$I(z) = I_0 \frac{z_1^2}{(\sqrt{a^2 + z_1^2} + \sqrt{a^2 + z^2})^2}. \quad (2)$$

At a point which is symmetrical with respect to the screen ($z_1 = z$) the intensity is given by

$$I(z) = \frac{I_0}{4} \cdot \frac{z^2}{a^2 + z^2}.$$

It follows that a bright axial spot will appear for points which are not too close to the screen. This result, which is in conflict with the hypothesis of the rectilinear propagation of light, was first theoretically predicted by Poisson in 1818. He used this result as an argument against Fresnel's theory of diffraction and the wave theory of light as a whole. However, experiments carried out by Arago and Fresnel confirmed the presence of the bright spot which appears as the result of the symmetry of the screen. Waves from the zone surrounding the periphery of the disc arrive at the axial point with equal phases. It is clear that all points lying along the axial line will have this property, and at such points the intensity will be much greater than at neighbouring points which do not lie on the z -axis.

440. Using the Babinet principle (*cf.* VIII.27), we have for $z = z_1 \gg a$

$$I = I_0 \sin^2 \frac{ka^2}{2z},$$

where I_0 is the intensity of the primary wave at the aperture.

441. When $z \gg a$, $I = 4I_0 \sin^2 (ka^2/4z)$. The intensity along the axis of the circular aperture oscillates, decreasing to zero for $z \rightarrow \infty$. The reduction in the intensity is due to the fact that the parallel incident beam becomes divergent after diffraction at the aperture.

442. Using (VIII.26) we have, in the case of Fraunhofer diffraction,

$$dI = I_0 \frac{[aJ_1(ak\alpha) - bJ_1(bk\alpha)]^2}{a^2} d\Omega,$$

where α is the angle of diffraction, and I_0 is the intensity of the incident light. For a circular aperture

$$dI = I'_0 \frac{J_1^2(ak\alpha)}{\pi\alpha^2} d\Omega,$$

where $I'_0 \sim \pi a^2 |u_0|^2$ is the total intensity of the radiation incident

on the aperture.

443. The diffracted wave is given by

$$u_P = \frac{u_0 e^{ikR_0}}{2\pi i R_0} \int e^{i\mathbf{q}_{\parallel} \cdot \mathbf{r}} dS_n,$$

where $\mathbf{k}' - \mathbf{k} = \mathbf{q}$; $\mathbf{q}_{\parallel}$ and $\mathbf{q}_{\perp}$ are respectively the components of $\mathbf{q}$ in the plane of the screen and in the perpendicular direction.

It is convenient to use polar coordinates with the origin at the centre of the aperture and the polar axis in the direction of $\mathbf{q}_{\parallel}$. This yields

$$u_P = \frac{u_0 e^{ikR_0} k \cos \theta}{2\pi i R_0} \int e^{-iq_{\parallel} r \cos \varphi} r dr d\varphi,$$

where θ is the angle of incidence. Using Eqns. (9) and (11) of Appendix III, we have

$$dI = |u_P|^2 R_0^2 d\Omega = I_0 \frac{J_1^2(q_{\parallel} a)}{\pi q_{\parallel}^2} d\Omega,$$

where $I_0 \sim |u_0|^2 \pi a^2 \cos \theta$ is the total intensity of the light incident on the aperture.

Assuming that the angle of diffraction α , *i.e.* the angle between $\mathbf{k}$ and $\mathbf{k}'$, is small, we have

$$q_{\parallel} = kx \sqrt{1 - \sin^2 \theta \cos^2 \alpha'},$$

$$dI = I_0 \frac{J_1^2(ka x \sqrt{1 - \sin^2 \theta \cos^2 \alpha'})}{\pi x^2 (1 - \sin^2 \theta \cos^2 \alpha')} d\Omega,$$

where α' is the azimuthal angle between $\mathbf{q}$ and the plane of incidence. The latter formula does not hold at glancing angles ($\theta \simeq \pi/2$).

444. Application of Kirchhoff's formula in vector form (VIII.28) yields the following expressions for the radiated field

$$E_y = H_x = -ikabE_0 \frac{e^{ikR}}{\pi R} \left(\frac{\sin k'_x a}{k'_x a} \right) \left(\frac{\sin k'_y b}{k'_y b} \right) (1 + \cos \theta) \sin \alpha,$$

$$E_x = -H_y = -ikabE_0 \frac{e^{ikR}}{\pi R} \left(\frac{\sin k'_x a}{k'_x a} \right) \left(\frac{\sin k'_y b}{k'_y b} \right) (1 + \cos \theta) \cos \alpha,$$

where θ , α are the polar angles measured from an axis perpendicular to the plane of the aperture; $k'_x = k \sin \theta \cos \alpha$ and $k'_y = k \sin \theta \sin \alpha$ are the components of the wave vector of the diffracted wave.

The angular distribution of the radiation is given by

$$dI = I_0 \frac{abk^2}{4\pi^2} \left(\frac{\sin k'_x a}{k'_x a} \right)^2 \left(\frac{\sin k'_y b}{k'_y b} \right)^2 (1 + \cos \vartheta)^2 d\Omega,$$

where $I_0 = (cab/2\pi)E_0^2$ is the intensity of the incident wave.

445. Let the x -, y -, and z -axes lie in the direction of the vectors $\mathbf{E}_0$, $\mathbf{H}_0$, and $\mathbf{k}$ respectively. The radiated field will then be given by

$$\begin{aligned} E_\vartheta = H_\alpha &= -\frac{ika^2 E_0}{2} \cdot \frac{e^{ikR}}{R} \left(\frac{J_1(ka \sin \vartheta)}{ka \sin \vartheta} \right) (1 + \cos \vartheta) \cos \alpha, \\ E_\alpha = -H_\vartheta &= \frac{ika^2 E_0}{2} \cdot \frac{e^{ikR}}{R} \left(\frac{J_1(ka \sin \vartheta)}{ka \sin \vartheta} \right) (1 + \cos \vartheta) \sin \alpha, \\ dI &= \frac{1}{4} I_0 \left(\frac{J_1(ka \sin \vartheta)}{\sin \vartheta} \right)^2 (1 + \cos \vartheta)^2 d\Omega, \end{aligned}$$

where $I_0 = (ca^2/8)E_0^2$ is the intensity of the incident wave. When $\vartheta \ll 1$, we have

$$dI = I_0 \frac{J_1^2(ka\vartheta)}{\pi\vartheta^2} d\Omega.$$

This result was obtained in the solution of Problem 442 with the aid of the scalar diffraction formula.

3. Plane waves in anisotropic and gyrotropic media

$$446. \quad \cos \alpha = \frac{(\epsilon_{\parallel} - \epsilon_{\perp}) \sin \theta \cos \vartheta}{\sqrt{\epsilon_{\parallel}^2 \cos^2 \theta + \epsilon_{\perp}^2 \sin^2 \theta}}, \quad \tan \vartheta = \frac{\epsilon_{\perp}}{\epsilon_{\parallel}} \tan \theta.$$

447. The boundary conditions for the field vectors will be satisfied at all points on the boundary of separation when the tangential components of the wave vectors of the incident, reflected and both refracted waves, are equal at the boundary of separation. For the ordinary wave this gives

$$k_0 \sin \theta_0 = k_1 \sin \theta'_2, \quad \frac{\sin \theta'_2}{\sin \theta_0} = \sqrt{\epsilon_{\perp} \epsilon_{\parallel} \mu}.$$

The ray direction (the direction of the Poynting vector) in the ordinary wave is the same as the direction of the wave vector, and is therefore at an angle θ'_2 to the normal to the boundary. In the case of the extraordinary wave

$$k_0 \sin \theta_0 = k_2 \sin \theta''_2 = k_0 \sin \theta''_2 \sqrt{\frac{\epsilon_{\perp} \epsilon_{\parallel} \mu}{\epsilon_{\perp} \sin^2 \theta''_2 + \epsilon_{\parallel} \cos^2 \theta''_2}}$$

[cf. (VIII.30)]. Hence,

$$\sin^2 \theta_2'' = \frac{\epsilon_{\parallel} \sin^2 \theta_0}{\epsilon_{\perp} \epsilon_{\parallel} \mu + (\epsilon_{\parallel} - \epsilon_{\perp}) \sin^2 \theta_0}.$$

The angle ϑ'' between the ray and the optic axis (which is parallel to the normal to the separation boundary) is, in accordance with the result of the preceding problem, given by

$$\tan \vartheta'' = \frac{\epsilon_{\perp}}{\epsilon_{\parallel}} \tan \theta_2'' = \frac{\sqrt{\epsilon_{\perp}} \sin \theta_0}{\sqrt{\epsilon_{\parallel} (\epsilon_{\parallel} \mu - \sin^2 \theta_0)}}.$$

The angle of reflection from the crystal is equal to the angle of incidence, just as for an isotropic medium.

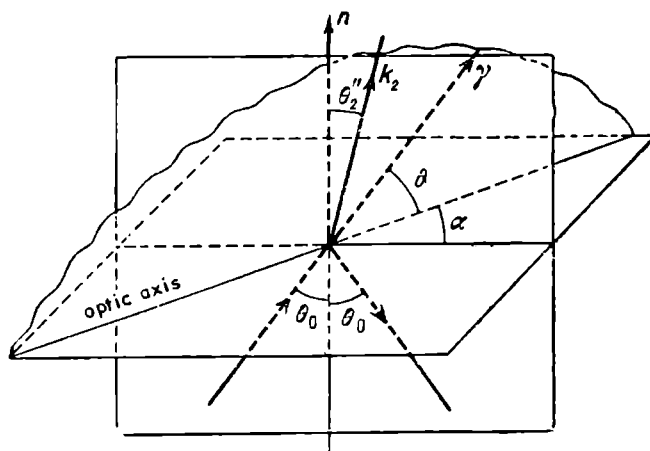


Fig. 70.

448. The ordinary ray lies in the plane of incidence, and makes an angle θ_2' with the normal to the surface, where

$$\sin \theta_2' = \sqrt{\epsilon_{\perp} \mu} \sin \theta_0.$$

The wave vector $\mathbf{k}_2$ of the extraordinary wave lies in the plane of incidence and makes an angle θ_2'' with the normal, which is given by

$$\sin^2 \theta_2'' = \frac{\epsilon_{\perp} \sin^2 \theta_0}{\epsilon_{\perp} \epsilon_{\parallel} \mu + (\epsilon_{\perp} - \epsilon_{\parallel}) \sin^2 \theta_0 \cos^2 \alpha}$$

The ray direction in the extraordinary wave does not lie in the plane of incidence. The ray direction lies in the same plane as $\mathbf{k}_2$ and the optic axis, and makes an angle ϑ with the latter, where (Fig. 70)

$$\tan \vartheta = \frac{\sqrt{\epsilon_{\perp}^2 \epsilon_{\parallel} \mu + \epsilon_{\perp} (\epsilon_{\perp} - \epsilon_{\parallel}) \sin^2 \theta_0 \cos^2 \alpha}}{\epsilon_{\parallel} \sin \theta_0 \cos \alpha}.$$

449. Substitution of the plane wave expressions for $\mathbf{E}$ and $\mathbf{H}$ into Maxwell's equations (VIII.1)–(VIII.4) produces the equation for the amplitudes and the wave vectors of waves which may propagate in the given medium:

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{H}_0) = -\frac{\omega^2 \epsilon}{c^2} \hat{\mu} \mathbf{H}_0. \quad (1)$$

Let θ be the angle between the wave vector $\mathbf{k}$ and the z -axis, and consider the components of (1) along the coordinate axes.

A biquadratic equation in k is obtained by equating the determinant of the system to zero. The solution of this equation is

$$k_{1,2}^2 = \frac{\omega^2 \epsilon \mu_{\parallel}}{2c^2} \cdot \frac{\mu \sin^2 \theta + (2\mu_{\perp}/\mu_{\parallel}) \pm \sqrt{\mu^2 \sin^4 \theta + (2\mu_a/\mu_{\parallel})^2 \cos^2 \theta}}{(\mu_{\perp}/\mu_{\parallel} - 1) \sin^2 \theta + 1}, \quad (2)$$

where

$$\mu = \frac{\mu_{\perp}^2 - \mu_a^2 - \mu_{\perp} \mu_{\parallel}}{\mu_{\parallel}^2}.$$

Two waves can propagate in each direction with different phase velocities $v_{1,2} = \omega/k_{1,2}$. These velocities are functions of the angle θ . There are no directions for which the two phase velocities are equal, since the expression under the square root in (2) does not vanish for any value of θ . With $\mu_a = 0$ in Eqn. (2), the latter will give the phase velocities of waves which can propagate in a non-gyrotropic but magnetically anisotropic crystal:

$$k_1^2 = \frac{\omega^2}{c^2} \epsilon \mu_{\perp}, \quad k_2^2 = \frac{\omega^2}{c^2} \cdot \frac{\epsilon \mu_{\perp} \epsilon_{\parallel}}{\mu_{\parallel} \cos^2 \theta + \mu_{\perp} \sin^2 \theta}.$$

The first of these waves (ordinary waves) has a velocity $v_1 = c/(\epsilon \mu_{\perp})^{1/2}$ which is independent of the direction of propagation. The velocity of the second wave (extraordinary wave) is a function of the angle between the axis of symmetry of the crystal and the direction of propagation. The two velocities are equal, and the two waves become indistinguishable, when the direction of propagation is along the axis of symmetry ($\theta = 0$).

450. Two waves can propagate in any direction with phase velocities $v_{1,2} = \omega/k_{1,2}$, where $k_{1,2}$ is given by Eqn. (2) in the solution of the preceding problem, in which the magnetic quantities should be replaced (for the purposes of this problem) by the corresponding electrical quantities.

451. A plane wave propagating in the direction of a constant magnetic field will split into two waves with opposite circular

polarisations and different phase velocities $v_{\pm} = c[\epsilon(\mu_{\perp} \pm \mu_Q)]^{-1/2}$. When the propagation takes place at right angles to the constant magnetic field, one of the waves [velocity $v = c/(\epsilon\mu_{\parallel})^{1/2}$] will be purely transverse ($\mathbf{E} \perp \mathbf{k}$, $\mathbf{H} \perp \mathbf{k}$). It is analogous to the waves which propagate in an isotropic medium with scalar parameters ϵ and $\mu = \mu_{\parallel}$. In the second wave {velocity $v = c[\mu_{\perp}/\epsilon(\mu_{\perp}^2 - \mu_Q^2)]^{1/2}$ } the vector $\mathbf{E}$ will be parallel to the constant magnetic field, while the vector $\mathbf{H}$ will have a component in the direction of propagation. Thus, a wave with an arbitrary polarisation will split into two linearly polarised waves.

All the results obtained in this problem remain valid when ϵ is a Hermitian tensor and μ is a scalar. All that is required is that the magnetic quantities should be replaced by the corresponding electrical quantities, and *vice versa*.

452. It was shown in the solution of the preceding problem that two waves with different phase velocities and opposite circular polarisations can propagate in the direction of a magnetic field. It follows that a wave whose polarisation is not circular will split into two circularly polarised waves. Since the phase velocities of the two waves are different, the phase difference between them will vary from point to point, and hence the polarisation of the resultant wave will be different at different points.

A detailed calculation will show that the polarisation of the resultant wave will remain linear, but the plane of polarisation will rotate through an angle $\chi = (k_+ - k_-)(z/2)$. This is known as the Faraday effect. The quantities k_+ and k_- are the wave vectors of the two circularly polarised waves, and may be found from the solutions of Problems 451 and 318. For a weak magnetic field

$$\chi = VHz,$$

where V is the Verdet constant. When the atoms of the material may be looked upon as harmonic oscillators, the constant V is given by

$$V = \frac{2\pi e^3 N}{nm^2 c} \cdot \frac{\omega^2}{(\omega^2 - \omega_0^2)^2},$$

where $n = \epsilon^{1/2}$ is the refractive index in the absence of the magnetic field.

453. It follows from symmetry considerations that the wave vectors of the reflected and transmitted waves should be perpendicular to the separation boundary. Both waves will be circularly

polarised in the same direction as the incident wave. The amplitude of the reflected wave is

$$H_1 = \frac{V_{\epsilon} - \sqrt{\mu_{\perp} \pm \mu_a}}{V_{\epsilon} + \sqrt{\mu_{\perp} \pm \mu_a}} H_0,$$

where H_0 is the amplitude of the incident wave, ϵ is the permittivity, and $\mu_{\perp}$ and μ_a are the components of the permeability tensor of the ferrite (*cf.* Problem 449). The amplitude of the transmitted wave is given by

$$H_2 = \frac{2V_{\epsilon}}{V_{\epsilon} + \sqrt{\mu_{\perp} \pm \mu_a}} H_0,$$

where the signs + and - correspond to waves with right- and left-handed circular polarisations.

454. The wave vectors of the reflected and transmitted waves are perpendicular to the separation boundary. The reflected wave is elliptically polarised with semi-axes

$$H'_1 = H_0 \frac{\epsilon - \sqrt{\mu^2 - \mu_a^2}}{(V_{\epsilon} + \sqrt{\mu + \mu_a})(V_{\epsilon} + \sqrt{\mu - \mu_a})},$$

$$H''_1 = H_0 \frac{\sqrt{\epsilon(\mu - \mu_a)} - \sqrt{\epsilon(\mu + \mu_a)}}{(V_{\epsilon} + \sqrt{\mu + \mu_a})(V_{\epsilon} + \sqrt{\mu - \mu_a})}.$$

The direction of H'_1 is the same as the direction of polarisation of the vector $\mathbf{H}$ of the incident wave. The transmitted wave splits into two waves with amplitudes

$$H'_2 = \frac{H_0 V_{\epsilon}}{V_{\epsilon} + \sqrt{\mu + \mu_a}}, \quad H''_2 = \frac{H_0 V_{\epsilon}}{V_{\epsilon} + \sqrt{\mu - \mu_a}},$$

which are circularly polarised in opposite directions. Their velocities of propagation are different (*cf.* solution of Problem 451).

455. If the wavelength is much smaller than the radius of the discs and smaller than the distance between neighbouring discs, then the artificial dielectric may be looked upon as a continuous medium. The electric field of the incident wave is parallel to the planes of the discs. Hence, in the absence of the external magnetic field, $\mathbf{H}_0$, the polarisability of the dielectric will be $\alpha = N\beta_e$, where $\beta_e = 4a^3/3\pi$ is the longitudinal (relative to the plane of the disc) electrical polarisability of a disc, and N is the number of discs per unit volume.

The longitudinal magnetic polarisability of a disc β_m is zero (*cf.* Problem 378) and hence the magnetic susceptibility of the dielectric for the given direction of the magnetic field is also zero.

The presence of the external magnetic field $\mathbf{H}_0$ gives rise to the Hall effect. Thus, conduction electrons which make up the current in each disc will be deflected under the action of the field $\mathbf{H}_0$ and will give rise to an additional electric field $\mathbf{E}_H$ which should balance the deflecting effect of the magnetic field. This gives rise to an additional electric moment in each disc which in turn produces a change in the polarisation vector in the medium and in the electric induction. To calculate the change in the induction, it is convenient to consider the total density of the polarisation current $\partial \mathbf{P} / \partial t$ in the dielectric, rather than the current in an isolated disc.

On the first approximation in H_0 , the field $\mathbf{E}_H$ due to the Hall effect is given by

$$\mathbf{E}_H = R(\mathbf{H}_0 \times \mathbf{j}) = R\left(\mathbf{H}_0 \times \frac{\partial \mathbf{P}}{\partial t}\right), \quad (1)$$

where R is the Hall constant and $\mathbf{P} = \alpha \mathbf{E}$ is the polarisation on the zero-order approximation. Owing to the presence of the field $\mathbf{E}_H$, the polarisation vector will increase by

$$\Delta \mathbf{P} = \alpha \mathbf{E}_H = \alpha^2 R\left(\mathbf{H}_0 \times \frac{\partial \mathbf{E}}{\partial t}\right),$$

and hence the induction $\mathbf{D}$ will be given by

$$\mathbf{D} = \mathbf{E} + 4\pi(\mathbf{P} + \Delta \mathbf{P}) = \epsilon \mathbf{E} + 4\pi \alpha^2 R\left(\mathbf{H}_0 \times \frac{\partial \mathbf{E}}{\partial t}\right), \quad (2)$$

where $\epsilon = 1 + 4\pi N\beta_e$ is the permittivity in the absence of the external magnetic field.

When the electric field $\mathbf{E}$ is a simple harmonic function of time, Eqn. (2) will give the following relation between $\mathbf{D}$ and $\mathbf{E}$

$$\mathbf{D} = \epsilon \mathbf{E} + i(\mathbf{E} \times \mathbf{g}),$$

where $\mathbf{g} = 4\pi \alpha^2 \omega R \mathbf{H}_0$ is the gyration vector [*cf.* (VIII.32)]. Thus, the medium will be gyrotropic. It follows from the results of Problem 451 that two waves can propagate in the direction of the vector $\mathbf{g}$. The two waves are circularly polarised in opposite directions, and have different phase velocities $v_{\pm} = \omega/k_{\pm}$. On

calculating $k_{\pm}$ in the usual way we find that

$$k_{\pm}^2 = \frac{\omega^2}{c^2} (\varepsilon \pm g).$$

456. The solution of the Maxwell equations will be sought in the form of plane waves. The amplitude $\mathbf{E}_0$ of such waves must satisfy the following equations

$$\mathbf{k} \times \mathbf{E}_0 = \frac{\omega}{c} \mathbf{H}_0, \quad (1)$$

$$\mathbf{k} \times \mathbf{H}_0 = -\frac{\omega}{c} \varepsilon(\omega) \mathbf{E}_0.$$

However, for a longitudinal electric field $\mathbf{k} \times \mathbf{E}_0 = 0$, and hence $\mathbf{H}_0 = 0$ and $\varepsilon(\omega)\mathbf{E}_0 = 0$.

It follows from the latter result that a longitudinal electric field will exist provided

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)} = 0. \quad (2)$$

The frequency of longitudinal plasma waves is defined by this equation and turns out to be complex. When $\gamma \neq 0$, the oscillations are damped out. When $\gamma \ll \omega$, attenuation may be neglected and Eqn. (2) yields

$$\omega = \omega_p = \sqrt{\frac{4\pi e^2 N}{m}}. \quad (3)$$

According to this formula, the frequency ω is independent of the wave vector, and hence the group velocity of longitudinal plasma waves is zero. However, this result holds only on the first approximation, and is due to the fact that the spatial non-uniformity of the electric field is not taken into account. Longitudinal plasma waves consist of oscillations of the electron cloud relative to the ion cloud (the ions are assumed to be fixed on the present approximation).

Longitudinal oscillations may occur not only in the plasma, but also in other media at frequencies ω_{α} for which $\varepsilon(\omega_{\alpha}) = 0$. However, in the plasma, the attenuation of these oscillations may be small, while in other media it is usually large.

457. It follows from the solution of Problem 451 that two waves with opposite circular polarisations can propagate in the direction of the magnetic field. The wave vectors of these waves

are given by (*cf.* Problem 321)

$$\frac{k^2 c^2}{\omega^2} = \epsilon_{\pm} = 1 - \frac{\omega_p^2}{\omega \left(\omega + i\gamma \mp \omega_H - \frac{\omega_H \Omega_H}{\omega} \right)}.$$

When $\Omega_H \ll \omega$ the effect of the motion of the positive ions is very small and may be neglected. In the opposite case, namely $\Omega_H \gg \omega$ and $\gamma\omega \ll \omega_H \Omega_H$, the positive ions play the leading role in the process, and

$$\frac{k^2 c^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega_H \Omega_H} = 1 + \frac{4\pi N M c^2}{H_0^2}.$$

Both waves propagate with the same phase velocity v_φ which is equal to their group velocity v_g :

$$v_g = v_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{1 + \frac{4\pi N M c^2}{H_0^2}}} \quad (2)$$

or

$$v_g = v_\varphi = \frac{H_0}{\sqrt{4\pi N M}} = \frac{H_0}{\sqrt{4\pi\tau}}. \quad (3)$$

The latter result holds provided that the second term in the denominator of Eqn. (2) is $\gg 1$. The quantity $\tau = NM$ is the density of the gas (the electron mass can clearly be neglected). If the motion of the positive ions is not taken into account, then instead of the finite constant velocity given by (3), we would have a zero velocity for $\omega \rightarrow 0$, and the corresponding waves would not be able to propagate. Thus, mechanical oscillations of a gas and the oscillations of the electromagnetic field are in this case very closely related. Waves propagating with the velocity given by (3) are known as magnetohydrodynamic waves. They are of particular importance in astrophysical and other processes.

458. The linearised equation relating the amplitudes of the high-frequency components of the magnetisation ($\mathbf{m}_0$) and the magnetic field ($\mathbf{h}_0$) can be deduced from (VI.15) and (VI.16) and is

$$i\omega \mathbf{m}_0 = -\gamma(\mathbf{M}_0 \times \mathbf{h}_0) - \gamma(\mathbf{m}_0 \times \mathbf{H}_0) + \gamma q k^2 (\mathbf{M}_0 \times \mathbf{m}_0), \quad (1)$$

where $\mathbf{M}_0$ is the saturation magnetisation and is parallel to the magnetic field $\mathbf{H}_0$. If the z -axis is chosen to be parallel to $\mathbf{H}_0$

($z = x_3$), then the components of the tensor μ_{ik} are [cf. Eqn. (1)]:

$$\mu_{11} = \mu_{22} = 1 + \frac{\omega_M (\omega_0 + \alpha k^2)}{(\omega_0 + \alpha k^2) - \omega^2}, \quad (2)$$

$$\mu_{12} = -\mu_{21} = -i \frac{\omega \omega_M}{(\omega_0 + \alpha k^2) - \omega^2}, \quad \mu_{33} = 1,$$

where

$$\omega_0 = \gamma H_0, \quad \omega_M = 4\pi\gamma M_0, \quad \alpha = q\gamma M_0.$$

The remaining components of μ_{ik} are zero.

As can be seen from (2), the magnetic permeability is now a function of both frequency and the wave vector. This is due to the fact that the magnetisation at each point depends on the magnitude of the magnetic field not only at that particular point but also at the neighbouring points (this is due to the term $q\nabla^2 \mathbf{M}$ in the expression for $\mathbf{H}_{\text{eff}}$). The dependence of the permittivity or the magnetic permeability on the wave vector is called spatial dispersion. The dependence of μ on $\mathbf{k}$ is significant only for very non-uniform fields (short wavelengths).

459. Consider the plane wave solutions of the Maxwell equations and the equation of motion of the magnetisation vector (VI.15):

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad \mathbf{H} = \mathbf{H}_0 + \mathbf{h}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad \mathbf{M} = \mathbf{M}_0 + \mathbf{m}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}. \quad (1)$$

The amplitudes of the fields and of the magnetisation satisfy the following system of equations

$$c(\mathbf{k} \times \mathbf{h}_0) = -\omega \varepsilon \mathbf{E}_0, \quad c(\mathbf{k} \times \mathbf{E}_0) = \omega(\mathbf{h}_0 + 4\pi \mathbf{m}_0), \quad \mathbf{k} \cdot (\mathbf{h}_0 + 4\pi \mathbf{m}_0) = 0, \quad (2)$$

$$i\omega \mathbf{m}_0 = -\gamma(\mathbf{M}_0 \times \mathbf{h}_0) - \gamma(\mathbf{m}_0 \times \mathbf{H}_0) + \gamma q k^2 (\mathbf{M}_0 \times \mathbf{m}_0). \quad (3)$$

Eliminating $\mathbf{E}_0$ and $\mathbf{h}_0$ from (2) and (3) and substituting

$$u = \frac{\omega_M}{\Omega}, \quad x = \frac{\omega}{\Omega}, \quad \xi = \frac{ck}{\Omega \sqrt{\varepsilon}}, \quad \Omega = \omega_0 + \omega_1 + \omega_M,$$

$$\omega_0 = \gamma H_0, \quad \omega_1 = \gamma q k^2 M_0, \quad \omega_M = 4\pi\gamma M_0,$$

we have

$$ix\mathbf{m}_0 = \frac{u}{x^2 - \xi^2} [x^2(\mathbf{e}_z \times \mathbf{m}_0) + \xi^2(\mathbf{n} \cdot \mathbf{m}_0)(\mathbf{e}_z \times \mathbf{n})] + (1 - u)(\mathbf{e}_z \times \mathbf{m}_0), \quad (4)$$

where $\mathbf{n} = \mathbf{k}/k$, $\mathbf{e}_z$ is a unit vector in the direction of $\mathbf{H}_0$, and $\mathbf{M}_0$ is parallel to $\mathbf{H}_0$.

Take the x -axis in the $\mathbf{n}$, $\mathbf{e}_z$ plane and denote the angle between $\mathbf{e}_z$ and $\mathbf{n}$ by θ . The following system of linear equations is then obtained from Eqn. (4):

$$\begin{aligned} ixm_{0x} + \left(1 + \frac{u_z^2}{x^2 - \xi^2}\right)m_{0y} &= 0, \\ \left(1 + \frac{u_z^2}{x^2 - \xi^2} \cos^2 \theta\right)m_{0x} - ixm_{0y} &= 0. \end{aligned}$$

The condition that these equations should be consistent yields the required dispersion relation:

$$\left(1 + \frac{u_z^2}{x^2 - \xi^2}\right)\left(1 + \frac{u_z^2}{x^2 - \xi^2} \cos^2 \theta\right) - x^2 = 0. \quad (5)$$

This equation is of the third degree in ω^2 ($\omega^2 = \Omega^2 x^2$; Ω is independent of ω) and hence three types of wave with different dispersion relations can propagate in the medium under consideration. Two of these dispersion relations were investigated in the solution of Problem 449, where it was assumed that $\omega_1 = 0$. They correspond to ordinary electromagnetic waves propagating in a gyrotropic medium. To investigate the third type of wave, let us suppose that $\omega^2 \epsilon / c^2 k^2 \ll 1$ ($x^2 \ll \xi^2$). Neglecting x^2 in comparison with ξ^2 in the denominator of (4), we have the third dispersion relation

$$\omega^2 = (\omega_0 + \omega_1)(\omega_0 + \omega_1 + \omega_M \sin^2 \theta), \quad (6)$$

where $\omega_1 = q\gamma k^2 M_0$. Since $\omega^2 \epsilon \ll c^2 k^2$ we find, assuming that ω_0 , ω_1 , and ω_M are comparable in magnitude, that the dispersion law (6) will hold provided $\xi^2 \gg 1$.

Consider now the relative magnitudes of E_0 and h_0 for waves with dispersion relation (6). Using the Maxwell equation (2) and the condition $\omega^2 \epsilon / c^2 k^2 \ll 1$ we have

$$E_0 \simeq \frac{4\pi\omega}{ck^2} (\mathbf{k} \times \mathbf{m}); \quad h_0 \simeq 4\pi\mathbf{n} (\mathbf{n} \cdot \mathbf{m}).$$

Thus, $E_0 \ll h_0$. The waves under consideration are purely magnetic oscillations of the magnetisation vector, for which the electric field does not appear. They are known as spin waves, and determine many magnetic, thermal, and electrical properties of ferromagnetics.

460. Let the y -axis be perpendicular to the surface of the metal (positive in the inward direction) and let the z -axis lie along the magnetic field. Since the impedance ξ is independent

of the angle of incidence of the wave, we shall consider normal incidence only. Solution of Maxwell's equations and the definition of surface impedance yield

$$\zeta_{xx} = (1 - i) \sqrt{\frac{\omega \mu_{\parallel}}{8\pi\sigma}}, \quad \zeta_{zz} = (1 - i) \sqrt{\frac{\omega \mu_{\perp}}{8\pi\sigma_3}}, \quad \zeta_{xz} = \zeta_{zx} = 0,$$

where

$$\sigma = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1}, \quad \mu = \frac{\mu_{\perp}^2 - \mu_a^2}{\mu_{\perp}}.$$

The dependence of ζ_{zz} on frequency has a resonance character (*cf.* Problem 329, in which the components of μ_{ik} are calculated). The component ζ_{xx} does not exhibit resonance properties, since $\mu_{\parallel} = 1$.

$$461. \quad \zeta_{\pm} = \pm \frac{E_{\pm 1}}{h_{\pm 1}} = -(1 - i) \sqrt{\frac{\omega \mu^{\pm}}{8\pi\sigma^{\pm}}},$$

where $\mu^{\pm} = \mu_{\perp} \pm \mu_a$, $\sigma^{\pm} = \sigma_1 \pm \sigma_2$; $E_{\pm 1}$ and $h_{\pm 1}$ are the cyclic components of $\mathbf{E}$ and $\mathbf{h}$ [$h_{\pm 1} = \mp (h_x \pm ih_y)/\sqrt{2}$].

Chapter

9

ELECTROMAGNETIC OSCILLATIONS IN BOUNDED BODIES

462. For E -waves

$$\mathcal{E}_z = \mathcal{E}_0 \sin \kappa_1 x \sin \kappa_2 y,$$

where

$$\kappa_1 = \frac{n_1 \pi}{a}, \quad \kappa_2 = \frac{n_2 \pi}{b}, \quad n_1, n_2 = 1, 2, \dots$$

For H -waves

$$\mathcal{H}_z = \mathcal{H}_0 \cos \kappa_1 x \cos \kappa_2 y$$

with the same κ_1, κ_2 . It follows that in transverse directions the field is of the standing wave type. The relation between the propagation constant k and ω is of the form

$$k^2 = \frac{\omega^2}{c^2} - \pi^2 \left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} \right).$$

The transverse field components may be expressed in terms of $\mathcal{E}_z, \mathcal{H}_z$ with the aid of Maxwell's equations.

463. For E -waves

$$\alpha = \frac{2\zeta' \omega}{ck\kappa^2 ab} (\kappa_1^2 b + \kappa_2^2 a),$$

where

$$\kappa^2 = \kappa_1^2 + \kappa_2^2, \quad \zeta' = \operatorname{Re} \zeta.$$

For H_{n0} waves

$$\alpha = \frac{\zeta' \omega}{ckab} \left(a + \frac{2\kappa^2 b}{k^2 + \kappa^2} \right).$$

For $H_{n_1 n_2}$ ($n_1, n_2 \neq 0$):

$$\alpha = \frac{2c\kappa^2\zeta'}{\omega k a b} \left[a + b + \frac{k^2}{\kappa^4} (\kappa_1^2 a + \kappa_2^2 b) \right].$$

The notation is the same as in the preceding problem.

464. *E*-waves.

(a) Even solutions [$\mathcal{E}_x(x) = \mathcal{E}_x(-x)$, $\mathcal{H}_y(x) = \mathcal{H}_y(-x)$, $\mathcal{E}_z(x) = -\mathcal{E}_z(-x)$]:

when $x > a$

$$\mathcal{E}_z = A e^{-sx}, \quad \mathcal{E}_x = \frac{ik}{s} A e^{-sx}, \quad \mathcal{H}_y = \frac{i\omega}{sc} A e^{-sx};$$

when $-a \leq x \leq a$

$$\mathcal{E}_z = B \sin \kappa x, \quad \mathcal{E}_x = \frac{ik}{\kappa} B \cos \kappa x, \quad \mathcal{H}_y = \frac{i\omega\epsilon}{\kappa c} B \cos \kappa x; \quad (1)$$

when $x < -a$

$$\mathcal{E}_x = -A e^{sx}, \quad \mathcal{E}_z = \frac{ik}{s} A e^{sx}, \quad \mathcal{H}_y = \frac{i\omega}{sc} A e^{sx},$$

where $A = B \exp(sa) \sin \kappa a$; the remaining components of $\mathcal{E}$ and $\mathcal{H}$ are zero. The parameters κ and s may be determined from the following equations

$$(\kappa a)^2 + (sa)^2 = \frac{\omega^2 a^2}{c^2} (\epsilon\mu - 1), \quad (2)$$

$$sa = \frac{1}{\epsilon} \kappa a \tan \kappa a. \quad (3)$$

These equations can easily be solved graphically. The possible values of κ and s correspond to the points of intersection of the curves defined by (3) with the circle of radius $r = (\omega a/c)(\epsilon\mu - 1)^{1/2}$ (Fig. 71). For given ω , a , ϵ , and μ there is a finite number of points of intersection, *i.e.* a finite number of types of wave, for which the field distribution is described by the above equations. In particular, when $r < \pi$, there is a single E_{00} wave.

Consider now the propagation constant

$$k = \sqrt{\frac{\omega^2 \epsilon \mu}{c^2} - \kappa^2} = \sqrt{\frac{\omega^2}{c^2} + s^2} \quad (4)$$

for given parameters of the dielectric layer and given types of wave. It is clear from Fig. 71 that for frequencies near the

limiting frequency at which the given type of wave appears, the parameter s approaches zero, while k tends to ω/c . At such frequencies the wave has the same propagation constant as in a vacuum, and the field penetrates to large distances in the layer.

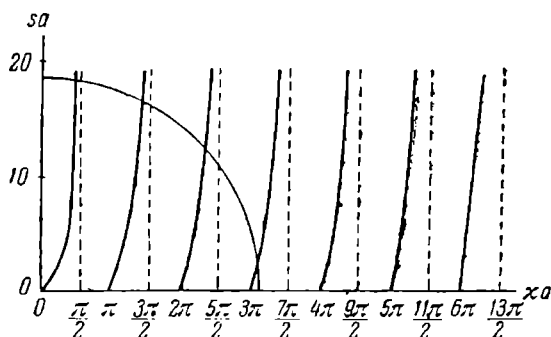


Fig. 71.

The parameter s increases with increasing ω while κ remains bounded. At the same time, k tends to $(\omega/c)(\epsilon\mu)^{1/2}$, i.e. to a value which corresponds to a wave propagating in an infinite dielectric medium with parameters ϵ, μ . For sufficiently large ω , and therefore large s , the field is almost entirely concentrated within the dielectric layer.

(b) Odd solutions [$\mathcal{E}_x(x) = -\mathcal{E}_x(-x)$, $\mathcal{H}_y(x) = -\mathcal{H}_y(-x)$, $\mathcal{E}_z(x) = \mathcal{E}_z(-x)$]:

when $x > a$

$$\mathcal{E}_z = Ae^{-sx}, \quad \mathcal{E}_x = -\frac{ik}{s} Ae^{-sx}, \quad \mathcal{H}_y = \frac{i\omega}{sc} Ae^{-sx}; \quad (5)$$

when $-a \leq x \leq a$

$$\mathcal{E}_z = B \cos \kappa x, \quad \mathcal{E}_x = -\frac{ik}{\kappa} B \sin \kappa x, \quad \mathcal{H}_y = -\frac{i\omega\epsilon}{\kappa c} B \sin \kappa x;$$

when $x < -a$

$$\mathcal{E}_z = Ae^{sx}, \quad \mathcal{E}_x = -\frac{ik}{s} Ae^{sx}, \quad \mathcal{H}_y = -\frac{i\omega}{sc} Ae^{sx},$$

where $A = B \exp(sa) \cos \kappa a$; the remaining components of $\mathcal{E}$ and $\mathcal{H}$ are zero. The parameters s and κ may be determined from the equations

$$(\kappa a)^2 + (sa)^2 = \frac{\omega^2 a^2}{c^2} (\epsilon\mu - 1), \quad sa = -\frac{1}{\epsilon} \kappa a \cotan \kappa a. \quad (6)$$

The propagation constant k is related to κ and ϵ by Eqn. (4).

It is easily shown by a graphical method that odd electric waves cannot exist when $r < \pi/2$. The remaining properties are the same as for even waves.

H-waves may be analysed in a similar way.

465. Even *E*-waves and odd *H*-waves can propagate along the layer, and their characteristics (propagation constant, field configuration in the region $x > 0$, and so on) are the same as in the preceding problem.

466. *E*-waves.

In order to determine these waves, it is necessary to solve the following equation for the longitudinal component of the electric field:

$$\frac{\partial^2 \mathcal{E}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{E}_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \mathcal{E}_z}{\partial \alpha^2} + \kappa^2 \mathcal{E}_z = 0. \quad (1)$$

This equation can be integrated by the method of separation of variables. Special solutions are of the form

$$\mathcal{E}_z(r, \alpha) = J_m(\kappa r) \sin(m\alpha + \psi_m), \quad (2)$$

where J_m is the Bessel function of order m and ψ_m is an arbitrary constant. In order that the field should return to its original value when α changes by 2π , the quantity m must be an integer ($m = 0, 1, 2, \dots$).

The transverse components of the electric and magnetic fields can be expressed in terms of $\mathcal{E}_z$ with the aid of Maxwell's equations:

$$\begin{aligned} \mathcal{E}_r &= \frac{ik}{\kappa} J'_m(\kappa r) \sin(m\alpha + \psi_m), \\ \mathcal{E}_\alpha &= \frac{imk}{\kappa^2 r} J_m(\kappa r) \cos(m\alpha + \psi_m), \\ \mathcal{H}_r &= -\frac{im\omega}{\kappa^2 cr} J_m(\kappa r) \cos(m\alpha + \psi_m), \\ \mathcal{H}_\alpha &= \frac{i\omega}{\kappa c} J'_m(\kappa r) \sin(m\alpha + \psi_m). \end{aligned}$$

The possible values of κ are determined by the boundary conditions at the waveguide wall:

$$\mathcal{E}_r|_{r=a} = 0, \quad \mathcal{E}_\alpha|_{r=a} = 0.$$

This yields $\kappa_{mn}a = \alpha_{mn}$ where α_{mn} is the n -th root of the Bessel function: $J_m(\alpha_{mn}) = 0$, $n = 1, 2, 3, \dots$

Thus, the above waves are characterised by two subscripts, namely, m and n . When $m = 0$ the field exhibits rotational symmetry with respect to the z -axis. For an ideal waveguide, the phases ψ_m are determined by the conditions of excitation. In reality, however, they are very dependent on defects in the waveguide walls (departures from circular cross-section, longitudinal cracks, and so on).

The wave propagation along the waveguide is possible when $k = [(\omega/c)^2 - \kappa^2]^{1/2}$ is real. Hence, a wave characterised by m, n will propagate in the waveguide, provided its frequency is such that

$$\omega^2 \gg \frac{c^2 \alpha_{mn}^2}{a^2}.$$

The minimum possible frequency for a $(0, 1)$ wave is

$$\omega_0 = \frac{c \alpha_{01}}{a} \simeq 2.4 \frac{c}{a}.$$

The corresponding wavelength is

$$\lambda_0 = \frac{2\pi c}{\omega_0} \simeq 2.6 a,$$

which is of the order of the radius of the waveguide.

The H -waves are

$$\mathcal{H}_z = J_m(\kappa r) \sin(m\alpha + \psi_m) \quad (m = 0, 1, 2; \dots).$$

The propagation constant is given by

$$k^2 = \frac{\omega^2}{c^2} - \frac{\beta_{mn}^2}{a^2} \quad (n = 1, 2, \dots),$$

where β_{mn} is the n -th root of the equation $J'_m(\beta_{mn}) = 0$. The smallest root is $\beta_{11} \simeq 1.8$, and this corresponds to a limiting frequency $\omega_0 \simeq 1.8c/a$ and a limiting wavelength $\lambda_0 = 2\pi c/\omega_0 \simeq 3.5a$.

The limiting frequency is lower for H -waves than for E -waves. When the frequency lies within the limits $\omega_{0,E} > \omega > \omega_{0,H}$ then only H_{11} waves will propagate.

467. E -waves:

$$\alpha = \frac{\omega_s^2}{c a k}.$$

H -waves of the (m, n) type

$$\alpha = \frac{c \zeta' x^2}{\omega a k} \left[1 + \frac{m^2 \omega^2}{c^2 x^2 (a^2 x^2 - m^2)} \right],$$

where $\zeta' = \operatorname{Re} \zeta$.

468. The wave vector k and the frequency ω of the waves in the waveguide are related by

$$\frac{\omega^2}{c^2} = k^2 + x^2,$$

where x is a constant which depends on the type of waves and the transverse dimensions of the waveguide. Usual formulae give

$$v_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\lambda/\lambda_0)^2}},$$

$$v_g = \frac{d\omega}{dk} = c \sqrt{1 - (\lambda/\lambda_0)^2},$$

where λ_0 is the limiting wavelength.

It is clear from the above formulae that $v_\varphi > c$, $v_g < c$, and $v_\varphi \cdot v_g = c^2$. This result holds for a waveguide in a vacuum (the dielectric properties of air may be neglected for the range of phenomena under consideration).

If the waveguide is filled with a dielectric and the dispersion of ϵ and μ may be neglected, then all the above formulae remain valid, provided c is replaced by $v = c/(\epsilon\mu)^{1/2}$. Hence, for this type of waveguide $v_\varphi = c/(\epsilon\mu)^{1/2} [1 - (\lambda/\lambda_0)^2]^{1/2}$ may be less than c and the wave is "slowed down" (cf. Problem 474).

$$469. \quad H_z = \frac{1}{2} \mathcal{H}_0 [e^{i(x_1 x + kz)} + e^{i(-x_1 x + kz)}] e^{-i\omega t}.$$

The directions of propagation of the two plane waves into which H_{10} is resolved are at an angle θ to the axis of the waveguide (Fig. 72), where

$$\cos \theta = \frac{k}{\sqrt{k^2 + x_1^2}} = \sqrt{1 - \left(\frac{\lambda}{\lambda_0}\right)^2}.$$

The phase plane I moves with a velocity c in the direction making an angle θ with the z -axis. However, its velocity along the axis of the waveguide will be larger:

$$v = \frac{c}{\cos \theta} = \frac{c}{\sqrt{1 - (\lambda/\lambda_0)^2}} = v_\varphi.$$

This is, in fact, the phase velocity of the wave in the waveguide.

The group velocity is equal to the rate of transport of energy. However, for a plane wave propagating in a vacuum, the energy

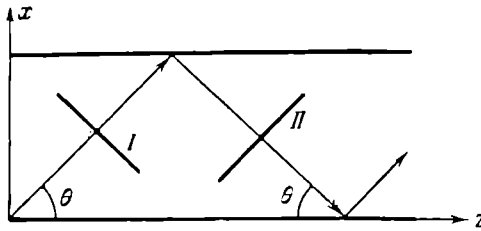


Fig. 72.

is transported with a velocity c in the direction of propagation of the wave. Each component plane wave of H_{10} will undergo multiple reflections from the walls of the waveguide, and will therefore travel over a zigzag path. The resultant velocity along the axis

of the waveguide will then be

$$v = c \cos \theta = c \sqrt{1 - \left(\frac{\lambda}{\lambda_0}\right)^2},$$

which is equal to the group velocity v_g .

$$470. \quad H_a = E_r = \frac{A}{r} e^{i\omega\left(\frac{z}{c} - t\right)}, \quad (1)$$

where A is a constant, and the remaining components are all equal to zero.

The energy flux is given by

$$\bar{\gamma} = \frac{A^2 c}{4} \ln \frac{b}{a}. \quad (2)$$

For a single perfect conductor, the field outside the conductor is given by (1) and the total energy flux through the $z = \text{const}$ plane is infinite ($\gamma \rightarrow \infty$ when $b \rightarrow \infty$). It follows that this type of wave cannot be maintained by a source of finite power, and therefore this case is of no physical significance.

471. E -waves:

$$\mathcal{E}_z = [A_{mn} J_m(x_{mn} r) + B_{mn} N_m(x_{mn} r)] \sin(m\alpha + \psi_m), \quad m = 0, 1, 2, \dots,$$

where x_{mn} is the n -th root of the equation

$$J_m(xa) N_m(xb) - J_m(xb) N_m(xa) = 0.$$

In this expression N_m and J_m are the cylindrical functions (cf. Appendix III) and A_{mn} and B_{mn} are constants which are related by

$$A_{mn} J_m(x_{mn} a) + B_{mn} N_m(x_{mn} a) = 0.$$

H -waves:

$$\mathcal{H}_z = [C_{mn} J_m(x_{mn} r) + D_{mn} N_m(x_{mn} r)] \sin(m\alpha + \psi_m), \quad m = 0, 1, 2, \dots,$$

where κ_{mn} is the n -th root of the equation

$$J'_m(\kappa a) N'_m(\kappa b) - N'_m(\kappa a) J'_m(\kappa b) = 0,$$

and C_{mn} and D_{mn} are related by

$$C_{mn} J'_m(\kappa_{mn} a) + D_{mn} N'_m(\kappa_{mn} a) = 0.$$

The remaining components of the electric and magnetic fields may be expressed in terms of $\mathcal{E}_z$ and $\mathcal{H}_z$ by means of Maxwell's equations.

$$472. \quad a = \frac{\zeta'(a+b)}{2ab \ln(b/a)},$$

where $\zeta' = \operatorname{Re} \zeta$.

473. If the field is symmetric with respect to the axis of the conductor, then the longitudinal component $\mathcal{E}_z$ will satisfy the following differential equation:

$$\frac{d^2 \mathcal{E}_z}{dr^2} + \frac{1}{r} \frac{d \mathcal{E}_z}{dr} + \kappa^2 \mathcal{E}_z = 0. \quad (1)$$

Since the conductor has a finite conductivity, the parameters k and κ will be complex. The sign of κ will be defined so that $\operatorname{Im} \kappa = \kappa'' > 0$.

The general solution of (1) may be written in the form

$$\mathcal{E}_z(r) = A' H_0^{(1)}(\kappa r) + B' H_0^{(2)}(\kappa r),$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions. It follows from the asymptotic behaviour of these functions (*cf.* Appendix III) and the condition $\operatorname{Im} \kappa > 0$ that $B' = 0$ since otherwise the field would diverge at infinity. The remaining components of $\mathcal{E}$ and $\mathcal{H}$ can be expressed in terms of $\mathcal{E}_z$ with the aid of Maxwell's equations:

$$\begin{aligned} \mathcal{E}_z &= A' H_0^{(1)}(\kappa r), \quad \mathcal{E}_r = \frac{ik}{\kappa} A' H_1^{(1)}(\kappa r), \\ H_\alpha &= \frac{i\omega}{\kappa c} A' H_1^{(1)}(\kappa r). \end{aligned} \quad (2)$$

For sufficiently large κr , the functions $H_0^{(1)}$ and $H_1^{(1)}$ are proportional to $(\kappa r)^{-1/2} \exp(-\kappa'' r)$, and therefore the electromagnetic field is damped out exponentially at large distances from the conductor. The maximum concentration of the field exists near the conductor, and hence the wave is of the surface type.

The boundary condition

$$\mathcal{E}_z = \zeta \mathcal{H}_\alpha$$

at the surface of the conductor leads to the following characteristic equation for κa :

$$\kappa a \frac{H_0^{(1)}(\kappa a)}{H_1^{(1)}(\kappa a)} = i\zeta \frac{\omega}{c} a,$$

where ζ is the surface impedance of the metal. For a good conductor $|\zeta| \ll 1$, and hence the latter equation can only be satisfied for small κa . Using the approximate formulae for $H_0^{(1)}$ and $H_1^{(1)}$ (Appendix III) we have

$$(\kappa a)^2 \ln \left(\frac{\gamma \kappa a}{2i} \right) = i\zeta \frac{\omega}{c} a, \quad \ln \gamma = 0.5772. \quad (3)$$

This transcendental equation cannot be solved by a graphical method because it involves complex quantities. It was solved by Sommerfeld, who used an iteration method based on the fact that $\ln \kappa a$ varies much more slowly than κa . Let $(\gamma \kappa a / 2i)^2 = u$, $-i\gamma^2 \omega \zeta a / 2c = v$. Eqn. (3) can then be written in the form

$$u \ln u = v.$$

If an approximate solution u_n (n -th order approximation) is known, then a more accurate value u_{n+1} [$(n+1)$ -th approximation] may be obtained from the formula

$$u_{n+1} \ln u_n = v.$$

In the zero-order approximation $u_0 = v$, and hence

$$u_1 = \frac{v}{\ln v}, \quad u_2 = \frac{v}{\ln(v/\ln v)}, \quad u_3 = \frac{v}{\ln \frac{v}{\ln(v/\ln v)}}, \text{ and so on.}$$

For decimeter waves ($\lambda = 2\pi c/\omega = 30$ cm), which propagate along a copper conductor of radius equal to 1 mm (the conductivity of copper is $5.2 \times 10^{17} \text{ sec}^{-1}$), the above method and equations (VIII.9)–(VIII.11) yield

$$u \simeq (4.2 + 4.5i) \cdot 10^{-8},$$

and hence

$$k = \frac{\omega}{c} [1 + (6.0 + 6.4i) \cdot 10^{-5}].$$

The phase velocity is given by

$$v_\varphi = \frac{\omega}{\text{Re } k} = (1 - 6 \cdot 10^{-5}) c < c,$$

and hence the wave is slightly slowed down.

This result can be understood from the following considerations. For a perfect conductor, the transverse electromagnetic wave has a phase velocity c and the field inside the conductor is zero. In the case of a finite conductivity a proportion of the energy will propagate within the conductor, and since the velocity of propagation in the metal is appreciably less than c , it follows that the electromagnetic wave will, on the average, be slowed down. Moreover, damping will appear.

Consider the nature of the field in the limiting case $\xi \rightarrow 0$ (perfect conductor). It follows from (3) that $\kappa \rightarrow 0$, $k \rightarrow \omega/c$. Using the expressions for $H_0^{(1)}$ and $H_1^{(1)}$ for small arguments of these functions, we have from (2)

$$\mathcal{E}_z = \lim_{\kappa \rightarrow 0} \frac{2iA'}{\pi} \ln\left(\frac{\gamma\kappa r}{2i}\right), \quad \mathcal{E}_r = \lim_{\kappa \rightarrow 0} \frac{2kA'}{\pi\kappa^2} \frac{1}{r},$$

$$\mathcal{H}_\alpha = \lim_{\kappa \rightarrow 0} \frac{2kA'}{\pi\kappa^2} \cdot \frac{1}{r}.$$

Since the field components cannot become infinite, it must be assumed that A' is proportional to κ^2 . Let $A' = A\kappa^2/k$, so that

$$\mathcal{E}_r = \mathcal{H}_\alpha = \frac{A}{r}, \quad \mathcal{E}_z = 0.$$

This is a purely transverse electromagnetic wave propagating with a velocity c .

474. The components of the electromagnetic field in the waveguide are given by the following expressions:

when $r \leq a$

$$\mathcal{E}_z = \mathcal{E}_0 J_0(\kappa_1 r), \quad \mathcal{E}_r = -i \frac{k}{\kappa_1} \mathcal{E}_0 J_1(\kappa_1 r), \quad \mathcal{H}_\alpha = -i \frac{\omega}{c\kappa_1} \mathcal{E}_0 J_1(\kappa_1 r);$$

when $a \leq r \leq b$

$$\mathcal{E}_z = AJ_0(\kappa_2 r) + BN_0(\kappa_2 r), \quad \mathcal{E}_r = -i \frac{k}{\kappa_2} [AJ_1(\kappa_2 r) + BN_1(\kappa_2 r)],$$

$$\mathcal{H}_\alpha = -i \frac{\epsilon\omega}{c\kappa_2} [AJ_1(\kappa_2 r) + BN_1(\kappa_2 r)],$$

where

$$\kappa_1 = \sqrt{\frac{\omega^2}{c^2} - k^2}, \quad \kappa_2 = \sqrt{\frac{\epsilon\omega^2}{c^2} - k^2},$$

and $\mathcal{E}_0$, A , and B are constants.

The boundary conditions may be written in the form

$$\mathcal{E}_z|_{r=b} = 0, \quad \mathcal{E}_z|_{r=a-0} = \mathcal{E}_z|_{r=a+0}, \quad \mathcal{H}_\alpha|_{r=a-0} = \mathcal{H}_\alpha|_{r=a+0},$$

when the boundary condition for $\mathcal{E}_\alpha$ will be satisfied automatically. When the constants A , B , and $\mathcal{E}_0$ are eliminated, we have the following transcendental equation relating k and ω :

$$\frac{\epsilon x_1}{x_2} \cdot \frac{J_0(x_1 a)}{J_1(x_1 a)} = \frac{J_0(x_2 a) N_0(x_2 b) - N_0(x_2 a) J_0(x_2 b)}{J_1(x_2 a) N_0(x_2 b) - N_1(x_2 a) J_0(x_2 b)}. \quad (1)$$

This equation can be considerably simplified when $a \ll b$. Consider the wave which has the maximum k . If the waveguide were fully filled with a dielectric ($a = 0$), then the corresponding value of x_2 would be $x_{02} = \alpha_{01}/b$ where $\alpha_{01} = 2.4$, $J_0(\alpha_{01}) = 0$ (cf. Problem 466).

We shall seek a solution which is not very different from x_{02} :

$$x_2 = x_{02} + x'_2 = \frac{\alpha_{01}}{b} + \frac{\Delta a}{b},$$

where the order of magnitude of Δa is not lower than that of a/b . Assuming that $\alpha_{01}(a/b) \ll 1$, and using the approximate formulae for J_0 , N_0 , J_1 , and N_1 (Appendix III), we have instead of (1)

$$\epsilon \left[(x_2 a)^2 N_0(x_2 b) + \frac{2}{\pi} J_0(x_2 b) \right] = (x_2 a)^2 \left[N_0(x_2 b) + \frac{2}{\pi} \ln \frac{2}{\gamma x_2 a} J_0(x_2 b) \right].$$

Substituting

$$N_0(x_2 b) = N_0(\alpha_{01} + \Delta a) \simeq N_0(\alpha_{01}), \quad J_0(x_2 b) = -J_1(\alpha_{01}) \Delta a,$$

and neglecting the small logarithmic term, we have

$$\Delta a = \left(1 - \frac{1}{\epsilon}\right) \frac{\pi \alpha_{01}^2}{4} \cdot \frac{N_0(\alpha_{01})}{J_1(\alpha_{01})} \cdot \left(\frac{a}{b}\right)^2.$$

The phase velocity is then given by

$$v_\varphi = \frac{\omega}{k} = \frac{\omega}{\sqrt{\frac{\epsilon \omega^2}{c^2} - \frac{(\alpha_{01}^2 + 2\alpha_{01} \Delta a)}{b^2}}}.$$

Substituting $\omega_0 = \alpha_{01}(c/b) \simeq 2.4(c/b)$ (minimum frequency for a waveguide not containing a dielectric) and using tabulated values for $N_0(\alpha_{01})$ and $J_1(\alpha_{01})$ we have

$$v_\varphi = c \left\{ \epsilon - \left(\frac{\omega_0}{\omega}\right)^2 \left[1 + 3.7 \left(1 - \frac{1}{\epsilon}\right) \frac{a^2}{b^2} \right] \right\}^{-\frac{1}{2}}. \quad (2)$$

When the waveguide is completely filled with the dielectric ($a = 0$), the phase velocity is given by

$$v_{\varphi} = c \left[\epsilon - \left(\frac{\omega_0}{\omega} \right)^2 \right]^{-\frac{1}{2}}.$$

The limiting frequency of a partially filled waveguide is given by

$$\omega_1 = \frac{\omega_0}{\sqrt{\epsilon}} \left[1 + 1.85 \left(1 - \frac{1}{\epsilon} \right) \frac{a^2}{b^2} \right]$$

and lies between the limiting frequencies of an empty and fully-filled waveguide:

$$\frac{\omega_0}{\sqrt{\epsilon}} < \omega_1 < \omega_0.$$

The phase velocity given by (2) becomes smaller than c at frequencies

$$\omega > \frac{\omega_0}{\sqrt{\epsilon - 1}} \left[1 + 1.85 \left(1 - \frac{1}{\epsilon} \right) \frac{a^2}{b^2} \right].$$

Thus, a partially or fully-filled waveguide is a retarding system in which the phase velocity of electromagnetic waves may be less than c . An important property of slow waves is the fact that they can effectively interact with charged particle beams. This interaction may be used both for the generation and the amplification of ultra-high frequency electromagnetic oscillations (backward wave oscillator, magnetron) and for the acceleration of particles (linear accelerator).

475. E -waves cannot exist in this instance. The H -waves are given by

$$\begin{aligned} \mathcal{H}_z &= \frac{i\mathcal{E}_0 c}{\omega \mu} \left(x \cos x x - k \frac{\mu_a}{\mu_{\perp}} \sin x x \right), \\ \mathcal{H}_x &= \frac{\mathcal{E}_0 c}{\omega \mu} \left(k \sin x x - x \frac{\mu_a}{\mu_{\perp}} \cos x x \right), \quad \mathcal{E}_z = \mathcal{E}_0 \sin x x, \\ \mathcal{E}_x &= \mathcal{E}_y = \mathcal{H}_y = 0, \end{aligned}$$

where

$$\begin{aligned} x &= \frac{n\pi}{a}, \quad k = \sqrt{\frac{\omega^2 \epsilon_{\parallel} \mu}{c^2} - \left(\frac{n\pi}{a} \right)^2}, \quad n = 1, 2, 3, \dots, \\ \mu &= \mu_{\perp} - \frac{\mu_a^2}{\mu_{\perp}}. \end{aligned}$$

The limiting frequency is $\omega_0^{(n)} = c\kappa_n/\epsilon_{||}\mu$.

It follows from the above formulae for $\mathcal{H}_z$ and $\mathcal{H}_x$ that the configuration of the magnetic field for this type of wave depends on the sign of k , *i.e.* on the direction of propagation of the wave, and on the sign of μ_Q , *i.e.* on the direction of the constant magnetic field. This effect is associated with the gyrotropy of the medium filling the waveguide.

476. Maxwell's equations for the complex conjugate amplitudes $\mathcal{E}_0^*$, $\mathcal{H}_0^*$ are of the form

$$\begin{aligned}\operatorname{curl} \mathcal{E}_0^* - ik_0(\mathbf{e}_z \times \mathcal{E}_0^*) &= -\frac{i\omega}{c} \mathcal{H}_0^*, \\ \operatorname{curl} \mathcal{H}_0^* - ik_0(\mathbf{e}_z \times \mathcal{H}_0^*) &= \frac{i\omega}{c} \mathcal{E}_0^*.\end{aligned}\quad (1)$$

The amplitudes $\mathcal{E}$, $\mathcal{H}$ satisfy the equations

$$\begin{aligned}\operatorname{curl} \mathcal{E} + ik(\mathbf{e}_z \times \mathcal{E}) &= \frac{i\omega}{c} \hat{\mu}' \mathcal{H}, \\ \operatorname{curl} \mathcal{H} + ik(\mathbf{e}_z \times \mathcal{H}) &= -\frac{i\omega}{c} \hat{\epsilon}' \mathcal{E},\end{aligned}\quad (2)$$

where $\hat{\mu}' \mathcal{H}$, $\hat{\epsilon}' \mathcal{E}$ are vectors with components $\mu'_{ik} \mathcal{H}_k$, $\epsilon'_{ik} \mathcal{E}_k$; $\mu'_{ik} = \epsilon'_{ik} = \delta_{ik}$ outside the region filled by the dielectric. Inside the region, $\mu'_{ik} = \mu_{ik}$, $\epsilon'_{ik} = \epsilon_{ik}$.

It follows from the equations for $\operatorname{curl} \mathcal{E}_0^*$ and $\operatorname{curl} \mathcal{H}$ that

$$\begin{aligned}\mathcal{H} \operatorname{curl} \mathcal{E}_0^* - \mathcal{E}_0^* \operatorname{curl} \mathcal{H} + l(k - k_0)(\mathbf{e}_z \times \mathcal{E}_0^*) \mathcal{H} &= \\ &= -i\omega(\mathcal{H} \cdot \mathcal{H}_0^* - \hat{\epsilon}' \mathcal{E} \cdot \mathcal{E}_0^*).\end{aligned}\quad (3)$$

Integrate both parts of this equation over the cross-section of the waveguide S . The first two terms may be transformed as follows:

$$\int_S (\mathcal{H} \operatorname{curl} \mathcal{E}_0^* - \mathcal{E}_0^* \operatorname{curl} \mathcal{H}) dS = \frac{1}{l} \int_V \operatorname{div}(\mathcal{E}_0^* \times \mathcal{H}) dV.$$

The integral on the right-hand side of this expression is evaluated over the volume bounded by the walls of the waveguide and two cross-sections at a distance l from each other (the integration is independent of z).

Next, from Gauss' theorem, we have

$$\int \operatorname{div}(\mathcal{E}_0^* \times \mathcal{H}) dV = \int (\mathcal{E}_0^* \times \mathcal{H}) \cdot \mathbf{n} dS = \int (\mathbf{n} \times \mathcal{E}_0^*) \cdot \mathcal{H} dS.$$

On the wall of the waveguide $\mathbf{n} \times \mathcal{E}_0^* = 0$ in view of the boundary condition $\mathcal{E}_{0\tau} = 0$, while the integrals over the cross-sections

have opposite signs and cancel out. Hence,

$$\int (\mathbf{n} \times \mathbf{E}_0^*) \cdot \mathbf{H} dS = 0$$

and Eqn. (3) gives

$$(k - k_0) \int_S (\mathbf{E}_0^* \times \mathbf{H}) \cdot \mathbf{e}_z dS = -\omega \left[\int_S (\mathbf{H} \cdot \mathbf{H}_0^*) dS - \int_S (\mathbf{E} \cdot \mathbf{E}_0^*) dS - \int_{\Delta S} \Delta \hat{\epsilon} \mathbf{E} \cdot \mathbf{E}_0^* dS \right] \quad (4)$$

where $\Delta \hat{\epsilon} = \hat{\epsilon} - \hat{1}$ and ΔS is the cross-sectional area of the region occupied by the dielectric.

Similarly, using the equations for $\text{curl } \mathbf{E}$ and $\text{curl } \mathbf{H}_0^*$ we have

$$(k - k_0) \int_S (\mathbf{E} \times \mathbf{H}_0^*) \cdot \mathbf{e}_z dS = \frac{\omega}{c} \left[\int_S (\mathbf{H} \cdot \mathbf{H}_0^*) dS - \int_S (\mathbf{E} \cdot \mathbf{E}_0^*) dS + \int_{\Delta S} \Delta \hat{\mu} \mathbf{H} \cdot \mathbf{H}_0^* dS \right], \quad (5)$$

where $\Delta \hat{\mu} = \hat{\mu} - \hat{1}$.

Combining Eqns. (4) and (5) we obtain the formula given in the problem. This formula represents the exact relation between the change in the propagation constant and the field amplitudes. However, in the majority of situations, the exact solution of the problem of a waveguide partially filled with a dielectric cannot be obtained. The amplitudes of the disturbed field $\mathbf{E}$ and $\mathbf{H}$ can only be determined approximately, provided that the transverse dimensions of the region occupied by the dielectric are small. When this is so, the formula for the change in the propagation constant may be used to calculate the value of Δk , which is an important characteristic of the wave propagating in the waveguide. Examples of calculations based on this method are given in the solution of Problems 477 to 479.

477. For a thin plate, the amplitudes of the disturbed fields may be approximately expressed in terms of the undisturbed amplitudes, which for an H_{10} wave are of the form

$$\begin{aligned} \mathcal{H}_{0z} &= \mathcal{H}_0 \cos \frac{\pi x}{a}, & \mathcal{H}_{0x} &= -\frac{ik_0 a}{\pi} \mathcal{H}_0 \sin \frac{\pi x}{a}, \\ \mathcal{E}_{0y} &= \frac{i\omega a}{\pi c} \mathcal{H}_0 \sin \frac{\pi x}{a}, & \mathcal{E}_{0x} &= \mathcal{E}_{0z} = \mathcal{H}_{0y} = 0. \end{aligned}$$

These expressions may be obtained from the solutions of Problem 462. Let us now neglect the change in the field amplitudes outside the volume occupied by the plate, and also the variation in the fields within the plate. This is equivalent to neglecting terms of the order of d^2 or higher. The boundary conditions which must

be satisfied on the surface of the plate are

$$\mathcal{E}_y = \mathcal{E}_{0y}, \quad \mathcal{H}_z = \mathcal{H}_{0z}, \quad \mu_{\perp} \mathcal{H}_x - i\mu_a \mathcal{H}_y = \mathcal{H}_{0x}, \quad \mathcal{H}_y = \mathcal{H}_{0y} = 0,$$

where the undisturbed amplitudes on the right-hand side of the equations should be taken at $x = x_1$. These equations define the amplitudes of the disturbed field in the plate.

The integral in the numerator of the expression for Δk (*cf.* preceding problem) is equal to the product of the integrand and the cross-sectional area ba of the plate, since the field is independent of y and the dependence on x is neglected.

The undisturbed amplitudes may be substituted into the integral in the denominator.. The final result is

$$\Delta k = \frac{d}{k_0 a} \left\{ \left[\frac{(\epsilon - 1)\omega^2}{c^2} + \left(1 - \frac{1}{\mu_{\perp}}\right) k_0^2 \right] \sin^2 \frac{\pi x_1}{a} + (\mu_{\parallel} - 1) \left(\frac{\pi}{a}\right)^2 \cos^2 \frac{\pi x_1}{a} \right\}.$$

Since $\mu_{\perp}$ depends on the magnitude of the constant magnetising field H_0 (*cf.* Problem 329), it follows that Δk will also depend on this field. A change in H_0 gives rise to a change in the phase of the wave. Devices which are based on these phenomena are widely used in electronics for phase transformation.

$$478. \quad \Delta k = \frac{\omega d}{4\pi c} \cdot \frac{b-a}{ab \ln \frac{b}{a}} \left(\epsilon - \frac{1}{\mu_{\perp}} \right).$$

$$479. \quad (a) \quad \Delta k = \frac{\omega d}{4\pi c} \cdot \frac{b-a}{ab \ln \frac{b}{a}} \left(\epsilon - \frac{1}{\mu_{\parallel}} \right);$$

$$(b) \quad \Delta k = \frac{\omega d}{4\pi c} \cdot \frac{b-a}{ab \ln \frac{b}{a}} \left(\epsilon - \frac{1}{\mu_{\perp}} \right).$$

In (a) the change in the propagation constant Δk is practically independent of the magnitude of the constant magnetic field H_0 since $\mu_{\parallel} \approx 1$ (*cf.* Problem 329). This is explained by the fact that inside the plate the high-frequency magnetic field has the same direction as the constant field, and does not maintain the precession of the magnetisation vector $\mathbf{M}$. In (b) the high-frequency magnetic field inside the plate is perpendicular to the constant field, $\mu_{\perp}$ depends on H_0 , and this dependence is of the resonance type.

480. Integrate the differential equations (IX.1) subject to the boundary condition $E_\tau = 0$, *i. e.*

$$\begin{aligned} E_y = E_z = 0 & \quad x = 0, \quad x = a, \\ E_z = E_x = 0 & \quad y = 0, \quad y = b, \\ E_x = E_y = 0 & \quad z = 0, \quad z = h. \end{aligned}$$

The equations and the boundary conditions are satisfied by the functions

$$\begin{aligned} E_x &= A_1 \cos k_1 x \sin k_2 y \sin k_3 z \cdot e^{-i\omega t}, \\ H_x &= -i \frac{c}{\omega} (A_3 k_2 - A_2 k_3) \sin k_1 x \cos k_2 y \cos k_3 z \cdot e^{-i\omega t}, \end{aligned}$$

and similarly for the other components. The quantities entering into these expressions are given by

$$k_1 = \frac{n_1 \pi}{a}, \quad k_2 = \frac{n_2 \pi}{b}, \quad k_3 = \frac{n_3 \pi}{h}, \quad \omega^2 = c^2 (k_1^2 + k_2^2 + k_3^2),$$

where n_1 , n_2 , and n_3 can assume integral positive values, including zero. Negative values of n_i do not correspond to new types of oscillations, since the direction of the fields and their configuration remain unaltered when $k_i \rightarrow -k_i$. The equation $\text{div } \mathbf{E} = 0$ leads to the following relation between the constants A_i

$$A_1 k_1 + A_2 k_2 + A_3 k_3 = 0.$$

Two of these constants are independent, and hence the proper oscillations (for $n_1, n_2, n_3 \neq 0$) are, in general, doubly degenerate, *i. e.* to each value of the proper frequency there corresponds two oscillations with different configurations of the electric and magnetic fields. If one of the numbers n_1, n_2, n_3 is zero, then there is no degeneracy. However, if the dimensions of the resonator a, b, h are in the ratio of integers, then the order of the degeneracy is increased.

$$481. \quad \Delta N = \frac{V}{\pi^2 c^3} \omega^2 \Delta \omega.$$

482.

$$\begin{aligned} \Delta \omega' &= \frac{\pi^2 c \zeta''}{2} \left[a \left(\frac{1}{b^2} + \frac{1}{h^2} \right) + 2 \left(\frac{1}{b} + \frac{1}{h} \right) \right], \\ \Delta \omega'' &= - \frac{\pi^2 c \zeta'''}{2} \left[a \left(\frac{1}{b^2} + \frac{1}{h^2} \right) + 2 \left(\frac{1}{b} + \frac{1}{h} \right) \right]. \end{aligned}$$

483. The E -waves are

$$E_z = \mathcal{E}_0 J_m(\chi r) \sin(m\alpha + \psi_m) \cos kze^{-i\omega t}, \quad H_z = 0,$$

$$E_r = -\frac{k}{\chi} \mathcal{E}_0 J'_m(\chi r) \sin(m\alpha + \psi_m) \sin kze^{-i\omega t},$$

$$E_\alpha = -\frac{mk}{\chi^2 r} \mathcal{E}_0 J_m(\chi r) \cos(m\alpha + \psi_m) \sin kze^{-i\omega t},$$

$$H_r = -\frac{im\omega}{\chi^2 cr} \mathcal{E}_0 J_m(\chi r) \cos(m\alpha + \psi_m) \cos kze^{-i\omega t},$$

$$H_\alpha = \frac{i\omega}{\chi c} \mathcal{E}_0 J'_m(\chi r) \sin(m\alpha + \psi_m) \cos kze^{-i\omega t},$$

where $k = l\pi/h$, $l = 0, 1, 2, \dots$, $\chi_{mn} = \alpha_{mn}/a$, and α_{mn} are the roots of the equation

$$J_m(\alpha_{mn}) = 0, \quad \omega^2 = c^2(\chi_{mn}^2 + k^2).$$

The H -waves are

$$H_z = \mathcal{H}_0 J_m(\chi r) \sin(m\alpha + \psi_m) \sin kze^{-i\omega t},$$

where $k = l\pi/h$, $l = 1, 2, \dots$; the value $l = 0$ is impossible; $\chi_{mn} = \beta_{mn}/a$ where β_{mn} is a root of the equation $J'_m(\beta_{mn}) = 0$; $\omega^2 = c^2(\chi_{mn}^2 + k^2)$. The remaining field components may be expressed in terms of H_z with the aid of Maxwell's equations.

When $m \neq 0$, both E - and H -waves are in general doubly degenerate, since to each proper frequency there correspond two proper functions, for example,

$$H_z = \mathcal{H}_0 J_m(\chi r) \sin m\alpha \sin kze^{-i\omega t},$$

$$H_z = \mathcal{H}_0 J_m(\chi r) \cos m\alpha \sin kze^{-i\omega t}.$$

Chapter 10

SPECIAL THEORY OF RELATIVITY

1. Lorentz transformation

$$\begin{aligned}
 484. \quad x - x_0 &= \frac{x' - x'_0 + V(t' - t'_0)}{\sqrt{1 - \beta^2}}, \\
 y - y_0 &= y' - y'_0, \\
 z - z_0 &= z' - z'_0, \\
 t - t_0 &= \frac{t' - t'_0 + \frac{V}{c^2}(x' - x'_0)}{\sqrt{1 - \beta^2}}.
 \end{aligned}$$

487. The coordinates of clocks indicating equal times $t = t'$ in S and S' are

$$x = \frac{c^2}{V} \left(1 - \frac{1}{\gamma}\right)t, \quad x' = -\frac{c^2}{V} \left(1 - \frac{1}{\gamma}\right)t.$$

It is clear from these formulae that the point at which $t = t'$ executes a uniform motion in each of the systems. In the system in which this point is at rest, S and S' move in opposite directions with equal velocities $V_0 = (c^2/V)(1 - 1/\gamma)$ where V_0 is the relativistic "half" of the velocity V in the sense that the relativistic addition of the two velocities V_0 yields V .

488. (a) No. 12 h 00 min can be indicated simultaneously by two clocks in one of the systems and only one clock in the other system.

(b) The readings of spatially coincident clocks are independent of the choice of the reference frame, so that

$$\begin{aligned}
 t_{A'} &= 12 \text{ h } 00 \text{ min} + l_0/V = 13 \text{ h } 00 \text{ min}; \\
 t_A &= 12 \text{ h } 00 \text{ min} + l_0 \sqrt{1 - V^2/c^2}/V = 12 \text{ h } 36 \text{ min}.
 \end{aligned}$$

In view of the relativity of simultaneity, the readings of the clocks

B and B' will depend on the choice of the reference frame.

For an observer on the platform (Fig. 73*a*)

$$t_{B'} = 12 \text{ h } 21.6 \text{ min}, t_B = t_A = 12 \text{ h } 36 \text{ min}.$$

For an observer on the train (Fig. 73*b*)

$$t_{B'} = t_{A'} = 13 \text{ h } 00 \text{ min}, t_B = 13 \text{ h } 14.4 \text{ min}.$$

(c) For an observer on the platform

$$t_A = 13 \text{ h } 00 \text{ min} = t_B, t_{B'} = 12 \text{ h } 36 \text{ min}, t_{A'} = 13 \text{ h } 14.4 \text{ min}.$$

For an observer on the train

$$t_A = 12 \text{ h } 21.6 \text{ min}, t_{A'} = t_{B'} = 12 \text{ h } 36 \text{ min}, t_B = 13 \text{ h } 00 \text{ min}.$$

The clocks whose readings are compared with the readings of the two clocks in the other reference frames will always lag behind.

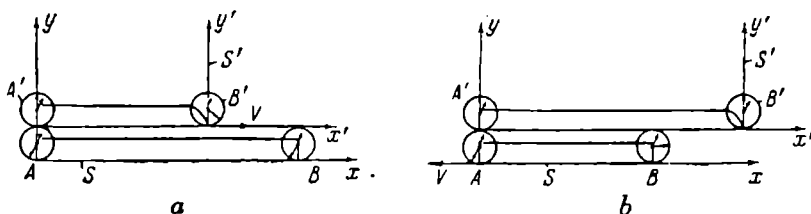


Fig. 73.

489. According to the clocks on the earth $\Delta t = 8$ years. In estimating the necessary supplies, the time interval must be taken to be $\Delta t_0 = 0.01 \Delta t \approx 1$ month (according to clocks on the rocket);

$$T = mc^2(\gamma - 1) \approx 2.5 \times 10^{16} \text{ kWh}.$$

This amount of energy exceeds the annual world output of electrical energy by a factor of 10 000.

$$490. \quad v = \frac{2l_0 \Delta t}{(\Delta t)^2 + l_0^2/c^2}.$$

According to the observer moving with the first ruler (Fig. 74*a*), the left-hand ends will pass each other first. For an observer at

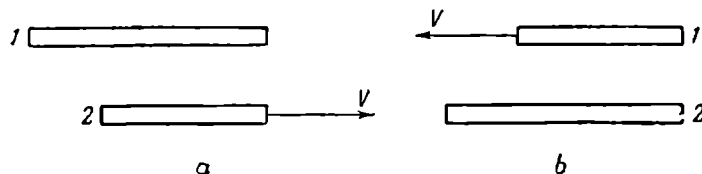


Fig. 74.

rest relative to the second ruler (Fig. 74*b*), the right-hand ends will meet first. According to an observer relative to whom the two rulers move with equal speed, the ends will coincide simultaneously.

491. Consider the transverse and longitudinal components of the position vector $\mathbf{r}$:

$$\begin{aligned} r_{\parallel} &= V \frac{\mathbf{r} \cdot \mathbf{V}}{V^2}, & r'_{\parallel} &= V \frac{\mathbf{r}' \cdot \mathbf{V}}{V^2}; \\ r_{\perp} &= \mathbf{r} - r_{\parallel} \mathbf{V}, & r'_{\perp} &= \mathbf{r}' - r'_{\parallel} \mathbf{V}. \end{aligned}$$

On applying the Lorentz transformation (X.1) to $\mathbf{r}_{\parallel}$ and $\mathbf{r}_{\perp}$ we have

$$r_{\parallel} = \gamma (r'_{\parallel} + V t'), \quad r_{\perp} = r'_{\perp}.$$

Finally,

$$\begin{aligned} \mathbf{r} &= \gamma (\mathbf{r}' + V t' \mathbf{V}) + (\gamma - 1) \frac{(\mathbf{r}' \times \mathbf{V}) \times \mathbf{V}}{V^2}, \\ t &= \gamma \left(t' + \frac{\mathbf{r}' \cdot \mathbf{V}}{c^2} \right). \end{aligned}$$

$$\begin{aligned} 492. \quad \mathbf{A} &= \gamma \left(\mathbf{A}' - i \frac{\mathbf{V}}{c} A'_4 \right) + (\gamma - 1) \frac{(\mathbf{A}' \times \mathbf{V}) \times \mathbf{V}}{V^2}, \\ A_4 &= \gamma \left(A'_4 + i \frac{\mathbf{A}' \cdot \mathbf{V}}{c^2} \right). \end{aligned}$$

$$493. \quad \mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} = \frac{\mathbf{v}' + \mathbf{V} + (\gamma - 1) \frac{\mathbf{V}}{V^2} [(\mathbf{v}' \cdot \mathbf{V}) + V^2]}{\gamma \left(1 + \frac{\mathbf{v}' \cdot \mathbf{V}}{c^2} \right)}.$$

It is also possible to differentiate with respect to time the radius vector $\mathbf{r}$ taken as a function of $\mathbf{r}'$ and t' as given by the formula obtained in Problem 491.

$$497. \quad l = l_0 \frac{1 - v^2/c^2}{1 + v^2/c^2}.$$

498. $l = l_0 (1 - v^2/c^2)$; the relative velocity of the rulers can conveniently be obtained with the aid of the formula given in the solution of Problem 493.

$$499. \quad (a) \ V = 2 \times 0.9c = 1.8c; \quad (b) \ V = 0.994c.$$

500. The relative velocity of the two particles in the system in which one of them is at rest is $V = 2v/(1 + v^2/c^2)$. Hence,

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{V^2}{c^2}}} = mc^2 \left[2 \left(\frac{\mathcal{E}_0}{mc^2} \right)^2 - 1 \right].$$

In the ultra-relativistic case $\mathcal{E}_0 \gg mc^2$, and hence $\mathcal{E} = 2\mathcal{E}_0^2/mc^2$. Suppose that the particles under acceleration are electrons ($mc^2 = 0.5$ MeV). Then for an energy $\mathcal{E}_0 = 50$ MeV, the gain in the power

supplied by the accelerator is a factor of 200: $\mathcal{E} = 10\,000$ MeV.

501. This problem is similar to Problem 493 in that it can be solved in two ways. The result is

$$\dot{\mathbf{v}} = \frac{1}{\gamma^2 s^2} \dot{\mathbf{v}}' - \frac{(\gamma-1)(\dot{\mathbf{v}}' \cdot \mathbf{V})\mathbf{V}}{\gamma^3 s^3 V^2} - \frac{(\dot{\mathbf{v}}' \cdot \mathbf{V})\mathbf{v}'}{\gamma^2 s^3 c^2},$$

where

$$s = 1 + \frac{\mathbf{v}' \cdot \mathbf{V}}{c^2}.$$

It is clear from these formulae that if the particle moves with a constant acceleration $\mathbf{v}'$ in one system, then the acceleration $\mathbf{v}$ in another reference system will in general be a function of time (because the variable velocity $\mathbf{v}'$ enters into the transformation formulae).

502.

$$w_i^2 = \gamma^6 \left[\dot{\mathbf{v}}^2 - \left(\dot{\mathbf{v}} \times \frac{\mathbf{v}}{c} \right)^2 \right] = \gamma^4 \left[\dot{\mathbf{v}}^2 + \gamma^2 v^2 \frac{\dot{\mathbf{v}}^2}{c^2} \right] > 0,$$

i.e. the 4-dimensional acceleration is a space-like vector.

503. Let S' be the system in which the particle is instantaneously at rest. In accordance with the solution of Problem 501

$$\dot{\mathbf{v}}' = \gamma^2 \left[\dot{\mathbf{v}} + \frac{\gamma-1}{v^2} (\dot{\mathbf{v}} \cdot \mathbf{v}) \mathbf{v} \right]. \quad (1)$$

Hence, the square of the acceleration is given by

$$\dot{\mathbf{v}}'^2 = \gamma^4 \left[\dot{\mathbf{v}}^2 + \frac{\gamma^2 (\dot{\mathbf{v}} \cdot \mathbf{v})^2}{c^2} \right] = \gamma^6 \left[\dot{\mathbf{v}}^2 - \left(\dot{\mathbf{v}} \times \frac{\mathbf{v}}{c} \right)^2 \right]. \quad (2)$$

If the velocity of the particle varies in magnitude only, then $\dot{\mathbf{v}} \parallel \mathbf{v}$ and

$$\dot{\mathbf{v}}' = \gamma^3 \dot{\mathbf{v}}. \quad (3)$$

If the velocity of the particle varies only in direction, then $\mathbf{v} \perp \dot{\mathbf{v}}$ and $\mathbf{v} \cdot \dot{\mathbf{v}} = 0$, so that

$$\dot{\mathbf{v}}' = \gamma^2 \dot{\mathbf{v}}. \quad (4)$$

Eqn. (2) can also be obtained by another simpler method which involves the use of the expression for the square of the 4-dimensional acceleration which was found in the solution to the preceding problem. The quantity w_i^2 is a 4-invariant. This means that the evaluation of w_i^2 should yield the same result in S as in S' .

Since the velocity of the particle $\mathbf{v}' = 0$, this leads directly to Eqn. (2).

504. According to clocks in the stationary frame

$$T = \frac{1}{|\dot{\mathbf{v}}|} \int_0^v \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \frac{v}{|\dot{\mathbf{v}}| \sqrt{1 - \frac{v^2}{c^2}}} = 47.5 \text{ years},$$

while according to the clocks at rest in the rocket

$$\tau = \frac{c}{2|\dot{\mathbf{v}}|} \ln \left| \frac{1 + v/c}{1 - v/c} \right| = 2.5 \text{ years}.$$

505. In S

$$\cos \alpha = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1| |\mathbf{v}_2|},$$

while in S'

$$\cos \alpha' = \frac{(\mathbf{v}_1 - \mathbf{V}) \cdot (\mathbf{v}_2 - \mathbf{V}) - \frac{1}{c^2} (\mathbf{v}_1 \times \mathbf{V}) \cdot (\mathbf{v}_2 \times \mathbf{V})}{\sqrt{(\mathbf{v}_1 - \mathbf{V})^2 - \frac{1}{c^2} (\mathbf{v}_1 \times \mathbf{V})^2} \sqrt{(\mathbf{v}_2 - \mathbf{V})^2 - \frac{1}{c^2} (\mathbf{v}_2 \times \mathbf{V})^2}}.$$

506. In S' the angle tends to zero. In order to show this, let $\mathbf{V} = \mathbf{V}_0 c$, where $|\mathbf{V}_0| = 1$. Using the formula for $\cos \alpha'$ derived in the preceding problem, and the result

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a}_1 \times \mathbf{b}_1) = (\mathbf{a} \cdot \mathbf{a}_1)(\mathbf{b} \cdot \mathbf{b}_1) - (\mathbf{a} \cdot \mathbf{b}_1)(\mathbf{a}_1 \cdot \mathbf{b}),$$

we have

$$\cos \alpha' = \frac{c^2 - \mathbf{v}_1 \cdot \mathbf{V} - \mathbf{v}_2 \cdot \mathbf{V} + \frac{1}{c^2} (\mathbf{v}_1 \cdot \mathbf{V})(\mathbf{v}_2 \cdot \mathbf{V})}{\sqrt{\left(c - \frac{\mathbf{v}_1 \cdot \mathbf{V}}{c}\right)^2} \cdot \sqrt{\left(c - \frac{\mathbf{v}_2 \cdot \mathbf{V}}{c}\right)^2}} = 1,$$

and hence $\alpha' = 0$. This contraction of the angular distribution is a specifically relativistic effect which appears in many phenomena.

507. The determination of the angle of aberration may be reduced to the determination of the angle α_1 between the direction of the ray AC (Fig. 75) and the direction of the velocity of the earth $\mathbf{v}$ in its first position, and the angle α_2 between the ray direction BC and the direction of the velocity of the earth $\mathbf{v}'$ in its second position (after six months). The angle of aberration δ is then given by $\delta = (\pi - \alpha_2) - \alpha_1 = \pi - \alpha_1 - \alpha_2$. The angles α_1 and α_2 may be computed from (X.15) by expressing them in terms of the angle ϑ which is observed in the reference frame in which the

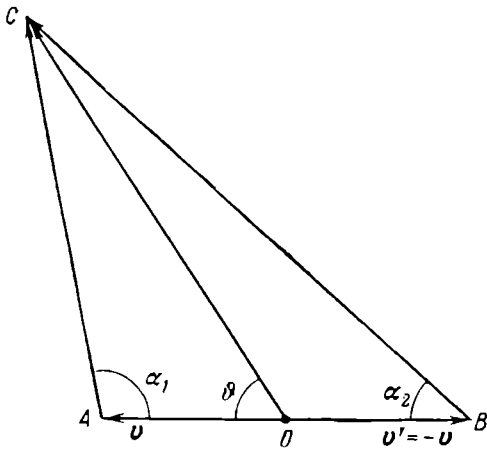


Fig. 75.

sun is at rest, and is equal to the angle between the light ray OC and the velocity of the earth:

$$\tan(\pi - \alpha_1) = \frac{1}{\gamma} \cdot \frac{\sin \vartheta}{\cos \vartheta - \beta},$$

$$\tan(\pi - \alpha_2) = -\frac{1}{\gamma} \cdot \frac{\sin \vartheta}{\cos \vartheta + \beta},$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Hence,

$$\tan \frac{\delta}{2} = \sqrt{\frac{1 - \cos \delta}{1 + \cos \delta}} = \beta \gamma \sin \vartheta.$$

We note that the three angles in Fig. 75 refer to different reference frames, and the drawing itself is

conventional (for example, $AC = CO = CB = c$). It is clear from these results that the angle of aberration δ depends only on the relative velocity v of the earth and the sun, and is independent of the velocity of the solar system relative to the star.

508. If the position of the earth on the orbit is defined by the azimuth angle φ and $\mathbf{a} = (0, a_\vartheta, a_\alpha)$ is a vector drawn from the point (ϑ, α) on the celestial sphere to the point of the apparent position of the star on this sphere, then

$$a_\vartheta = -\beta \cos \vartheta \cdot \sin(\alpha - \varphi),$$

$$a_\alpha = -\beta \cos(\alpha - \varphi).$$

It is clear that the apparent position of the star on the celestial sphere will, in the course of a year, describe an ellipse with semi-axes $\beta \cos \vartheta$ and β .

509. The beam which in S lies in the solid angle $d\Omega = \sin \vartheta d\vartheta d\alpha$ will be observed in S' in the solid angle $d\Omega' = \sin \vartheta' d\vartheta' d\alpha'$. Since $\alpha = \alpha'$ and $\cos \vartheta' = (\cos \vartheta - \beta)/(1 - \beta \cos \vartheta)$ it follows that

$$d\Omega' = \sin \vartheta' d\vartheta' d\alpha' = \frac{1 - \beta^2}{(1 - \beta \cos \vartheta)^2} d\Omega,$$

where

$$\int d\Omega' = \int d\Omega = 4\pi.$$

$$510. \quad \frac{dN}{d\Omega'} = \frac{N_0}{4\pi} \cdot \frac{d\Omega}{d\Omega'} = \frac{N_0}{4\pi} \cdot \frac{1 - \beta^2}{(1 - \beta \cos \vartheta')^2},$$

where N_0 is the total number of visible stars.

511.

$$\omega = \gamma \omega' \left(1 + \frac{\mathbf{n}' \cdot \mathbf{V}}{c} \right) \quad \text{or} \quad \omega = \frac{\omega'}{\gamma \left(1 - \frac{\mathbf{n} \cdot \mathbf{V}}{c} \right)},$$

$$\mathbf{k} = \gamma \left(\mathbf{k}' + \frac{\mathbf{V} \omega'}{c^2} \right) + (\gamma - 1) (\mathbf{k}' \times \mathbf{V}) \times \frac{\mathbf{V}}{V^2},$$

where

$$\mathbf{n} = \frac{\mathbf{k}}{k}, \quad \mathbf{n}' = \frac{\mathbf{k}'}{k'}, \quad k = \frac{\omega}{c}.$$

512. If ω_0 is the frequency in the frame in which the source is at rest, and V is the velocity of the source relative to the detector, then the detector will record the (lower) frequency $\omega = \omega_0 (1 - V^2/c^2)^{1/2}$ (this is the so-called red shift). The angle α

between the ray and the direction of motion of the source in the frame in which it is at rest is given by

$$\cos \alpha = \frac{V}{c}.$$

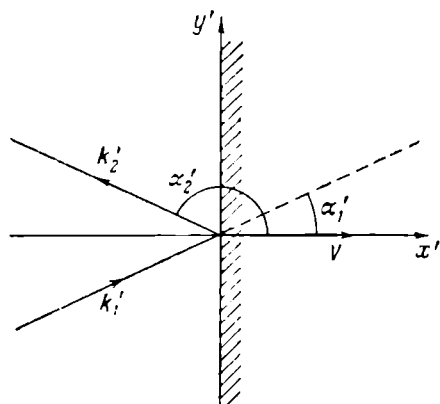


Fig. 76.

$$513. \quad (a) \lambda = \lambda_0 \sqrt{\frac{1 - V/c}{1 + V/c}};$$

$$(b) \lambda = \lambda_0 \sqrt{\frac{1 + V/c}{1 - V/c}}.$$

514. Consider the system S' in which the mirror is at rest,

and the laboratory system S . Let α'_1 and α'_2 be the angles between the wave vectors $\mathbf{k}'_1$ and $\mathbf{k}'_2$ of the incident and reflected waves and the direction of the velocity $\mathbf{V}$ of the mirror (Fig. 76). Let the frequencies before and after reflection be denoted by ω'_1 and ω'_2 . The analogous quantities in S will be denoted by the same symbols but without primes. According to the laws of reflection, in S' we have $\omega'_1 = \omega'_2 = \omega'$ and $\alpha'_2 = \pi - \alpha'_1$. Hence $\cos \alpha'_2 = -\cos \alpha'_1$. We can now express ω' in terms of ω , and $\cos \alpha'$ in terms of $\cos \alpha$, with the aid of (X.4) and (X.14). Solution of the resulting equations for ω_2 and $\cos \alpha_2$ yields

$$\cos \alpha_2 = -\frac{(1 + \beta^2) \cos \alpha_1 - 2\beta}{1 - 2\beta \cos \alpha_1 + \beta^2},$$

$$\omega_2 = \omega_1 \frac{1 - 2\beta \cos \alpha_1 + \beta^2}{1 - \beta^2}.$$

When $\beta \rightarrow 1$, then for normal incidence on a receding mirror $\omega_2 \rightarrow 0$, while for normal incidence on an approaching mirror $\omega_2 \rightarrow \infty$.

515. $\omega_1 = \omega_2$. The angle of incidence is equal to the angle of reflection.

516. Suppose that a plane wave of frequency ω propagates in the direction $\mathbf{n}_0$ in a medium moving with a velocity $\mathbf{V} \parallel O\mathbf{x}$ (S' system). The field components are proportional to $\exp(ik_i x_i)$ where $k_i = [(\omega/v) \cos \alpha, (\omega/v) \sin \alpha, 0, i(\omega/c)]$. The direction of propagation $\mathbf{n}_0$ remains perpendicular to the z -axis and is at an angle α to the x -axis; v is the phase velocity of the plane wave. Since the phase $k_i x_i$ is invariant under the Lorentz transformation, it follows that k_i is a 4-vector. Using (X.4) and (X.14) we have

$$\omega = \gamma \omega' (1 + \beta n \cos \alpha'), \quad (1)$$

$$\tan \alpha = \frac{\sin \alpha'}{\gamma \left(\cos \alpha' + \frac{\beta}{n} \right)}, \quad (2)$$

$$v = c \frac{1 + \beta n \cos \alpha'}{\sqrt{(n \cos \alpha' + \beta)^2 + n^2 \sin^2 \alpha' (1 - \beta^2)}}, \quad (3)$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. It is clear from (3) that the phase velocity in a moving medium depends on the direction of propagation. Thus, a particular form of anisotropy appears in a moving medium.

517. The required velocity may be found from Eqn. (3) of the preceding problem ($\alpha' = 0$):

$$v = c \frac{1 + \beta n(\lambda')}{n(\lambda') + \beta} \simeq \frac{c}{n(\lambda')} + V \left(1 - \frac{1}{n^2(\lambda')} \right),$$

where $\lambda' = 2\pi c/\omega'$ and ω' is the frequency observed in the frame S' in which the medium is at rest. Using Eqn. (1) of the preceding solution, we have, to within terms of the first order in V/c ,

$$\frac{\lambda'}{\lambda} = \frac{\omega}{\omega'} = 1 + \frac{nV}{c},$$

whence,

$$\frac{c}{n(\lambda')} = \frac{c}{n(\lambda)} - \frac{c}{n^2} \cdot \frac{dn}{d\lambda} \lambda \frac{nV}{c},$$

and finally,

$$v = \frac{c}{n(\lambda)} + V \left(1 - \frac{1}{n^2(\lambda)} - \frac{\lambda}{n(\lambda)} \frac{dn(\lambda)}{d\lambda} \right).$$

2. Four-dimensional vectors and tensors

519. One 3-dimensional tensor of rank 2 [$A_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3$)], two 3-dimensional vectors [$A_{4\alpha}$ and $A_{\alpha 4}$ ($\alpha = 1, 2, 3$)], and a 3-dimensional scalar [A_{44}].

520. The skew-symmetric 4-tensor A_{ik} may be written in the form

$$A_{ik} = \begin{pmatrix} 0 & A_3 & -A_2 & B_1 \\ -A_3 & 0 & A_1 & B_2 \\ A_2 & -A_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix},$$

where $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ are 3-dimensional vectors, or more precisely, $\mathbf{B}$ is a polar vector and $\mathbf{A}$ an axial vector.

524. (a) a vector, (b) a tensor of rank 2, and (c) a vector.

525. The condition for A_i and B_i to be parallel may be written in the form

$$\frac{a_{1i}A_1}{a_{1i}B_1} = \frac{a_{2i}A_2}{a_{2i}B_2} = \frac{a_{3i}A_3}{a_{3i}B_3} = \frac{a_{4i}A_4}{a_{4i}B_4},$$

where the numerator and denominator of each fraction is multiplied by the same number. Hence, using the properties of equal fractions, we have

$$\frac{A_i}{B_i} = \frac{a_{1i}A_1 + a_{2i}A_2 + a_{3i}A_3 + a_{4i}A_4}{a_{1i}B_1 + a_{2i}B_2 + a_{3i}B_3 + a_{4i}B_4} = \frac{A'_i}{B'_i}.$$

526. There are four different components. They are identical, except for the sign, with the components of the vector

$$A_i = \frac{1}{6} \epsilon_{iklm} A_{klm} \text{ and hence, } A_4 = -A_{123} = A_{213} = \dots, A_1 = A_{234} = -A_{324} = \dots, A_2 = A_{314} = -A_{134} = \dots, A_3 = A_{124} = -A_{214} = \dots$$

The remaining components A_{ikl} are all zero (they have identical indices). It follows that the non-zero components of A_{ikl} transform on 4-dimensional rotations and reflections as the components of a 4-dimensional pseudo-vector.

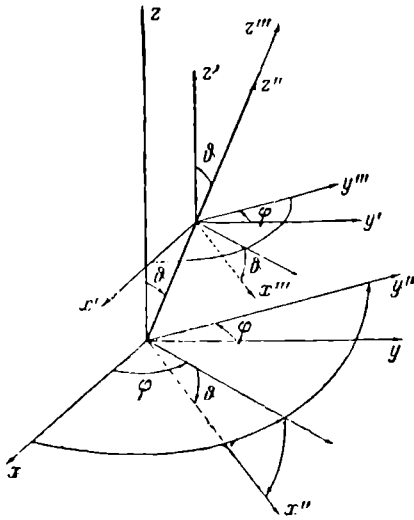


Fig. 77

527. The invariance of $d\Omega$ follows from the fact that the Jacobian for the proper transformation is

$$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(x'_1, x'_2, x'_3, x'_4)} = |\alpha_{ik}| = 1.$$

528. If $x_i = \alpha_{ik} x'_k$ then the matrix $\hat{\alpha}$ is of the form (the coordinate x_0 is inserted in the fourth place)

$$\hat{\alpha} = \begin{pmatrix} \cosh \alpha & 0 & 0 & \sinh \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \alpha & 0 & 0 & \cosh \alpha \end{pmatrix}.$$

529. The required matrix $\hat{g}$ may be written in the form of the product of three matrices:

$$\hat{g} = \hat{g}(\vartheta, \varphi) \hat{g}(\alpha) \hat{g}^{-1}(\vartheta, \varphi).$$

The matrix

$$\hat{g}(\vartheta, \varphi) = \begin{pmatrix} \cos \vartheta \cos \varphi & -\sin \varphi & \sin \vartheta \cos \varphi & 0 \\ \cos \vartheta \sin \varphi & \cos \varphi & \sin \vartheta \sin \varphi & 0 \\ -\sin \vartheta & 0 & \cos \vartheta & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

describes spatial rotation of the reference frame (Fig. 77):

$$x_i = \sum_k g_{ik}(\vartheta, \varphi) x''_k.$$

The matrix

$$\hat{g}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \alpha & \sinh \alpha \\ 0 & 0 & \sinh \alpha & \cosh \alpha \end{pmatrix}$$

corresponds to the transition from S''' to S'' , where the former moves along the x''_3 -axis with the velocity $V = c \tanh \alpha$ [cf. the special form of the Lorentz transformation (X.1)]. Finally, the matrix $g^{-1}(\vartheta, \varphi)$ describes a rotation as a result of which the frame S' transforms into S''' (Fig. 77), and is identical with the

transpose of $\hat{g}(\vartheta, \varphi)$. On multiplying the matrices, we have

$$\hat{g} = \begin{pmatrix} 1 + \omega_1^2 (\cosh \alpha - 1) & \omega_1 \omega_2 (\cosh \alpha - 1) & \omega_1 \omega_3 (\cosh \alpha - 1) & \omega_1 \sinh \alpha \\ \omega_1 \omega_2 (\cosh \alpha - 1) & 1 + \omega_2^2 (\cosh \alpha - 1) & \omega_2 \omega_3 (\cosh \alpha - 1) & \omega_2 \sinh \alpha \\ \omega_1 \omega_3 (\cosh \alpha - 1) & \omega_2 \omega_3 (\cosh \alpha - 1) & 1 + \omega_3^2 (\cosh \alpha - 1) & \omega_3 \sinh \alpha \\ \omega_1 \sinh \alpha & \omega_2 \sinh \alpha & \omega_3 \sinh \alpha & \cosh \alpha \end{pmatrix},$$

where

$$\omega_1 = \sin \vartheta \cos \varphi, \quad \omega_2 = \sin \vartheta \sin \varphi, \quad \omega_3 = \cos \vartheta.$$

3. Relativistic electrodynamics

530. In a vacuum

$$\begin{aligned} \mathbf{E} &= \gamma \left(\mathbf{E}' - \frac{\mathbf{V}}{c} \times \mathbf{H}' \right) - (\gamma - 1) \mathbf{V} \frac{(\mathbf{V} \cdot \mathbf{E}')}{V^2}; \\ \mathbf{H} &= \gamma \left(\mathbf{H}' + \frac{\mathbf{V}}{c} \times \mathbf{E}' \right) - (\gamma - 1) \mathbf{V} \frac{(\mathbf{V} \cdot \mathbf{H}')}{V^2}. \end{aligned}$$

In a medium

$$\begin{aligned} \mathbf{P} &= \gamma \left(\mathbf{P}' + \frac{\mathbf{V}}{c} \times \mathbf{M}' \right) - (\gamma - 1) \mathbf{V} \frac{(\mathbf{V} \cdot \mathbf{P}')}{V^2}; \\ \mathbf{M} &= \gamma \left(\mathbf{M}' - \frac{\mathbf{V}}{c} \times \mathbf{P}' \right) - (\gamma - 1) \mathbf{V} \frac{(\mathbf{V} \cdot \mathbf{M}')}{V^2}. \end{aligned}$$

The transformation formulae for the pairs of vectors $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{D}$, $\mathbf{H}$ are analogous to the transformation formulae for $\mathbf{E}$, $\mathbf{H}$ in a vacuum.

531. The problem has an infinite number of solutions. If the system S' (moving with a velocity $\mathbf{V}$) in which $\mathbf{E}'$ is parallel to $\mathbf{H}'$ has been found, then in any other reference frame, which moves relative to S' along this common direction, the vectors $\mathbf{E}$ and $\mathbf{H}$ will again be parallel [this follows from (X.25)]. In view of this, let us try to find the reference frame S' which moves at right angles to the plane $\mathbf{E}$, $\mathbf{H}$. From the condition for the vectors $\mathbf{E}'$ and $\mathbf{H}'$ to be parallel, *i.e.* $\mathbf{E}' \times \mathbf{H}' = 0$, and the transformation formulae of Problem 530, we have

$$\frac{\mathbf{V}}{c} = \mathbf{E} \times \mathbf{H} \frac{E^2 + H^2 - \sqrt{(E^2 - H^2)^2 + 4(\mathbf{E} \cdot \mathbf{H})^2}}{2(\mathbf{E} \times \mathbf{H})^2}.$$

Using the field invariants we have

$$\begin{aligned} E'^2 &= \frac{1}{2} [E^2 - H^2 + \sqrt{(E^2 - H^2)^2 + 4(\mathbf{E} \cdot \mathbf{H})^2}], \\ H'^2 &= \frac{1}{2} [H^2 - E^2 + \sqrt{(E^2 - H^2)^2 + 4(\mathbf{E} \cdot \mathbf{H})^2}]. \end{aligned}$$

532. When $E > H$, a reference frame should exist in which $H' = 0$ and $E' = (E^2 - H^2)^{1/2}$. When $E < H$, there exists a reference frame in which $E' = 0$ and $H' = (H^2 - E^2)^{1/2}$.

When $E > H$ we have

$$\mathbf{V} = c \frac{\mathbf{E} \times \mathbf{H}}{E^2}, \quad E' = \frac{E}{\gamma} \sqrt{E^2 - H^2}.$$

In any frame S'' moving in the direction of $\mathbf{E}'$ with an arbitrary velocity, the magnetic field will also be zero.

When $E < H$

$$\mathbf{V} = c \frac{\mathbf{H} \times \mathbf{E}}{H^2}, \quad H' = \frac{H}{\gamma} \sqrt{H^2 - E^2}.$$

533. When $\kappa < \mathcal{G}/c$ and the reference frame moves with velocity $V = c^2 \kappa / \mathcal{G}$ parallel to the axis of the cylinder and to $\mathbf{E} \times \mathbf{H}$, the electric field E' is zero and the magnetic field is given by $H' = (2\mathcal{G}/cr)[1 - (c^2 \kappa^2 / \mathcal{G}^2)]^{1/2}$.

When $\kappa > \mathcal{G}/c$ and the reference frame moves with velocity $V = \mathcal{G}/\kappa$ parallel to the axis of the cylinder and to $\mathbf{E} \times \mathbf{H}$, the magnetic field H' is zero and $E' = (2\kappa/r)[1 - (\mathcal{G}^2/c^2 \kappa^2)]^{1/2}$.

When $\kappa = \mathcal{G}/c$, there is no reference frame in which there is only an electric or only a magnetic field. As can be seen from the above formulae, when $\kappa \rightarrow \mathcal{G}/c$, the velocity of such a reference frame tends to the velocity of light, and the magnitude of the two fields tend to zero.

535.

$$\varphi = \frac{e}{R^*}, \quad \mathbf{A} = e \frac{\mathbf{V}}{cR^*},$$

$$\mathbf{E} = \frac{e\mathbf{R}}{\gamma^2 R^{*3}} = \frac{e\mathbf{R}(1 - V^2/c^2)}{R^3 \left(1 - \frac{V^2}{c^2} \sin^2 \vartheta\right)^{3/2}}, \quad \mathbf{H} = \frac{\mathbf{V}}{c} \times \mathbf{E},$$

where $R^* = [(x - vt)^2 + (1 - \beta^2)(y^2 + z^2)]^{1/2}$, $(vt, 0, 0)$ are the coordinates of the moving charge at time t , $\mathbf{R}(x - vt, y, z)$ is the position vector of the charge relative to the point of observation at time t , and ϑ is the angle between $\mathbf{R}$ and $\mathbf{v}$.

536. It follows from the formulae derived in the preceding solution that along the line of motion of the charge ($\vartheta = 0, \pi$) the field E is smaller than the Coulomb field $E_{\text{Coul}} = e/R^2$ by a factor $1 - V^2/c^2$, while in the perpendicular direction ($\vartheta = \pi/2$) the field E is greater by a factor $(1 - V^2/c^2)^{-1/2}$. When $V \sim c$ the field is large only within the angular range $\Delta\vartheta \sim (1 - V^2/c^2)^{1/2}$ near the equatorial plane.

The condition $E_{||} = E'_{||}$ refers to the same points in the 4-space. However, if in the rest system of the charge, a point A lies on the x -axis at a distance R from the charge, then in the laboratory system the same point will lie at a distance $R(1 - \beta^2)^{1/2}$. Comparing the values of $E_{||}$ at the point $R(1 - \beta^2)^{1/2}$ and $E'_{||}$ at the point R , we have

$$E_{||} = \frac{eR\sqrt{1-\beta^2}(1-\beta^2)}{(R\sqrt{1-\beta^2})^3} = \frac{e}{R^2} = E'_{||},$$

as was to be expected.

537.

$$\varphi = \frac{\mathbf{p}_0 \cdot \mathbf{r}^*}{\gamma r^{*3}}, \quad \mathbf{A} = \frac{\mathbf{v}}{c} \varphi,$$

$$\mathbf{E} = \frac{3\mathbf{R}(\mathbf{p}_0 \cdot \mathbf{r}^*) - p_0 r^{*2}}{\gamma^2 r^{*5}}, \quad \mathbf{H} = \frac{\mathbf{v}}{c} \times \mathbf{E},$$

where $\mathbf{R} = (x - vt, y, z)$, $\mathbf{r}^* = (x - vt, y/\gamma, z/\gamma)$; the dipole moves along the x -axis and its position vector at time t is $\mathbf{v}t$.

538.

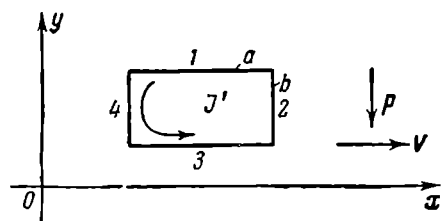


Fig. 78.

$$\mathbf{p} = \mathbf{p}' + \frac{\mathbf{v}}{c} \times \mathbf{m} - (\gamma - 1) \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{p}'}{\gamma V^2},$$

$$\mathbf{m} = \mathbf{m}' - \frac{\mathbf{v}}{c} \times \mathbf{p}' - (\gamma - 1) \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{m}'}{\gamma V^2},$$

where $\mathbf{p}'$ and $\mathbf{m}'$ are the dipole moments in the rest system.

539. Using the formulae for the transformation of the 4-dimensional current density, we find that the sides 2 and 4 of the rectangle in Fig. 78 are uncharged, while the sides 1 and 3 carry charges $q_1 = -q_3 = -(V/c)(\mathcal{J}'a/c)$ where $\mathcal{J}'$ is the current in the system S' in which the loop is at rest. Hence (or from the result of Problem 538), it follows that the electric dipole moment of the loop as observed in S' is $p = q_3 b = (V/c)m'$ where $m' = \mathcal{J}'ab/c$ is the magnetic moment of the loop as observed in S' .

540. Let u_i be the 4-dimensional velocity of the medium. Consider the 4-invariant [cf. Eqn. (X.37)]

$$f_i u_i = \gamma(\mathbf{f} \cdot \mathbf{V}) - \gamma(Q + \mathbf{f} \cdot \mathbf{V}) = -\gamma Q = \text{inv.}$$

If Q_0 is the amount of heat released per unit volume of the medium per unit time in the system in which the medium is at rest, then $Q = Q_0(1 - \beta^2)^{1/2}$.

542. $T_{ii} = 0$.

543. The momentum and energy of the field in the volume V

at time $t = x_4/ic$ are given by the integrals $\int T_{4\alpha} dS$ and $\int T_{44} dS$ respectively, where dS is an element of the hypersurface $x_4 = \text{const}$ (it is evident that $dS = dV$). The momentum and energy

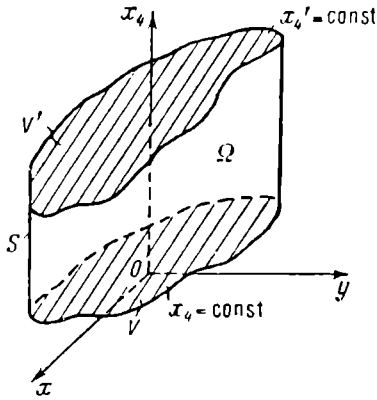


Fig. 79.

of the field at the time $t' = x'_4/ic$ is given by analogous integrals. Consider an arbitrary auxiliary constant 4-vector a_i and evaluate the sum $T_{4i} a_i$. Next, consider the 4-volume Ω which is bounded by a cylindrical hypersurface S , whose generator is parallel to the x_4 -axis, and the two hyperplanes $x_4 = \text{const}$ and $x'_4 = \text{const}$ (Fig. 79)†.

Applying the 4-dimensional form of Gauss' theorem to the surface integral of the function

$T_{4i} a_i$ over this hypersurface, we have

$$\oint T_{4i} a_i dS = \int_{\Omega} \frac{\partial T_{4i}}{\partial x_i} a_i d\Omega = 0,$$

since $\partial T_{4i}/\partial x_i = 0$ in the absence of charges. On the cylindrical hypersurface $T_{4i} = 0$, since the field is zero on the boundary of the volume V . Hence (taking into account the direction of the normal), we have

$$a_i \int T_{4i} dV = a'_i \int T'_{4i} dV'.$$

In other words, the quantity $a_i \int T_{4i} dV$ is invariant with respect to the Lorentz transformation, and hence $\int T_{4i} dV$ should be a 4-vector (cf. Problems 524 and 4).

544. Consider the change in K_{ik} during a time dt , where K_{ik} is a function of the space-like hypersurface $t = \text{const}$. It will be necessary to compare the values of K_{ik} at two hyperplanes $t = \text{const}$ and $t + dt = \text{const}$. Since the field at infinity is zero, the difference between integrals over these hyperplanes may be transformed into an integral over a closed hypersurface S which is formed by adding an infinitely distant lateral hyperplane to the above two hyperplanes. The resulting integral may be

† Note that such figures are only conventional, since the Minkowski 4-space is pseudo-Euclidean, i.e. the x_4 -axis is not fully equivalent to the remaining axes, since imaginary quantities are plotted along it.

transformed in accordance with Gauss' theorem:

$$\oint_S A_{ikl} dS_l = \int_\Omega \frac{\partial A_{ikl}}{\partial x_l} d\Omega,$$

where Ω is the volume enclosed by the closed hypersurface S . The integrand on the right-hand side may be transformed as follows:

$$\frac{\partial A_{ikl}}{\partial x_l} = \frac{\partial}{\partial x_l} (x_l T_{kl} - x_k T_{ll}) = T_{kl} - T_{lk} + x_l \frac{\partial T_{kl}}{\partial x_l} - x_k \frac{\partial T_{ll}}{\partial x_l},$$

where $T_{ik} = T_{ki}$, owing to the symmetry of the stress 4-tensor.

Consider now the integral

$$\int x_l \frac{\partial T_{kl}}{\partial x_l} d\Omega = -\frac{1}{c} \int x_l F_{kl} j_l d\Omega.$$

Since we are concerned with a system of point particles, we have

$$\int x_l F_{kl} j_l dV = \sum e x_l F_{kl} \frac{dx_l}{dt},$$

where the right-hand side of this expression involves the coordinates of the particles and functions of them at time t . From the equations of motion of the particles, $(e/c) F_{kl} dx_l/dt = dp_k/dt$. The integral $\int x_k (\partial T_{il}/\partial x_l) d\Omega$ may be considered in a similar way. Thus, the integral with respect to Ω becomes equal to $-\Sigma (x_i dp_k/dt - x_k dp_i/dt) dt$ and is cancelled by an equal sum over the particles. Hence, $dK_{ik}/dt = 0$ and $K_{ik} = \text{const.}$

545. The total angular momentum of the particles and the field within the volume is

$$K_{\alpha\beta}(t) = \sum k_{\alpha\beta} - \frac{i}{c} \int (x_\alpha T_{\beta\gamma} - x_\beta T_{\alpha\gamma}) dS_\gamma,$$

where $k_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha$ is the momentum of one of the particles, and the integral is evaluated over the part of the hyperplane $t = \text{const}$ whose projection on to 3-dimensional space is V . $K_{\alpha\beta}(t + dt)$ may be written down in a similar way. The angular momentum lost by the system in a time dt is

$$-dK_{\alpha\beta} = K_{\alpha\beta}(t) - K_{\alpha\beta}(t + dt) = -\sum dk_{\alpha\beta} + \frac{i}{c} \int_{t+dt} \dots - \frac{i}{c} \int_t \dots$$

The difference between integrals over near hyperplanes may be rewritten in another form, since $\int_t + \int_{t+dt} + \int_{S_{\text{side}}} = \oint$ over a

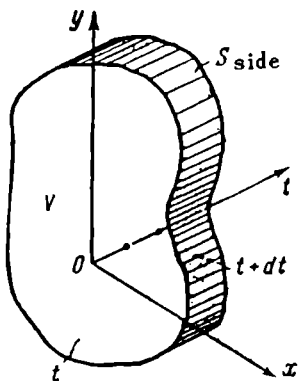


Fig. 80.

closed cylindrical hypersurface (cf. Fig. 80) whose generators are parallel to the time axis†. As in the previous solution, it can be shown that $\oint$ is cancelled by

$-\sum dk_{\alpha\beta}$, and hence

$$-dK_{\alpha\beta} = \frac{i}{c} \int_{S_{\text{side}}} (x_{\alpha} T_{\beta\gamma} - x_{\beta} T_{\alpha\gamma}) dS_{\gamma}.$$

The elements of the hypersurface S_{side} are normal to the t -axis and may be written in the form $dS_{\gamma} = ic dt n_{\gamma} df$, where df is an ele-

ment of the ordinary surface enclosing the volume V and $\mathbf{n}$ is a unit normal to this element. Hence, the rate of loss of angular momentum of the system is

$$-\frac{dK_{\alpha\beta}}{dt} = \int (-x_{\alpha} T_{\beta\gamma} + x_{\beta} T_{\alpha\gamma}) n_{\gamma} df. \quad (1)$$

Consider now the tensor $\mathfrak{R}_{\alpha\beta\gamma} = x_{\beta} T_{\alpha\gamma} - x_{\alpha} T_{\beta\gamma}$ which is skew-symmetric with respect to the subscripts α, β . This tensor may be interpreted as the angular momentum flux density, which is evident from Eqn. (1). The component $\mathfrak{R}_{\alpha\beta\gamma}$ is equal to the $\alpha\beta$ -component of the total angular momentum $K_{\alpha\beta}$ which flows per unit time through a unit area at right angles to the x_{γ} -axis. Similarly to the way in which $K_{\alpha\beta}$ may be replaced by the pseudo-vector $\mathbf{K}$, it is possible to introduce a pseudo-vector equivalent of $\mathfrak{R}_{\alpha\beta\gamma} n_{\gamma}$. Eqn. (1) will then be of the form

$$-\frac{d\mathbf{K}}{dt} = \int \mathfrak{R} df, \quad (2)$$

$$\mathfrak{R} = \frac{E^2 + H^2}{8\pi} (\mathbf{r} \times \mathbf{n}) - \frac{1}{4\pi} [(\mathbf{r} \times \mathbf{E})(\mathbf{n} \cdot \mathbf{E}) + (\mathbf{r} \times \mathbf{H})(\mathbf{n} \cdot \mathbf{H})]. \quad (3)$$

In deriving (3) use was made of the expression given by (X.29) for the components $T_{\alpha\beta}$.

† It must be remembered that such diagrams are purely conventional.

Chapter

11

RELATIVISTIC MECHANICS

1. Energy and momentum

$$546. \quad p = \frac{1}{c} \sqrt{T(T + 2mc^2)}.$$

$$547. \quad v = \frac{cp}{\sqrt{p^2 + m^2c^2}}.$$

$$548. \quad \beta = \frac{v}{c} = \sqrt{1 - \left(\frac{\mathcal{E}_0}{\mathcal{E}}\right)^2},$$

where $\mathcal{E}_0 = mc^2$. In the non-relativistic case $\beta \simeq (2T/\mathcal{E}_0)^{1/2}$; in the ultra-relativistic case $\beta = 1 - \frac{1}{2}(\mathcal{E}_0/\mathcal{E})^2$.

$$549. \quad (a) \quad T = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^2}{c^2} + \dots,$$

$$(b) \quad T = \frac{p^2}{2m} - \frac{1}{8} \cdot \frac{p^4}{m^3c^2} + \dots$$

$$550. \quad v = \sqrt{\frac{2eV}{m} \cdot \frac{1 + \frac{eV}{2mc^2}}{\left(1 + \frac{eV}{mc^2}\right)^2}}.$$

In particular, when $eV \ll mc^2$

$$v = \sqrt{\frac{2eV}{m} \left(1 - \frac{3}{4} \frac{eV}{mc^2}\right)} \ll c.$$

When $eV \gg mc^2$

$$v = c \left[1 - \frac{1}{2} \left(\frac{mc^2}{eV}\right)^2\right] \simeq c.$$

551. (a) $v = 0.0342c$; (b) $v = 0.9999985c$; (c) $0.81c$;
(d) $0.9956c$.

552. The length of the n -th tube is given by

$$L_n = \frac{v_n}{2v} = \frac{c}{2v} \sqrt{1 - \left(\frac{mc^2}{nV_{ee} + mc^2}\right)^2}.$$

where v_n is the particle velocity in the n -th tube. At the beginning of the acceleration $mc^2 \gg neV_e$ and $L_n \simeq (2eV_e \cdot n/m)^{1/2}/2v$. In the ultra-relativistic limit, $T_n \gg mc^2$, $v \simeq c$, and $L_n \simeq c/2v$.

The total length of the accelerator is given by

$$L = \sum_n L_n \simeq \frac{c}{2v} \int_0^N \sqrt{1 - \left(\frac{mc^2}{nV_e + mc^2} \right)^2} dn = \\ = \frac{c}{2veV_e} \left[\sqrt{(NeV_e + mc^2)^2 - m^2c^4} - mc^2 \cos^{-1} \frac{mc^2}{NeV_e + mc^2} \right].$$

553. The ratio of the intensities is given by

$$\frac{I_h}{I_0} = 2 \frac{h}{v\tau} \simeq 2 \frac{h}{v_0 c} \cdot \frac{m_\mu c^2}{\hbar} = 2,$$

where $\tau = \tau_0[1 - (v^2/c^2)]^{-1/2}$ is the half-life of the μ -meson moving with a velocity v . In the absence of time dilatation, the intensity ratio is given by (assuming $v = c$)

$$\frac{I'_h}{I'_0} \simeq 2 \frac{h}{v_0 c} = 2^5 = 32.$$

The first result ($I_h/I_0 = 2$) is in agreement with observation, and therefore constitutes a direct experimental proof of the existence of a relativistic slowing down of clocks.

$$554. \quad \tan \vartheta = \frac{1}{\gamma} \cdot \frac{p' \sin \vartheta'}{p' \cos \vartheta' + V \frac{\mathcal{E}'}{c^2}} = \frac{1}{\gamma} \cdot \frac{\sin \vartheta'}{\cos \vartheta' + \frac{V}{v'}},$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad \mathcal{E} = \gamma(\mathcal{E}' + p'V \cos \vartheta'),$$

and p, p' are the momenta of the particle in S and S' respectively.

The approximate formula given in the question may be used provided $\cos \frac{1}{2} \vartheta' \gg |1 - (V/v')|^{1/2}$, where $v' = p'c^2/\mathcal{E}'$ is the velocity of the particle in S' . The energy in the ultra-relativistic limit is $\mathcal{E} \simeq pc \simeq 2\gamma \mathcal{E}' \cos^2 \frac{1}{2} \vartheta'$.

555. Consider the dN particles moving within the solid angle $d\Omega'$ in S' . In S , these dN particles will move within a solid angle $d\Omega = \sin \vartheta d\vartheta d\alpha$, generated by the velocity vectors of these particles in S . The angular distribution in S which is described by the function $F(\vartheta, \alpha)$ is then given by

$$F(\vartheta, \alpha) d\Omega = F'(\vartheta', \alpha') d\Omega' = dW = \frac{dN}{N}. \quad (1)$$

The angle ϑ' expressed as a function of ϑ is given by the formula

$$\cos^2 \vartheta = \frac{1}{1 + \tan^2 \vartheta} = \frac{\left(\cos \vartheta' + \frac{V}{v'}\right)^2}{\left(\cos \vartheta' + \frac{V}{v'}\right)^2 + \frac{1}{\gamma^2} \sin^2 \vartheta'},$$

which follows from the solution of Problem 554 ($v' = p'c^2/\mathcal{E}'$ is the velocity of the particles in S'). Since $\alpha = \alpha'$, we have

$$F(\vartheta, \alpha) = F'[\vartheta'(\vartheta), \alpha] \frac{\gamma^2 \left[\left(\cos \vartheta' + \frac{V}{v'}\right)^2 + \frac{1}{\gamma^2} \sin^2 \vartheta' \right]^{\frac{3}{2}}}{1 + \frac{V}{v'} \cos \vartheta'}. \quad (2)$$

For ultra-relativistic particles $v' = c$, and the angular distribution in S assumes the simpler form (*cf.* Problem 509)

$$F(\vartheta, \alpha) = F'[\vartheta'(\vartheta), \alpha] \frac{1 - \frac{V^2}{c^2}}{\left(1 - \frac{V}{c} \cos \vartheta\right)^2}. \quad (3)$$

We note that particles moving at different angles ϑ in S have different energies, in spite of the fact that in S' the energies are the same.

$$556. \quad dW = \frac{d\Omega}{4\pi\gamma^2(1 - \beta \cos \vartheta)^2}, \quad \int_{4\pi} dW = 1,$$

where $\beta = v/c$.

557. $f = (1 + \beta)/(1 - \beta)$, and hence $\mathcal{E} = mc^2(1 + f)/2f^{1/2}$, where m is the rest mass of the π^0 -meson.

558. Since the momentum of the photon is $p = \mathcal{E}/c$, it follows that (*cf.* Problem 554)

$$\mathcal{E} = \frac{\mathcal{E}'}{\gamma(1 - \beta \cos \vartheta)}, \quad \mathcal{E}' = \frac{mc^2}{2}, \quad \beta = \frac{v}{c}.$$

From a comparison of the resulting expression $d\mathcal{E} = -\mathcal{E}'d(1 - \beta \cos \vartheta)/\gamma(1 - \beta \cos \vartheta)^2$ with the angular distribution of γ -rays found in the answer to Problem 556, the following probability distribution is found for the energies of photons produced in the disintegration

$$dW(\mathcal{E}) = \frac{|d\mathcal{E}|}{\mathcal{E}_{\max} - \mathcal{E}_{\min}},$$

where $\mathcal{E}_{\min} = \mathcal{E}'(1 - \beta)^{1/2}/(1 + \beta)^{1/2}$ is the minimum energy of the γ -rays (at $\vartheta = \pi$) and $\mathcal{E}_{\max} = \mathcal{E}'(1 + \beta)^{1/2}/(1 - \beta)^{1/2}$ is the maximum energy (at $\vartheta = 0$). Hence, it is evident that the spectrum of the γ -rays produced as a result of the disintegration is rectangular in the laboratory system, *i.e.* all energies between $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ are equally probable.

$$559. \quad m = \frac{2\sqrt{\mathcal{E}_1 \cdot \mathcal{E}_2}}{c^2}.$$

$$560. \quad v = \frac{c\sqrt{\mathcal{E}^2 - m_1^2 c^4}}{\mathcal{E} + m_2 c^2}.$$

561. According to the conservation of 4-momentum

$$p_{1i}^{(0)} + p_{2i}^{(0)} = p_{1i} + p_{2i}. \quad (1)$$

To determine the angle of scattering of the first particle, take p_{1i} to the left-hand side of the equation, and square both sides, so that

$$p_{1i}^{(0)2} + p_{2i}^{(0)2} + p_{1i}^2 + 2p_{1i}^{(0)}p_{2i}^{(0)} - 2p_{1i}^{(0)}p_{1i} - 2p_{2i}^{(0)}p_{1i} = p_{2i}^2. \quad (2)$$

In view of (XI.7), $p_{1i}^{(0)2} = p_{1i}^2 = -m_1^2 c^2$, $p_{2i}^{(0)2} = p_{2i}^2 = -m_2^2 c^2$. The scalar products are transformed as follows ($p_2^{(0)} = 0$):

$$p_{1i}^{(0)}p_{2i}^{(0)} = \mathbf{p}_1^{(0)} \cdot \mathbf{p}_2^{(0)} - \frac{1}{c^2} \mathcal{E}_1^{(0)}\mathcal{E}_2^{(0)} = -\mathcal{E}_0 m_2, \quad p_{2i}^{(0)}p_{1i} = -m_2 \mathcal{E}_1,$$

$$p_{1i}^{(0)}p_{1i} = \mathbf{p}_1^{(0)} \cdot \mathbf{p}_1 - \frac{1}{c^2} \mathcal{E}_1^{(0)}\mathcal{E}_1 = p_0 p_1 \cos \vartheta_1 - \frac{\mathcal{E}_0 \mathcal{E}_1}{c^2},$$

where $p_0 = (\mathcal{E}_0^2 - m_1^2 c^4)^{1/2}/c^2$. Substituting these expressions into (2), it is found that

$$\cos \vartheta_1 = \frac{\mathcal{E}_1(\mathcal{E}_0 + m_2 c^2) - \mathcal{E}_0 m_2 c^2 - m_1^2 c^4}{c^2 p_0 p_1}.$$

Similarly,

$$\cos \vartheta_2 = \frac{(\mathcal{E}_0 + m_2 c^2)(\mathcal{E}_2 - m_2 c^2)}{c^2 p_0 p_2}.$$

562.

$$\mathcal{E}_1 = m_1 c^2 \frac{\left(\gamma_0 + \frac{m_2}{m_1}\right)\left(1 + \gamma_0 \frac{m_2}{m_1}\right) \pm \cos \vartheta_1 (\gamma_0^2 - 1) \sqrt{\frac{m_2^2}{m_1^2} - \sin^2 \vartheta_1}}{\left(\gamma_0 + \frac{m_2}{m_1}\right)^2 - (\gamma_0^2 - 1) \cos^2 \vartheta_1}. \quad (1)$$

$$\mathcal{E}_2 = \frac{\left(\gamma_0 + \frac{m_2}{m_1}\right)^2 + (\gamma_0^2 - 1) \cos^2 \vartheta_2}{\left(\gamma_0 + \frac{m_2}{m_1}\right)^2 - (\gamma_0^2 - 1) \cos^2 \vartheta_2} m_2 c^2, \quad (2)$$

where $\gamma_0 = \mathcal{E}_0/m_1 c^2$.

It is evident from these formulae that when $m_1 > m_2$ the scattering angle must be such that $\vartheta_1 < \sin^{-1}(m_2/m_1)^{1/2}$ and the positive sign should be taken in (1). To each value of ϑ_1 there are two values of $\mathcal{E}_1$.

When $m_1 = m_2$, the scattering angle ϑ_1 will not exceed $\pi/2$, and to each value of ϑ_1 there is only one value of the energy which corresponds to the positive sign in (1). The negative sign would correspond to $\mathcal{E}_1 = m_1 c^2$ whatever the scattering angle, and this is unphysical.

When $m_1 < m_2$ all scattering angles are possible, and to each value of ϑ_1 there is one value of $\mathcal{E}_1$. The positive sign must be taken in (1) when $0 < \vartheta_1 < \pi/2$, and the negative sign when $\pi/2 < \vartheta_1 < \pi$. With this choice of signs, the scattering of incident particles through large angles will correspond to large energy losses, as expected.

$$563. \quad T_1 = \frac{T_0 \cos^2 \vartheta_1}{1 + \frac{1}{2} \left(\frac{T_0}{mc^2} \right) \sin^2 \vartheta_1}.$$

$$564. \quad T_1 = T_0 \left(\frac{m_1}{m_1 + m_2} \right)^2 \left[1 + \left(\frac{m_2}{m_1} \right)^2 - 2 \sin^2 \vartheta_1 \pm 2 \cos \vartheta_1 \sqrt{\left(\frac{m_2}{m_1} \right)^2 - \sin^2 \vartheta_1} \right];$$

$$T_2 = \frac{4m_1 m_2}{(m_1 + m_2)^2} T_0 \cos^2 \vartheta_2.$$

The signs should be chosen as in the solution to Problem 562.

565. The angle $\chi = \vartheta_1 + \vartheta_2$ between the directions of motion of the particles is given by

$$\tan \chi = \frac{(v'_1 + v'_2) \sqrt{1 - \frac{V^2}{c^2}} \sin \vartheta'}{\frac{V^2}{c^2} v'_1 \sin^2 \vartheta' + (V - v'_1)(1 - \cos \vartheta')}$$

(cf. Problem 505). When $m_1 = m_2$, we have $v'_1 = v'_2 = V$ and

$$\tan \chi = \frac{2c^2 \sqrt{1 - \frac{V^2}{c^2}}}{V^2 \sin \vartheta'}.$$

In this case, $\chi < 90^\circ$. In the non-relativistic limit $\chi \rightarrow 90^\circ$.

566. Proceeding as in the solution of Problem 561, we

find that

$$\omega = \frac{\omega_0 \left(\frac{\mathcal{E}_0}{c} - p_0 \cos \vartheta_0 \right)}{\frac{\mathcal{E}_0}{c} - p_0 \cos \vartheta_1 + \frac{\hbar \omega_0}{c} (1 - \cos \vartheta)},$$

where ϑ is the angle between the directions of motion of the primary and secondary photons, and ϑ_1 is the angle between the initial direction of motion of the electron and the direction of motion of the photon after scattering. If the electron is initially at rest, then

$$\omega = \frac{\omega_0}{1 + \frac{\hbar \omega_0}{mc^2} (1 - \cos \vartheta)}.$$

567. Let n_1 and n_2 be the numbers of scattered and scattering particles per unit volume, and consider the scattering process in the system S . The total number of particles dN scattered into a solid angle $d\Omega$ in a time t by target particles lying within a volume V is given by $dN = d\sigma_{12} J_{12} n_2 Vt$, where σ_{12} is the cross-section and $J_{12} = n_1 v_1$. In the S' system, the corresponding expression is $dN = d\sigma'_{12} J'_{12} n'_2 V't'$ where $J'_{12} = n'_1 |\mathbf{v}'_1 - \mathbf{v}'_2|$. In the latter system dN represents the number of particles scattered into the solid angle $d\Omega'$ which corresponds to $d\Omega$. Thus,

$$d\sigma_{12} n_1 n_2 v_1 Vt = d\sigma'_{12} n'_1 n'_2 |\mathbf{v}'_1 - \mathbf{v}'_2| V't'. \quad (1)$$

Since $n_i = \text{inv} [1 - (v_i^2/c^2)]^{-1/2}$, the set of 4 quantities $(n_i \mathbf{v}_i, in_i c)$ is a 4-vector which is proportional to the 4-velocity of the particle. It follows that

$$n_1 n_2 = n'_1 n'_2 \left(1 - \frac{\mathbf{v}'_1 \cdot \mathbf{v}'_2}{c^2} \right), \quad (2)$$

since the scalar product of 4-vectors is an invariant quantity. In view of (2) and the fact that the 4 volume is an invariant quantity, we have $Vt = V't'$, and hence finally,

$$d\sigma'_{12} = d\sigma_{12} \frac{v_1 \left(1 - \frac{\mathbf{v}'_1 \cdot \mathbf{v}'_2}{c^2} \right)}{|\mathbf{v}'_1 - \mathbf{v}'_2|}. \quad (3)$$

In the special case, where $\mathbf{v}'_1 \parallel \mathbf{v}'_2$

$$v_1 = \frac{v'_1 - v'_2}{1 - \frac{v'_1 \cdot v'_2}{c^2}}$$

(cf. Problem 493), and it follows from (3) that the cross-section is an invariant quantity, *i.e.*

$$d\sigma_{12} = d\sigma'_{12}. \quad (4)$$

This case occurs, for example, in the transformation from the laboratory system to the centre of mass system. We note that if the flux is defined by $J_{12} = n_1 \tilde{v}$ where $\tilde{v} = v_1 [1 - (\mathbf{v}'_1 \cdot \mathbf{v}'_2)/c^2]$, then the cross-section is invariant under any Lorentz transformation.

$$568. \quad m^2 = m_1^2 + m_2^2 + \frac{2}{c^2} [\sqrt{(p_1^2 + m_1^2 c^2)(p_2^2 + m_2^2 c^2)} - p_1 p_2 \cos \vartheta].$$

$$569. \quad m_1^2 = m^2 + m_2^2 - \frac{2}{c^2} [\sqrt{(p^2 + m^2 c^2)(p_2^2 + m_2^2 c^2)} - p p_2 \cos \vartheta_2].$$

$$570. \quad m^2 = \frac{1}{c^2} (\mathcal{E}^2 - c^2 p^2) = m_1^2 + m_2^2 + \frac{2m_1 m_2}{\sqrt{1 - \frac{v^2}{c^2}}},$$

$$V = \frac{pc^2}{\mathcal{E}} = \frac{m_1 v}{m_1 + m_2 \sqrt{1 - \frac{v^2}{c^2}}}.$$

571. Particles 1 and 2 assume the following kinetic energies:

$$T_1 = \mathcal{E}_1 - m_1 c^2 = (m_0 - m_1 - m_2)(m_0 - m_1 + m_2) \frac{c^2}{2m_0},$$

$$T_2 = \mathcal{E}_2 - m_2 c^2 = (m_0 - m_1 - m_2)(m_0 + m_1 - m_2) \frac{c^2}{2m_0},$$

where $T_1 + T_2 = \Delta \mathcal{E}$. It is evident from these expressions that a large fraction of the energy is taken up by the lighter particle.

572. (a) $T_\alpha/T_n = 58.5$ where T_α and T_n are the kinetic energies of the α -particle and the recoil nucleus (daughter nucleus) respectively.

(b) $T_\nu/T_\mu = 7.27$ where T_ν and T_μ are the kinetic energies of the neutrino and the μ -meson.

(c) $T_\gamma/T_n = 2mc^2/\Delta \mathcal{E}$ since $mc^2 \gg \Delta \mathcal{E}$ and the rest mass of the photon is zero. (T_γ is the energy of the γ -ray and m is the rest mass of the disintegrating nucleus.)

$$573. \quad \omega = \frac{\Delta \mathcal{E}}{\hbar} \left(1 - \frac{\Delta \mathcal{E}}{2mc^2}\right).$$

574. Consider the energy-momentum 4-vector p_i of the system of particles. It is conserved, *i.e.* the corresponding components of the energy-momentum before and after the reaction are equal. At the threshold kinetic energy, T_0 , the particles produced in the reaction are at rest in the centre of mass frame (we note that in the laboratory frame the particles cannot be at

rest for the threshold energy T_0 since this would violate the law of conservation of momentum). The total 4-momentum vector of the system before the reaction is (in the laboratory system)

$$p_i^{(0)} = \left(\mathbf{p}_0, i \frac{\mathcal{E}_0}{c} + i m_1 c \right),$$

where $\mathcal{E}_0$ is the total energy and $\mathbf{p}_0$ is the total momentum at the threshold.

After the reaction, the 4-momentum in the centre of mass system is $p'_i = (0, iMc)$. In view of the invariance of the square of a 4-vector, and the law of conservation of 4-momentum, we have $p_i^{(0)2} = p_i'^2$. Explicitly,

$$-M^2 c^2 = p_0^2 - \frac{\mathcal{E}_0^2}{c^2} - 2m_1 \mathcal{E}_0 - m_1^2 c^2,$$

and hence,

$$T_0 = \frac{c^2}{2m_1} (M - m_1 - m)(M + m_1 + m).$$

575. (a) $T_0 = 288$ MeV, (b) $T_0 = 160$ MeV, (c) $T_0 = 763$ MeV, and (d) $T_0 = 2(m_p/m)(m + 2m_p)c^2$ where m_p is the rest mass of the proton. In the special case of a collision with a proton $m = m_p$ and hence $T_0 = 6m_p c^2 = 5.63$ BeV. The approximate formula for the threshold energy is

$$T_0 = \frac{2(A+2)}{A} m_p c^2.$$

For large A this may be replaced by $T_0 \simeq 2m_p c^2$.

576. $T_0 = (1 + m/M)\Delta\mathcal{E}$. In case (a) we have $\Delta\mathcal{E} = T_0 = 2.18$ MeV ($m = 0$), in accordance with the above approximate formula. When the exact formula is used (*cf.* Problem 574), the result is larger by $|Q|^2/2Mc^2 \simeq 0.0012$ MeV, where $Q = -(M - m_1 - m)c^2$. In case (b) the approximate formula yields $T_0 = 2|Q| = 7.96$ MeV, and the difference between this result and the exact value is 0.003 MeV.

577. The equation for the reaction is $\gamma + (\text{particle}) \rightarrow e^+ + e^- + (\text{particle})$. The threshold energy may be found from the general formula (*cf.* Problem 574)

$$T_0 = \hbar\omega_0 = \frac{c^2}{2m_1} (m_1 + 2m - m_1)(m_1 + 2m + m_1) = 2mc^2 \left(1 + \frac{m}{m_1} \right).$$

where m is the rest mass of the electron (or positron).

When there is no particle, so that $m_1 \rightarrow 0$, the threshold energy T_0 tends to infinity, which means that the reaction is impossible. The latter result may also be obtained by showing that the relation $k_i = p_{+i} + p_{-i}$ cannot be satisfied, where k_i , p_{+i} , and p_{-i} are the 4-momenta of the photon, positron, and electron respectively.

580. When a particle moving in a medium with a 4-momentum p_{0i} emits a photon with a 4-momentum $k_i = (\hbar\omega n/c, i\hbar\omega/c)$, then the laws of conservation of energy and momentum may be described by the 4-dimensional equation

$$p_{0i} = p_i + k_i,$$

where p_i is the 4-momentum of the particle after the emission of the photon. Put k_i on the left-hand side of this equation and square both sides. After simple rearrangement it is found that

$$\cos \vartheta = \frac{1}{n\beta} \left[1 + \frac{\pi\Lambda}{n\lambda} (n^2 - 1) \sqrt{1 - \beta^2} \right], \quad (1)$$

where $\Lambda = \hbar/mc$ is the Compton wavelength of the particle, $\lambda = 2\pi c/\omega n$ is the wavelength of the photon, and $\beta = v/c$. The second term, which is of the order of Λ/λ , is usually very small. This term represents quantum corrections (Λ is proportional to $\hbar$). When it is neglected, the relation given by (1) reduces to the classical condition for the emission of Vavilov-Cherenkov radiation, which is

$$\cos \vartheta = \frac{1}{n\beta}.$$

582. Let p_{0i} and p_i represent the 4-momenta of the particle before and after the emission and k_i the "4-momentum" of the photon. The law of conservation of energy and momentum may then be written in the form

$$p_{0i} - k_i = p_i.$$

On squaring both sides of this equation, and neglecting terms containing $\hbar^2$, it is found that

$$(m^2 - m_0^2) c^2 - 2\mathbf{p} \cdot \mathbf{k} + \frac{2\mathcal{E}_0 k}{c} = 0,$$

where m_0 is the rest mass of the excited particle and m is the rest

mass of the particle in the ground state.

Since $c^2(m_0 - m)(m_0 + m) \simeq 2\hbar\omega_0 m$ we have

$$n(\omega)\beta \cos \vartheta = 1 - \frac{\omega_0}{\omega} \sqrt{1 - \beta^2}, \quad (1)$$

where $\beta = v/c$. When $\omega_0 \rightarrow 0$, Eqn. (1) becomes

$$n(\omega)\beta \cos \vartheta = 1,$$

which is the condition for the emission of the Vavilov-Cherenkov radiation. It follows that this radiation is not associated in any way with a change in the internal state of the particle.

When $\omega_0 \neq 0$, Eqn. (1) is conveniently rewritten in the form

$$\omega = \frac{\omega_0 \sqrt{1 - \beta^2}}{1 - n(\omega)\beta \cos \vartheta}, \quad (2)$$

which describes the Doppler effect in a refracting medium (*cf.* Problem 516). It may be used when $n(\omega)\beta \cos \vartheta < 1$ and differs from the corresponding formula for the Doppler effect in a vacuum only by the presence of $n(\omega)$ in the denominator. When $\beta \ll 1$ no fundamentally new effects arise, but when $\beta \simeq 1$ and there is dispersion in the medium, the phenomenon becomes more complicated.

In general, Eqn. (2) is a non-linear equation in ω (n is a function of ω !), and there may be more than one solution. Instead of a single shifted line, as in the usual Doppler effect, there are then several lines in the laboratory frame (composite Doppler effect).

583. Proceeding as in the solution of Problem 582, the following results are obtained.

Emission at a frequency ω , which is accompanied by the excitation of the particle, may occur when the velocity of the particle $v = \beta c$ exceeds the threshold value $c/n(\omega) \cos \vartheta$ where ϑ is the angle between the direction of the velocity v and the direction of the momentum of the photon. The energy necessary for this to occur is taken at the expense of the kinetic energy. This type of emission is observed at a fixed value of ω only in a certain range of acute angles ϑ within the Cherenkov cone whose surface is defined by $n\beta \cos \vartheta = 1$. The observed frequency ω is then given by

$$\omega = \frac{\omega_0 \sqrt{1 - \beta^2}}{n(\omega)\beta \cos \vartheta - 1} \quad [n(\omega)\beta \cos \vartheta > 1],$$

which is an equation in ω as in Problem 582. In general, this equation has several solutions (composite super-relativistic Doppler effect).

584.

$$F = \frac{\mathcal{J}}{ce} \sqrt{T(T + 2mc^2)}, \quad W = \frac{\mathcal{J}}{e} T.$$

585.

$$p = \frac{2mv^2N}{1 - \frac{v^2}{c^2}}.$$

The pressure in the system attached to the body and in the system attached to the gas is the same. This may be verified either by direct calculation of the pressure in each of the systems, or by applying the Lorentz transformation to the 4-force [cf. (XI.14)].

2. The motion of charged particles in an electromagnetic field

$$586. \quad \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dv}{dt} + \frac{mvv}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \frac{dv}{dt} = F;$$

$$a) \quad \frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \frac{dv}{dt} = F \quad v \parallel F; \quad b) \quad \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dv}{dt} = F \quad v \perp F;$$

$$c) \quad m \frac{dv}{dt} = F.$$

The quantities $m[1 - (v^2/c^2)]^{-3/2}$ and $m[1 - (v^2/c^2)]^{-1/2}$ are sometimes referred to as the longitudinal and transverse masses respectively.

$$587. \quad F = \frac{1}{\gamma} F' + \left(1 - \frac{1}{\gamma}\right) \frac{(\mathbf{v} \cdot \mathbf{F}') \cdot \mathbf{v}}{v^2},$$

$$F' = \gamma F - (\gamma - 1) \frac{(\mathbf{v} \cdot \mathbf{F}) \mathbf{v}}{v^2},$$

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$.

$$588. \quad F = \gamma^2 \frac{mv^2}{R}.$$

$$590. \quad \psi(\alpha) = - \frac{2\kappa(1 - \beta^2)}{\sqrt{(1 - \beta^2) \cos^2 \alpha + \sin^2 \alpha}} \ln r,$$

where $\beta = v/c$ and r is the distance of the point of observation from the conductor.

591. $F = 2e\kappa/\gamma r$. The problem may be solved in different ways: (a) the electromagnetic force on the moving point charge due to the linear charge and current can be computed directly (the Lorentz contraction must be taken into account!), (b) the force can be evaluated in the reference frame in which the magnetic field is zero, with subsequent application of the transformation formulae for the 4-force, and (c) one can use the convection potential ψ obtained in Problem 590 and defined by $\mathbf{F} = -e \text{grad} \psi$.

592. $F = e(1 - v^2/c^2)(2\mathcal{G}_r/vr)$ where r is the distance of the electron from the axis of the beam. The current through a circle of radius r is given by

$$\mathcal{G}_r = \frac{2\pi v}{\sqrt{1-\beta^2}} \int_0^r \rho(r) r dr$$

and the electron velocity is given by (cf. Problem 591)

$$v = \left(1 + \frac{eV}{mc^2}\right)^{-1} \left(1 + \frac{eV}{2mc^2}\right) \sqrt{\frac{2eV}{m}}$$

A surface electron experiences a force $F = e(1 - v^2/c^2)(2\mathcal{G}/va)$ where a is the radius of the beam.

593. The acceleration of an outer electron is normal to the axis of the beam and to the electron velocity, and hence in the laboratory system we have (cf. solutions to Problems 586 and 592)

$$\dot{v}_n = \frac{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{m} F = \frac{2e\mathcal{G}}{mav} \left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}.$$

The broadening of the beam is given by

$$\Delta a = \frac{\dot{v}_n t^2}{2} = \frac{\dot{v}_n L^2}{2v^2}.$$

Since $\Delta a \ll L$, we have $\dot{v}_n L/v \ll v$ or $\dot{v}_n t \ll v < c$. The broadening Δa can therefore be calculated from a non-relativistic formula.

The same result may be obtained for Δa by considering the broadening of the beam in the reference frame moving together with the electrons. In this frame the electrons experience only an electrostatic force.

594. Let the direction of the x -axis be parallel to $\mathbf{E}$. The differential equations of motion in the 4-dimensional form are

$$\frac{d^2x}{d\tau^2} = \frac{|e|E}{mc} \frac{d(ct)}{d\tau}, \quad \frac{d^2y}{d\tau^2} = 0, \quad \frac{d^2z}{d\tau^2} = 0, \quad \frac{d^2(ct)}{d\tau^2} = \frac{|e|E}{mc} \frac{dx}{d\tau}.$$

Integration of these equations subject to the initial conditions

$$x = y = z = ct = 0, \quad \frac{dx}{d\tau} = \frac{p_{0x}}{m}, \quad \frac{dy}{d\tau} = \frac{p_{0y}}{m},$$

$$\frac{dz}{d\tau} = 0, \quad c \frac{dt}{d\tau} = \frac{\mathcal{E}_0}{mc} \quad \text{at} \quad \tau = 0, \quad \text{where} \quad \mathcal{E}_0 = \sqrt{c^2 p_0^2 + m^2 c^4},$$

leads to the following equations for the particle trajectory in the 4-dimensional space:

$$\begin{aligned} x &= \frac{\mathcal{E}_0}{|e|E} \left(\cosh \frac{|e|E\tau}{mc} - 1 \right) + \frac{cp_{0x}}{|e|E} \sinh \frac{|e|E\tau}{mc}, \\ y &= \frac{p_{0y}\tau}{m}, \quad z = 0, \\ ct &= \frac{\mathcal{E}_0}{|e|E} \sinh \frac{|e|E\tau}{mc} + \frac{cp_{0x}}{|e|E} \left(\cosh \frac{|e|E\tau}{mc} - 1 \right). \end{aligned}$$

From the last equation we have

$$\tau = \frac{mc}{|e|E} \ln \frac{p_{0x} + |e|Et + \sqrt{(p_{0x} + |e|Et)^2 + m^2 c^2 + p_{0y}^2}}{p_{0x} + \frac{\mathcal{E}_0}{c}}.$$

Using this expression, and eliminating the sinh and cosh terms from the first and last equations, we obtain the following equations for the trajectory in the 3-dimensional form

$$\begin{aligned} x(t) &= \frac{c}{|e|E} \left[\sqrt{(p_{0x} + |e|Et)^2 + m^2 c^2 + p_{0y}^2} - \frac{\mathcal{E}_0}{c} \right]; \\ y(t) &= \frac{cp_{0y}}{|e|E} \ln \frac{p_{0x} + |e|Et + \sqrt{(p_{0x} + |e|Et)^2 + m^2 c^2 + p_{0y}^2}}{p_{0x} + \frac{\mathcal{E}_0}{c}}; \\ z(t) &= 0. \end{aligned}$$

When $p_0 \ll mc$ and $t \ll mc/|e|E$, the motion is non-relativistic. The expressions for x , y , and z then go over into the usual non-relativistic expressions

$$\begin{aligned} x(t) &= \frac{p_{0x}}{m} t + \frac{|e|E}{2m} t^2; \\ y(t) &= \frac{p_{0y}}{m} t. \end{aligned}$$

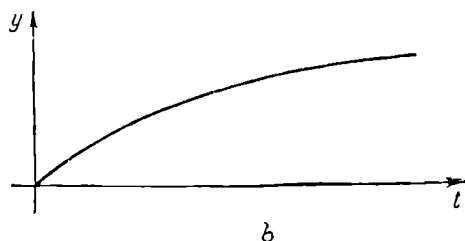
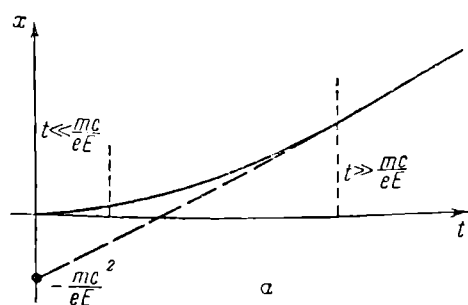


Fig. 81.

After a sufficiently long interval of time ($t \gg mc/|e|E$), the velocity of the particle will approach the velocity of light (even when it is initially small). Then

$$x(t) = ct - \frac{mc^2}{|e|E},$$

$$y(t) = \frac{cp_{0y}}{|e|E} \ln \frac{2|e|Et}{mc}$$

and the motion becomes uniform. The functions $x(t)$ and $y(t)$ are illustrated in Figs. 81a and 81b respectively. The motion for which $p_{0y} = 0$ (cf. Fig. 81a) is usually referred to as hyperbolic.

595. The particle trajectory is given by

$$x = \frac{\mathcal{E}_0}{|e|E} \left(\cosh \frac{|e|E}{cp_{0y}} y - 1 \right) + \frac{cp_{0x}}{|e|E} \sinh \frac{|e|E}{cp_{0y}} y.$$

In the non-relativistic limit $\mathcal{E}_0 = mc^2$, $p_0 \ll mc$, and $|e|Ey/cp_{0y} \ll 1$. The latter result follows from the fact that $|e|E\tau$, which is the momentum given to the particle, should be small in comparison with mc on the non-relativistic approximation. Thus,

$$x = \frac{m|e|Ey^2}{2p_{0y}^2} + \frac{p_{0x}}{p_{0y}} y.$$

596. $l = (\mathcal{E} - mc^2)/eE$.

597. Let the z -axis be parallel to $\mathbf{H}$ and consider the differential equations of motion in the 4-dimensional form†.

$$\frac{d^2x}{d\tau^2} = \omega_1 \frac{dy}{d\tau}, \quad \frac{d^2y}{d\tau^2} = -\omega_1 \frac{dx}{d\tau}, \quad \frac{d^2z}{d\tau^2} = 0, \quad \frac{d^2t}{d\tau^2} = 0,$$

where $\omega_1 = eH/mc$.

† It is possible to start with the 3-dimensional equation $\mathbf{p}/dt = e\mathbf{v} \times \mathbf{H}/c$ by making the substitution $\mathbf{p} = \mathcal{E}\mathbf{v}/c^2$ and using the fact that $\mathcal{E} = \text{const}$ (the magnetic field does no work on the particle).

The first two equations may be conveniently written in the form $d^2u/d\tau^2 + i\omega_1 du/d\tau = 0$, where $u = x + iy$. It follows from the latter equation that

$$ct = \frac{\mathcal{E}_0}{mc} \tau, \quad \mathcal{E}_0 = c \sqrt{p_0^2 + m^2 c^2}, \quad \mathcal{E} = mc^2 \frac{dt}{d\tau} = \mathcal{E}_0.$$

The energy of the particle is independent of time, since forces due to the magnetic field do no work. By integrating the equations for u and z , separating the real and imaginary parts of u , and expressing the proper time τ in terms of t , it is found that

$$\left. \begin{aligned} x &= R_1 \cos(\omega_2 t + \alpha) + \frac{cp_{0y}}{eH} + x_0, \\ y &= -R_1 \sin(\omega_2 t + \alpha) - \frac{cp_{0x}}{eH} + y_0, \\ z &= v_{0z}t. \end{aligned} \right\} \quad (1)$$

It is evident from Eqn. (1) that the particle will travel on a helix. The radius of the helix is $R = |R_1|$ where $R_1 = p_{0\perp} c/eH$ and $p_{0\perp} = (p_{0x}^2 + p_{0y}^2)^{1/2}$. The associated frequency is given by $\omega = |\omega_2|$ where $\omega_2 = eHc/\mathcal{E}$ (the sign of the charge may be negative). The pitch of the helix is given by

$$\frac{2\pi |v_{0z}|}{\omega} = \frac{2\pi \mathcal{E} |v_{0z}|}{|e| Hc},$$

where $v_{0z} = p_{0z} c^2/\mathcal{E}$.

$R = v_{0\perp}/\omega$ where $v_{0\perp} = p_{0\perp} c^2/\mathcal{E}$ is the velocity component perpendicular to the field. At low particle velocities $\mathcal{E} = mc^2$, and

$$R = \frac{mc v_{0\perp}}{|e| H}, \quad \omega = \frac{|e| H}{mc}.$$

The angle α is given by

$$\sin \alpha = -\frac{p_{0x}}{p_{0\perp}}, \quad \cos \alpha = -\frac{p_{0y}}{p_{0\perp}}.$$

$$598. \quad \left. \begin{aligned} x &= a \sin \omega t + \frac{cE_y}{H} t, \\ y &= a (\cos \omega t - 1), \\ z &= \frac{eE_z}{2m} t^2 + v_{0z} t, \end{aligned} \right\} \quad (1)$$

where $a = [v_{0x} - (cE_y/H)]/\omega$.

The motion in the direction of the z -axis takes the form of uniform acceleration under the action of the z component of the electric field. In the x, y plane, the orbit is circular, of radius a , and the centre of the circle moves in the direction perpendicular

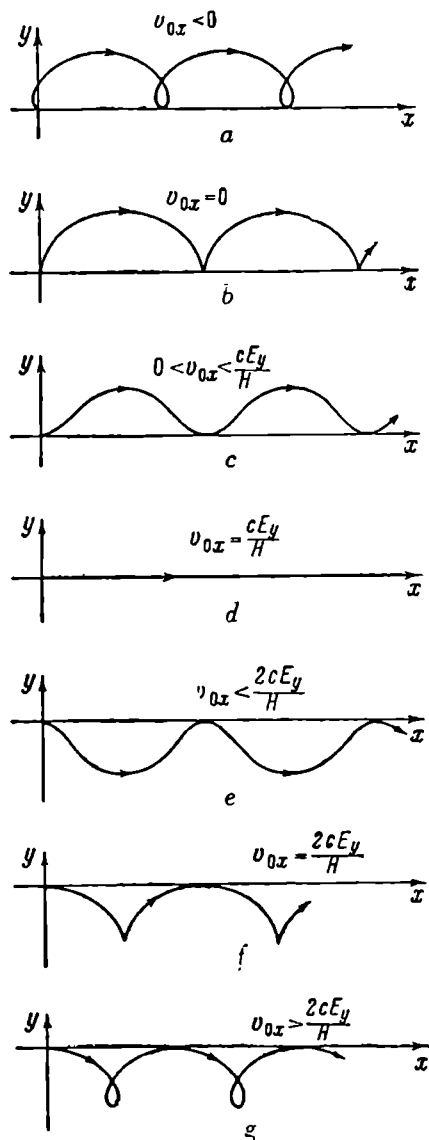


Fig. 82.

to the $(\mathbf{E}, \mathbf{H})$ plane. The drift velocity is $v_{dr} = cE_y/H$. The possible projections of the trajectories on the x, y plane are shown in Fig. 82. The curves shown in Figs. 82a, 82c, 82e, and 82g are trochoids, while those in Figs. 82b and 82f are cycloids. The motion is non-relativistic when $v_0 \ll c$, $E_y/H \ll 1$, and the time t is not too large, *i.e.*

$$t \ll \frac{mc}{eE_z} = \frac{H}{\omega E_z}.$$

599.

$$\begin{aligned} x &= \frac{p_{0x}c}{eH} \sin \kappa H\tau + \frac{p_{0y}c}{eH} (\cos \kappa H\tau - 1), \\ y &= \frac{p_{0x}c}{eH} (\cos \kappa H\tau - 1) + \frac{p_{0y}c}{eH} \sin \kappa H\tau, \\ z &= \frac{\mathcal{E}_0}{eH} (\cosh \kappa E\tau - 1) + \frac{p_{0z}c}{eH} \sinh \kappa E\tau, \\ ct &= \frac{p_{0z}c}{eE} (\cosh \kappa E\tau - 1) + \frac{\mathcal{E}_0}{eE} \sinh \kappa E\tau, \end{aligned}$$

where $\kappa = e/mc$.

600. (a) Let the electric field $\mathbf{E}$ be parallel to the y -axis, and the magnetic field $\mathbf{H}$ parallel to the z -axis (in the S system). At $t = 0$ the particle is at the point $x = y = z = 0$ and has a momentum $\mathbf{p}_0$. The motion is different when $E > H$ and when $H > E$. In the first case, $\mathbf{E} \cdot \mathbf{H} =$

$= 0$, $E^2 - H^2 > 0$, and hence there exists a frame S' in which the magnetic field is zero. It is clear from the Lorentz transformations for the field that S' should move relative to S with velocity $V = cH/E$ along the x -axis (*cf.* Problem 530). The required equations of motion for the particle in S may be obtained from the equations of motion in a uniform electric field $\mathbf{E}'$ with the aid of the Lorentz transformation $x = (x' + Vt')/(1 - V^2/c^2)^{1/2}$, etc. In this method E' , p'_{0x} , p'_{0y} , p'_{0z} should be expressed in terms of the corresponding quantities without primes. The final

result is

$$\left. \begin{aligned} x &= \frac{E(c p_{0x} E - \mathcal{E}_0 H)}{mc(E^2 - H^2)} \tau + \frac{H(\mathcal{E}_0 E - c p_{0x} H)}{e(E^2 - H^2)^{3/2}} \sinh x \sqrt{E^2 - H^2} \tau + \\ &\quad + \frac{c p_{0y} H}{e(E^2 - H^2)} (\cosh x \sqrt{E^2 - H^2} \tau - 1), \\ \text{where } x &= \frac{e}{mc}; \\ y &= \frac{\mathcal{E}_0 E - c p_{0x} H}{e(E^2 - H^2)} (\cosh x \sqrt{E^2 - H^2} \tau - 1) + \\ &\quad + \frac{p_{0y} c}{e \sqrt{E^2 - H^2}} \sinh x \sqrt{E^2 - H^2} \tau; \\ z &= \frac{p_{0z}}{m} \tau; \\ ct &= \frac{H(c p_{0x} E - \mathcal{E}_0 H)}{mc(E^2 - H^2)} \tau + \frac{E(\mathcal{E}_0 E - c p_{0x} H)}{e(E^2 - H^2)^{3/2}} \sinh x \sqrt{E^2 - H^2} \tau + \\ &\quad + \frac{c p_{0y} E}{e(E^2 - H^2)} (\cosh x \sqrt{E^2 - H^2} \tau - 1). \end{aligned} \right\} \quad (1)$$

When $H > E$, the transformation to the system in which there is only a magnetic field leads to the same result as that given by (1), except that E is replaced by H . In carrying out this replacement, it must be remembered that $\sinh i\alpha = i \sin \alpha$ and $\cosh i\alpha = \cos \alpha$.

The solution for $E = H$ may be obtained from the above formulae by letting $E \rightarrow H$. The result is

$$x = \frac{x^2 H^2}{6m} \left(\frac{\mathcal{E}_0}{c} - p_{0x} \right) \tau^3 + \frac{c p_{0y} x^2 H}{2e} \tau^2 + \frac{p_{0x}}{m} \tau$$

and so on. The solution for limit (b) is similar to the solutions used in Problems 594 and 597.

601. $T = mc^2(dt/d\tau - 1)$ and hence, for example, in the system considered in Problem 599, we have

$$T = \mathcal{E}_0 \cosh x E \tau + c p_{0z} \sinh x E \tau - mc^2.$$

602. Using the result obtained in Problem 530, and evaluating V/c to within terms of the first order in E/H , we find that $V/c = E_y/H$. The solution is similar to that for Problem 600. In all the calculations, second and higher orders of E_y/H , E_z/H , and v_0/c should be neglected. The final result is

$$\left. \begin{aligned} x &= a \sin \omega t + \frac{v_{0y}}{\omega} (\cos \omega t - 1) + c \frac{E_y}{H} t, \\ y &= a (\cos \omega t - 1) + \frac{v_{0y}}{\omega} \sin \omega t, \\ z &= \frac{e E_z t^2}{2m} + v_{0z} t, \end{aligned} \right\} \quad (1)$$

where

$$a = \frac{v_0 x - \frac{c E_y}{H}}{\omega} \quad \text{and} \quad \omega = \frac{e H}{m c}.$$

At $t = 0$ the particle is at the point $x = y = z = 0$. The formulae given by (1) contain the solution of Problem 598 as a special case.

603. Let the x -axis be parallel to the direction of propagation of the plane wave. The wave field can then be described by two functions of t' , for example, $E_y(t')$ and $E_z(t')$:

$$E = [0, E_y(t'), E_z(t')], \quad H = [0, -E_z(t'), E_y(t')].$$

Using (XI.15) we find that $t' = \tau$ and hence the parametric equations of motion are

$$\begin{aligned} x(\tau) &= \frac{1}{2m^2c} \int_0^\tau p_\perp^2 d\tau, & y(\tau) &= \frac{1}{m} \int_0^\tau p_y d\tau, \\ z(\tau) &= \frac{1}{m} \int_0^\tau p_z d\tau, & t(\tau) &= \tau + \frac{1}{2m^2c^2} \int_0^\tau p_\perp^2 d\tau, \end{aligned}$$

where $\mathbf{p}_\perp = e \int_0^\tau \mathbf{E}(t') dt' = \mathbf{e}_y p_y + \mathbf{e}_z p_z$ is the component of the momentum of the particle in the $\mathbf{E}, \mathbf{H}$ plane.

604. The coordinates of the particle are given by

$$x = x_0 \cos \omega t, \quad y = y_0 \cosh \omega t, \quad z = vt,$$

where $\omega^2 = 2ek/m$.

It follows from the form of $x(t)$ and $y(t)$ that this type of lens may be used to produce a charged particle beam in the form of a flat ribbon.

605.

$$\begin{aligned} \frac{d}{dt} \left(\frac{m\dot{r}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) &= \frac{m r \dot{\alpha}^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e E_r + \frac{e}{c} (-H_\alpha \dot{z} + H_z r \dot{\alpha}), \\ \frac{d}{dt} \left(\frac{m r^2 \dot{\alpha}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) &= e \left[E_\alpha + \frac{1}{c} (H_\alpha \dot{z} - H_z \dot{r}) \right] r, \\ \frac{d}{dt} \left(\frac{m \dot{z}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) &= e \left[E_z + \frac{1}{c} (H_\alpha \dot{r} - H_r r \dot{\alpha}) \right]. \end{aligned}$$

The first and last of these equations have the form of the usual Newtonian laws of motion, except that they contain the variable mass $m/(1 - v^2/c^2)^{1/2}$. Moreover, the right-hand side of the first equation contains the term $mr\dot{\alpha}^2/(1 - v^2/c^2)^{1/2}$ which is independent of the form of the electromagnetic forces (centripetal force). The second equation gives the time derivative of the angular momentum of the particle about the z -axis in terms of the z component of the moment of the Lorentz force.

606. When $H = 0$, the electron trajectories are rectilinear. As the magnetic field increases, the trajectories become more and more curved in the plane perpendicular to the axis. Consider the cylindrical coordinates r, α, z , where the z -axis lies along the axis of the cylinder. The electrons will not reach the anode if at $r = b$ their velocity is parallel to the surface of the anode, *i.e.* when $\dot{r}|_{r=b} = 0$. Moreover, $\dot{\alpha}|_{r=b} = v_{\max}/b$. Let us use the second of the equations given in the solution to Problem 605, which in the present situation assumes the form

$$\frac{d}{dt} \left(\frac{mr^2\dot{\alpha}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = -\frac{e}{c} H(r) r \frac{dr}{dt}. \quad (1)$$

Integrating (1) along the particle trajectory between $r = a$ and $r = b$, we have

$$\left. \frac{mr^2\dot{\alpha}}{\sqrt{1 - \frac{v^2}{c^2}}} \right|_{r=a}^{r=b} = -\frac{e}{2\pi c} \int_a^b 2\pi H r dr = -\frac{e\Phi}{2\pi c}.$$

Hence

$$\Phi_{\text{crit}} = \frac{2\pi cb}{|e|} p_{\max} = 2\pi cb \sqrt{\frac{2mV}{|e|} \left(1 + \frac{|e|V}{2mc^2} \right)}, \quad (2)$$

where we have used the result of Problem 546 and the fact that $T_{\max} = |e|V$.

When the potential difference is small, so that $|e|V \ll mc^2$, which is equivalent to saying that $v \ll c$, Eqn. (2) assumes the simpler form

$$\Phi_{\text{crit}} = 2\pi cb \sqrt{\frac{2mV}{|e|}}. \quad (3)$$

607. The potential difference V should be larger than

$$V_{\text{crit}} = \sqrt{\frac{4\mathcal{G}^2}{c^2} \ln^2 \frac{b}{a} + \frac{m^2 c^4}{e^2}} - \frac{mc^2}{|e|}.$$

When $|e|V \ll mc^2$ (non-relativistic electrons), we have

$$V_{\text{crit}} = \frac{2\mathcal{G}^2 |e|}{mc^4} \ln^2 \frac{b}{a}.$$

608. $b = a \exp(p_0 c^2 / \mathcal{G} |e|)$ where $p_0 = mv_0 / (1 - v_0^2/c^2)^{1/2}$.

610. Consider the cylindrical coordinates r, α whose origin coincides with the charge Ze and whose polar axis is parallel to the angular momentum of the particle. The motion will then take place in the $z = 0$ plane and r will be the distance between the charges $-e$ and Ze . The first two equations in the solution to Problem 605 will then be

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{mr\dot{r}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) &= \frac{mr\dot{\alpha}^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{Ze^2}{r^2}, \\ \frac{d}{dt} \left(\frac{mr^2\dot{\alpha}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) &= 0. \end{aligned} \right\} \quad (1)$$

It follows from the second equation that the angular momentum is an integral of the motion, so that

$$\frac{mr^2\dot{\alpha}}{\sqrt{1 - \frac{v^2}{c^2}}} = K = \text{const.} \quad (2)$$

The second integral of the motion is the total energy of the system:

$$\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{Ze^2}{r} = \mathcal{E} = \text{const.} \quad (3)$$

It follows from (3) that there are two main types of trajectories. At large values of r the total energy is $\mathcal{E} = mc^2 + T$ (T is the kinetic energy), since as $r \rightarrow \infty$, the potential energy $Ze^2/r \rightarrow 0$. Since $T \geq 0$, it follows that when $E < mc^2$ the particle cannot reach large distances from the attracting centre, and its trajectory will lie in a finite region (finite motion). When $E > mc^2$, there will be trajectories extending to infinity (infinite motion).

Let us now find the differential equation for the particle trajectory. It follows from (2) that

$$\frac{d}{dt} = \frac{K \sqrt{1 - \frac{v^2}{c^2}}}{mr^2} \frac{d}{d\alpha}. \quad (4)$$

Substituting (3) and (4) into the first equation in (1) we obtain the required differential equation:

$$\frac{d^2}{d\alpha^2} \left(\frac{1}{r} \right) + (1 - \rho^2) \frac{1}{r} = \frac{Ze^2 \mathcal{E}}{K^2 c^2}, \quad (5)$$

where $\rho = Ze^2/Kc$.

When $\rho \neq 1$, the integration of this equation yields

$$r = \frac{p}{1 + \epsilon \cos \sqrt{1 - \rho^2} \alpha}, \quad p = \frac{K^2 c^2 - Z^2 e^4}{Ze^2 \mathcal{E}}, \quad (6)$$

where ϵ is the integration constant. The second integration constant may be eliminated by a suitable choice of the origin of the angle α , and the quantity ϵ may be expressed in terms of $\mathcal{E}$ and K . The trajectories are symmetric with respect to the x -axis ($\alpha = 0$).

Consider the case $\rho < 1$ in greater detail. As can be seen from (6), the particle will not approach the centre to distances smaller than $r_{\min} = p/(1 + \epsilon)$ if it is assumed that $\epsilon > 0$. In

Eqn. (6) the origin for the angle α is chosen so that $r = r_{\min}$ when $\alpha = 0$. The particle can therefore reach the distance $r_{\min}$ from the centre several times. At all such points $\dot{r} = 0$ and the velocity is perpendicular to the radius vector $\mathbf{r}$. Hence, $K = mvr_{\min}/(1 - v^2/c^2)^{1/2}$. By eliminating v from this expression and from

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{Ze^2}{r_{\min}}$$

and expressing $r_{\min}$ in terms of ϵ it is found that

$$\epsilon = \frac{1}{\rho} \sqrt{1 - \frac{m^2 c^4}{\mathcal{E}^2} (1 - \rho^2)}. \quad (7)$$

It follows from (7) that $\epsilon < 1$ when $\mathcal{E} < mc^2$. The motion is then finite, and the trajectory is quasi-ellipsoidal (Fig. 83). It takes the form of a rosette lying

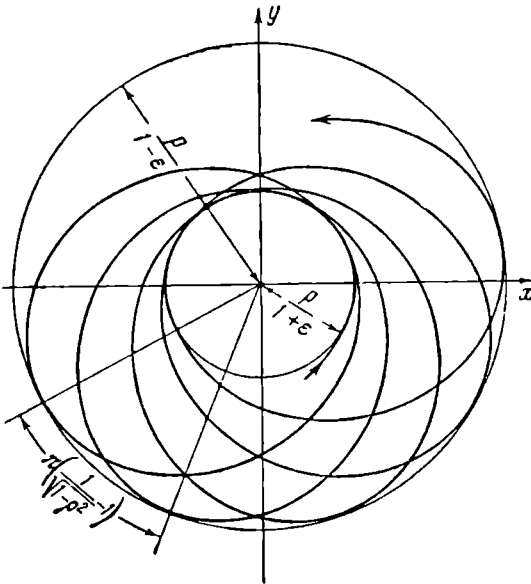


Fig. 83.

between circles of radii $p/(1 + \epsilon)$ and $p/(1 - \epsilon)$. The rosette may be obtained by rotating the non-relativistic elliptical trajectory in its own plane. A complete oscillation of r from the minimum value $r_{\min} = p/(1 + \epsilon)$ (perigee) to the maximum value $r_{\max} = p/(1 - \epsilon)$ (apogee) and back to a new minimum, occurs when α increases by $2\pi/(1 - \rho^2)^{1/2}$. The perigee of the orbit is thus rotated through an angle $2\pi[(1 - \rho^2)^{-1/2} - 1]$ in each period of r .

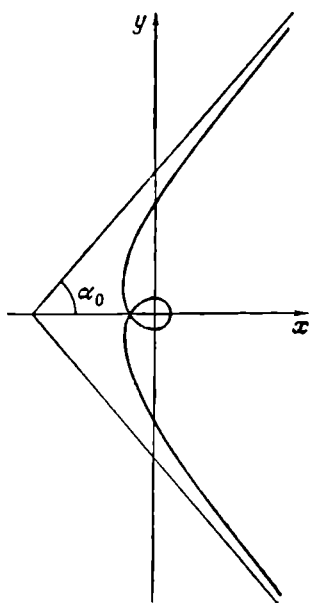


Fig. 84.

If $(1 - \rho^2)^{1/2}$ is a rational number, then the trajectory will close after a finite number of revolutions.

$\epsilon > 1$ when $\mathcal{E} > mc^2$. The motion is infinite and the trajectory is quasi-hyperbolic (Fig. 84). It has two branches extending to infinity at $\alpha = \pm \alpha_0$, where $\alpha_0 = \cos^{-1}(-1/\epsilon)/(1 - \rho^2)^{1/2}$. A particle approaching the charge Ze along one of these branches may execute a number of revolutions about the charge before it leaves it to escape to infinity along the other branch.

$\epsilon = 1$ corresponds to $\mathcal{E} = mc^2$.

Here, the motion is again infinite

and the trajectory is quasi-parabolic.

When $\rho \ll 1$, the above trajectories become elliptical ($\epsilon < 1$), hyperbolic ($\epsilon > 1$), and parabolic ($\epsilon = 1$), and correspond to the usual non-relativistic Keplerian solutions. This is inevitable, since $v/c \ll 1$ when $\rho \ll 1$ †.

611. When $\rho > 1$, the solution of Eqn. (5) in the preceding problem can conveniently be rewritten as

$$r = \frac{p_1}{-1 + \epsilon_1 \cosh \sqrt{\rho^2 - 1} \alpha}, \quad (1)$$

where

$$p_1 = \frac{-K^2 c^2 + Z^2 e^4}{Z e^2 \mathcal{E}},$$

$$\epsilon_1 = \sqrt{\frac{1}{\rho^2} + \frac{m^2 c^4}{\mathcal{E}^2} \left(1 - \frac{1}{\rho^2}\right)}.$$

† In the non-relativistic limit

$$\rho = \frac{Ze^2}{Kc} \sim \frac{Ze^2}{rmvc} \sim \frac{|U|}{mvc}.$$

According to the virial theorem $|U| = 2T \approx mv^2$, so that $\rho \approx v/c \ll 1$.

The trajectories described by (1) become spirals which pass through the origin when $\alpha \rightarrow \pm\infty$. The particle passes through the centre of force (in the non-relativistic limit this is only possible when $K = 0$, $\rho = \infty$). When $\mathcal{E} > mc^2$ we have $\varepsilon_1 < 1$ and the trajectory has two branches which reach out to infinity at $\alpha = \pm\alpha_0$, where $\alpha_0 = \cosh^{-1}(1/\varepsilon_1)/(\rho^2 - 1)^{1/2}$ (Fig. 85). When $\mathcal{E} < mc^2$ we have $\varepsilon_1 > 1$ and the trajectory is as shown in Fig. 86.

When $\rho = 1$, the solution given by (1) cannot be used, and the differential equation for the trajectory must be integrated again.

The result of this integration is

$$r = \frac{2Ze^2\mathcal{E}}{\mathcal{E}^2(\alpha^2 - 1) + m^2c^4}. \quad (3)$$

The trajectory is then also a spiral passing through the centre when $\alpha \rightarrow \pm\infty$, although the centre is approached much more slowly as compared with the case $\rho > 1$.

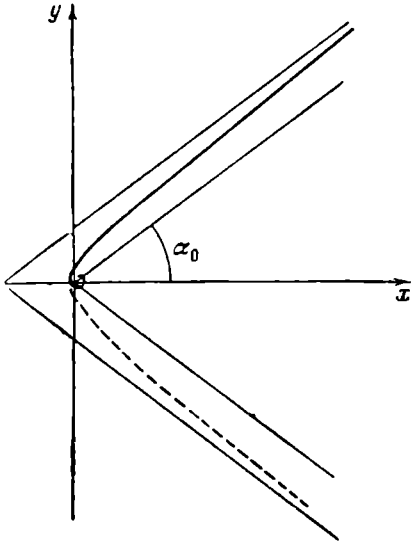


Fig. 85.

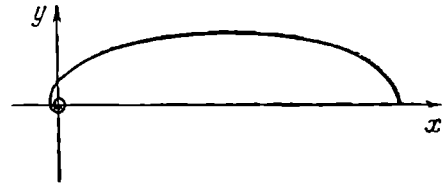


Fig. 86.

The general nature of the trajectory is similar to that illustrated in Figs. 85 and 86.

612. When $Ze^2/Kc < 1$,

$$r = \frac{p}{-1 + \varepsilon \cos \alpha \sqrt{1 - \frac{Z^2e^4}{K^2c^2}}},$$

where

$$p = -\frac{Z^2e^4 - K^2c^2}{Ze^2\mathcal{E}}, \quad \varepsilon = \frac{c}{Ze^2\mathcal{E}} \sqrt{K^2\mathcal{E}^2 - m^2c^2(K^2c^2 - Z^2e^4)} > 1.$$

The trajectory is quasi-hyperbolic (Fig. 87). Its two branches reach infinity at $\alpha = \pm\alpha_0$, where

$$\alpha_0 = \frac{1}{\sqrt{1 - \frac{Z^2e^4}{K^2c^2}}} \cos^{-1} \frac{1}{\varepsilon}.$$

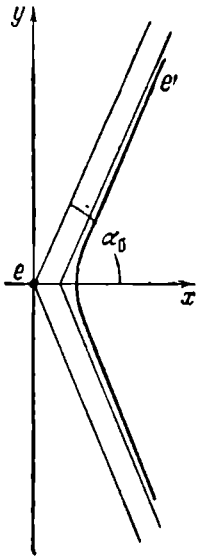


Fig. 87.

When $Ze^2/Kc \ll 1$, the particle moves on a hyperbola. This corresponds to non-relativistic motion with $v \ll c$ (cf. footnote to the solution of Problem 610).

When $Ze^2/Kc > 1$,

$$r = - \frac{p}{1 - \varepsilon \cosh \alpha \sqrt{\frac{Z^2 e^4}{K^2 c^2} - 1}},$$

where $\varepsilon < 1$. The general character of the trajectory is similar to that in the previous example.

The two branches reach infinity at

$$\alpha = \pm \frac{1}{\sqrt{\frac{Z^2 e^4}{K^2 c^2} - 1}} \cosh^{-1} \frac{1}{\varepsilon}.$$

When $Ze^2/Kc = 1$,

$$r = \frac{2Ze^2\mathcal{E}}{\mathcal{E}^2(1 - \alpha^2) - m^2c^4}.$$

The two branches of the trajectory reach infinity when

$$\alpha = \frac{\sqrt{\mathcal{E}^2 - m^2c^4}}{\mathcal{E}}.$$

614. When $ee' < 0$ (attraction)

$$r = \frac{a|\varepsilon^2 - 1|}{1 + \varepsilon \cos \alpha},$$

where

$$a = \left| \frac{ee'}{2\mathcal{E}} \right|, \quad \varepsilon = \sqrt{1 + \frac{2\mathcal{E}K^2}{\mu e^2 e'^2}}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2},$$

$K = \mu r^2 \dot{\alpha}$ is the angular momentum, $\mathcal{E} = ee'/r + \mu v^2/2$ is the total energy of the particle, and r, α are the polar coordinates. The particle trajectory is a conic section: it is an ellipse ($\varepsilon < 1$) when $\mathcal{E} < 0$, a hyperbola with the charge e' at one of its foci ($\varepsilon > 1$) when $\mathcal{E} > 0$, and a parabola ($\varepsilon = 1$) when $\mathcal{E} = 0$.

When $ee' > 0$ (repulsion)

$$r = \frac{a(\varepsilon^2 - 1)}{-1 + \varepsilon \cos \alpha}.$$

In this case, $\mathcal{E} \geq 0$ and $\varepsilon \geq 1$. When $\mathcal{E} > 0$ ($\varepsilon > 1$), the trajectory

is a hyperbola with the charge e' at the external focus, or a parabola when $\mathcal{E} = 0$ ($\epsilon = 1$).

615. The differential scattering cross-section may be calculated from the formula

$$\sigma(\theta) = \frac{sd s}{\sin \theta d \theta}, \quad (1)$$

where θ is the angle of scattering corresponding to a given value of the impact parameter s . The relation between s and θ may be found from the equation for the trajectory of the particle (cf. Problem 614). In the case of attraction ($ee' < 0$), $\cos \alpha > -\epsilon^{-1}$. The angle α varies between $-\alpha_0$ and α_0 (Fig. 88) as the particle traverses its orbit ($\cos \alpha_0 = -\epsilon^{-1}$). The scattering angle θ is the complement of the angle between the asymptotes of the hyperbolic trajectory. It follows from Fig. 88 that $\theta/2 = -\pi/2 + \alpha_0$ and hence $\cotan^2(\theta/2) = \operatorname{cosec}^2(\theta/2) - 1 = \sec^2 \alpha_0 - 1 = \epsilon^2 - 1 = 2\mathcal{E}K^2/me^2e'^2$. The angular momentum is $K = mv_0s$, so that

$$s^2 = \frac{e^2e'^2}{m^2v_0^4} \cotan^2 \frac{\theta}{2}.$$

Differentiation and substitution into (1) yields

$$\sigma(\theta) = \left(\frac{ee'}{2mv_0^2} \right)^2 \operatorname{cosec}^4(\theta/2).$$

This is the well-known Rutherford formula. The same result is obtained when $ee' > 0$.

616. For $ee' < 0$ (attraction),

$$\theta = \left(\frac{2cK}{\sqrt{c^2K^2 - Z^2e^4}} - 1 \right) \pi - \frac{2cK}{\sqrt{c^2K^2 - Z^2e^4}} \tan^{-1} \frac{v_0 \sqrt{c^2K^2 - Z^2e^4}}{cZe^2},$$

where v_0 is the velocity of the charge when $r \rightarrow \infty$.

When $ee' > 0$ (repulsion)

$$\theta = \pi - \frac{2cK}{\sqrt{c^2K^2 - Z^2e^4}} \tan^{-1} \frac{v_0 \sqrt{c^2K^2 - Z^2e^4}}{cZe^2}.$$

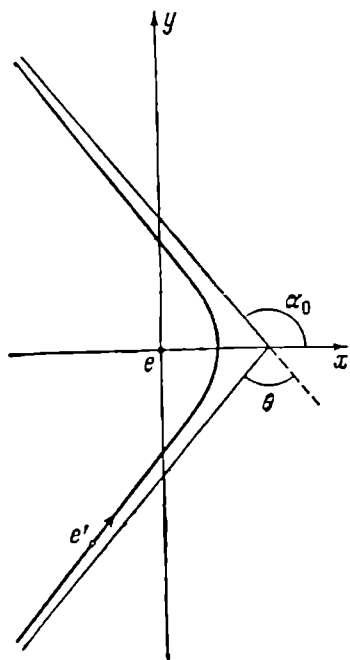


Fig. 88.

617. Large impact parameters s correspond to small angles of scattering. Hence, by putting $K = p_0 s$, where p_0 is the momentum at $r \rightarrow \infty$, it is possible to find the required dependence of the angle θ on s by letting $s \rightarrow \infty$ (evidently, $K > |ee'|/c$) in the general formulae derived in the preceding problem. On passing to the limit, the same result is obtained both for $ee' < 0$ and $ee' > 0$, namely,

$$\theta = \pi - 2 \tan^{-1} \frac{v_0 p_0 s}{|ee'|} = \frac{2Ze^2}{v_0 p_0 s} \ll 1,$$

and hence $s = 2|ee'|/v_0 p_0 \theta$ and

$$\sigma(\theta) = \frac{s ds}{\theta d\theta} = 4 \left(\frac{ee'}{v_0 p_0} \right)^2 \frac{1}{\theta^4}.$$

$$618. \quad x = vt = \frac{1}{3} \sqrt{\frac{2ma^3}{k}} = \frac{2}{3} \sqrt{\frac{\epsilon + 1}{\epsilon - 1} \cdot \frac{ma^3}{e^2}}.$$

619. The accelerating electric field is given by

$$E_a = \frac{1}{2\pi r c} \frac{d\Phi}{dt},$$

where r is the radius of the orbit of the electron, Φ is the magnetic flux intercepted by the orbit, and α is the azimuth angle of the electron.

When the electron travels along the orbit through a distance $r d\alpha$, the work done by the field E_a is

$$\delta A = E_a r d\alpha. \quad (1)$$

The acceleration of the electron takes place on an orbit of constant radius $r = cp/eH_0$ (cf. Problem 597), where H_0 is the magnetic field at the orbit. This field is perpendicular to the plane of the orbit and increases with time. From the condition $dr = 0$ we find that

$$dp = \frac{p}{H_0} dH_0. \quad (2)$$

The electron energy $\mathcal{E} = c(p^2 + m^2 c^2)^{1/2}$ is increased by

$$d\mathcal{E} = \frac{c^2 p dp}{\mathcal{E}} = \frac{c^2 p^2 dH_0}{\mathcal{E} H_0}, \quad (3)$$

in which Eqn. (2) has been used. It follows that

$$\delta A = d\mathcal{E}. \quad (4)$$

Substituting (1) and (3) into (4), and using the equation $c^2 p/\mathcal{E} = v = r d\alpha/dt$, we have, after integration,

$$\Phi = 2\Phi_0,$$

where $\Phi_0 = \pi r^2 H_0$.

The latter result is the required 2:1 rule.

620. The energy of interaction U between the two charged particles is given by (XI.19) into which the charge e_1 of one of the particles and the retarded potentials φ_2 , $\mathbf{A}_2$ for the other particle should be substituted. From the explanations put forward in Problem 658, we have

$$\varphi_2 = \frac{e_2}{R} + \frac{e_2}{2c^2} \frac{\partial^2 R}{\partial t^2}, \quad \mathbf{A}_2 = \frac{e_2 \mathbf{v}_2}{cR}, \quad (1)$$

where R is the distance between the particles. Writing χ in the form

$$\chi = \frac{e_2}{2c} \frac{\partial R}{\partial t},$$

and carrying out the gradient transformation of the potentials, it is found that the new potentials are

$$\varphi'_2 = \varphi_2 - \frac{1}{c} \frac{\partial \chi}{\partial t} = \frac{e_2}{R},$$

$$\mathbf{A}'_2 = \mathbf{A}_2 + \text{grad } \chi = \frac{e_2 [\mathbf{v}_2 + (\mathbf{n} \cdot \mathbf{v}_2) \mathbf{n}]}{2cR},$$

where $\mathbf{n} = \mathbf{R}/R$. Hence, the energy of interaction is given by the Breit formula

$$U = e_1 \varphi_2 - \frac{e_1}{c} (\mathbf{v}_1 \cdot \mathbf{A}_2) = \frac{e_1 e_2}{R} \left\{ 1 - \frac{1}{2c^2} [\mathbf{v}_1 \cdot \mathbf{v}_2 + (\mathbf{v}_1 \cdot \mathbf{n})(\mathbf{v}_2 \cdot \mathbf{n})] \right\}.$$

This formula gives an approximate representation of the fact that the force acting on one of the two interacting charged particles at a distance R from one another is determined by the previous position and state of motion of the other charge. Energy and momentum are communicated by the charges to the field and are transported by the field from one charge to the other in the time interval R/c . The particles and the field form a single system, and hence an exact description of the motion of a system of interacting particles cannot be given without considering the degrees of freedom of the field.

$$621. \quad L = \frac{m_1 v_1^2}{2} + \frac{m_1 v_1^4}{8c^2} + \frac{m_2 v_2^2}{2} + \frac{m_2 v_2^4}{8c^2} + \\ + \frac{e_1 e_2}{R} - \frac{e_1 e_2}{2c^2 R} [v_1 \cdot v_2 + (v_1 \cdot n)(v_2 \cdot n)].$$

622. The magnetic moment of the particle precesses about the direction of the magnetic field with an angular frequency $\omega = -\chi H$.

623. The electric dipole moment of the particle in the laboratory system is $\mathbf{p} = (\mathbf{v}/c) \times \mathbf{m}$, and hence in an electric field $\mathbf{E} = -(\mathbf{r}/r)d\varphi/dr$ the couple acting on the particle is $\mathbf{N} = \mathbf{p} \times \mathbf{E}$. If the resultant force acting on $\mathbf{p}$ is neglected, the vector $\mathbf{E}$ in the latter formula may be replaced by $(m/e)\dot{\mathbf{v}}$.

The mean value of the couple is given by

$$\bar{\mathbf{N}} = \frac{m}{e} \overline{\mathbf{p} \times \dot{\mathbf{v}}} = \overline{\frac{m}{ec} (\mathbf{v} \times \mathbf{m}) \times \dot{\mathbf{v}}}. \quad (1)$$

Differentiating the product $(\mathbf{v} \times \mathbf{m}) \times \mathbf{v}$ with respect to time, and using the fact that the average of the time derivative of a periodic function is zero, and also the fact that the term with $\dot{\mathbf{m}} = (e/2mc)\mathbf{p} \times \mathbf{E}$ is smaller than the terms with $\dot{\mathbf{v}}$, we have

$$\overline{(\mathbf{v} \times \mathbf{m}) \times \dot{\mathbf{v}}} + \overline{(\dot{\mathbf{v}} \times \mathbf{m}) \times \mathbf{v}} = 0.$$

Hence, using the identity

$$(\mathbf{v} \times \mathbf{m}) \times \dot{\mathbf{v}} + (\mathbf{m} \times \dot{\mathbf{v}}) \times \mathbf{v} + (\dot{\mathbf{v}} \times \mathbf{v}) \times \mathbf{m} = 0,$$

the average couple is found to be

$$\bar{\mathbf{N}} = \frac{1}{2} \mathbf{m} \times \bar{\mathbf{H}}', \quad (2)$$

where $\mathbf{H}' = -(\mathbf{v} \times \mathbf{E})/c$ is the magnetic field in the system in which the electron is instantaneously at rest.

The field $\mathbf{H}'$ may be expressed in terms of the angular momentum $\mathbf{l}$ associated with the orbital motion of the particle about the centre of force:

$$\mathbf{H}' = -\frac{1}{c} \mathbf{v} \times \mathbf{E} = \frac{1}{mcr} \frac{d\varphi}{dr} m\mathbf{v} \times \mathbf{r} = \frac{1}{mcr} \frac{d\varphi}{dr} \mathbf{l}. \quad (3)$$

The interaction of the magnetic moment of the particle with the field, which gives rise to the average couple given by (2), may be described in terms of the potential energy:

$$U = -\frac{1}{2} \mathbf{m} \cdot \bar{\mathbf{H}}' = -\frac{e}{2m^2c^2} \cdot \frac{1}{r} \frac{d\varphi}{dr} \mathbf{l} \cdot \mathbf{s}. \quad (4)$$

We note that this expression can also be used to describe the interaction of the electron spin with the orbital motion of the electron in the quantum-mechanical description of the atom.

624. The motion of $\mathbf{s}$ is described by

$$\overline{\frac{d\mathbf{s}}{dt}} = \frac{e}{2m^2c^2} \cdot \overline{\frac{1}{r} \frac{d\varphi}{dr}} \mathbf{s} \times \mathbf{j}, \quad (1)$$

where $\mathbf{j} = \mathbf{l} + \mathbf{s}$ is the total angular momentum. It follows from (1) that $\mathbf{s}$ will, on the average, precess about the direction of $\mathbf{j}$ with an angular velocity

$$\omega = \frac{e}{2m^2c^2} \cdot \overline{\frac{1}{r} \frac{d\varphi}{dr}} \mathbf{j}.$$

625. A reflection will occur for an anti-parallel orientation of the magnetic moment and the field, provided the glancing angle α is such that $\sin \alpha \leq (m_0 H / \mathcal{E})^{1/2}$.

626. The motion of the neutron along the conductor is uniform. The motion in the plane perpendicular to the conductor occurs in the potential $U = \pm 2m_0 \mathcal{J} / cr$. It follows that the projections of the neutron trajectories on this plane are similar to the trajectories of the uniform motion of two charges e and e' which interact in accordance with Coulomb's law (*cf.* Problem 614), provided ee' is replaced by $\pm 2m_0 \mathcal{J} / c$ and $\mathcal{E} = m\dot{r}^2/2 + K^2/2mr^2 + U(r)$ is regarded as the energy of the transverse motion ($K = m r^2 \dot{\alpha}$ is the angular momentum). In particular, when $\mathcal{E} < 0$ the neutron executes a finite motion about the conductor.

627.

$$l(\alpha) = \frac{m_0 \mathcal{J}}{cmv_0^2 \sin^2 \frac{\alpha}{2}}.$$

Chapter 12

EMISSION OF ELECTROMAGNETIC WAVES

1. The Hertz vector and the multipole expansion

630. $\Delta\varphi = -4\pi\rho,$

$$\Delta\mathbf{A} - \frac{1}{c^2} \cdot \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c} \operatorname{grad} \frac{\partial\varphi}{\partial t} - \frac{4\pi}{c} \mathbf{j}.$$

632. The angular momentum flux density is given by

$$\mathfrak{H} = \frac{(\mathbf{n} \times \ddot{\mathbf{p}})(\mathbf{n} \cdot \dot{\mathbf{p}})}{2\pi c^3 r^2}.$$

In evaluating the quantity $-d\mathbf{K}/dt = \int \mathfrak{H} r^2 d\Omega$ it is convenient to use the formula $\overline{n_i n_k} = \frac{1}{3} \delta_{ik}$ (cf. Chapter I). The result is

$$-\frac{d\mathbf{K}(t)}{dt} = \frac{2}{3c^2} \dot{\mathbf{p}} \times \ddot{\mathbf{p}} \Big|_{t'=t-\frac{r}{c}}.$$

633. The magnetic lines of force are in the form of circles whose planes are perpendicular to the z -axis, and whose centres lie on this axis. The electric lines of force are described by the following equations:

$$C_1 = \sin^2 \vartheta \left[\frac{1}{r} \cos(kr - \omega t) + k \sin(kr - \omega t) \right], \quad C_2 = \alpha,$$

where C_1 and C_2 are constants.

634.

$$\begin{aligned} \mathbf{H} &= \frac{1}{c} \cdot \frac{\partial \operatorname{curl} \mathbf{Z}}{\partial t} = \\ &= ea \left[\mathbf{e}_\vartheta \left(-i \frac{\omega^2}{c^2 r} + \frac{\omega}{cr^2} \right) + \mathbf{e}_\alpha \left(\frac{\omega^2}{c^2 r} + i \frac{\omega}{cr^2} \right) \cos \vartheta \right] e^{i(kr - \omega t + \alpha)}, \\ \mathbf{E} &= \operatorname{curl} \operatorname{curl} \mathbf{Z} = ea \left[\mathbf{e}_r \left(-\frac{i\omega}{cr^2} + \frac{1}{r^3} \right) 2 \sin \vartheta + \right. \\ &\quad \left. + \mathbf{e}_\vartheta \left(\frac{\omega^2}{c^2 r} + i \frac{\omega}{cr^2} - \frac{1}{r^3} \right) \cos \vartheta + \mathbf{e}_\alpha \left(i \frac{\omega^2}{c^2 r} - \frac{\omega}{cr^2} - \frac{i}{r^3} \right) \right] e^{i(kr - \omega t + \alpha)}. \end{aligned}$$

In the wave zone $r \gg \lambda = 2\pi c/\omega$ and the expressions for $\mathbf{E}$ and $\mathbf{H}$

assume the simpler form:

$$\mathbf{H} = ea \frac{\omega^2}{c^2 r} (-l \mathbf{e}_\vartheta + \mathbf{e}_\alpha \cos \vartheta) e^{l(kr - \omega t + \alpha)},$$

$$\mathbf{E} = ea \frac{\omega^2}{c^2 r} (\mathbf{e}_\vartheta \cos \vartheta + l \mathbf{e}_\alpha) e^{l(kr - \omega t + \alpha)} = \mathbf{H} \times \mathbf{n}.$$

When the emission takes place into the upper hemisphere ($\cos \vartheta > 0$), the polarisation is in general elliptical and left handed (it becomes circular when $\vartheta = 0$). When the radiation is emitted into the lower hemisphere ($\cos \vartheta < 0$), the elliptical polarisation is right handed, and the special case of circular polarisation occurs at $\vartheta = \pi$. Waves emitted in the equatorial plane are linearly polarised. The angular distribution and the total intensity are respectively given by

$$\frac{dI}{d\Omega} = \bar{\gamma} \cdot \mathbf{n} r^2 = \frac{e^2 \omega^4 a^2}{8\pi c^3} (1 + \cos^2 \vartheta), \quad \bar{I} = \frac{2\omega^4 e^2 a^2}{3c^3}.$$

The above situation obtains, for example, in the case of a charge moving in a magnetic field

635.

$$\mathbf{p} = \mathbf{m} = 0, \quad \mathbf{Q} \neq 0,$$

$$\begin{aligned} \mathbf{H} &= \frac{1}{c} \dot{\mathbf{A}} \times \mathbf{n} = \\ &= -\frac{4ea^2\omega^3}{c^3 r} \sin \vartheta [\mathbf{e}_\vartheta \cos(2\omega t' - 2\alpha) + \mathbf{e}_\alpha \cos \vartheta \sin(2\omega t' - 2\alpha)]. \end{aligned}$$

The frequency of the oscillations in the charge and current distributions, and therefore the frequency of the field, is equal to twice the orbital frequency ω . In general, the polarisation is elliptical, and becomes circular as $\vartheta \rightarrow 0, \pi$ and linear when $\vartheta = \pi/2$.

$$\begin{aligned} \frac{dI}{d\Omega} &= \frac{2e^2 a^4 \omega^6}{\pi c^5} \sin^2 \vartheta (1 + \cos^2 \vartheta), \\ \bar{I} &= \frac{32}{5} \cdot \frac{e^2 a^4 \omega^6}{c^5}. \end{aligned}$$

When one of the charges is removed, the intensity increases by a factor of the order of $(\lambda/a)^2$, *i.e.* very considerably, since the condition $a/\lambda \ll 1$ is satisfied.

636. If the angle between the position vectors of the charges is equal to $\pi - \varphi$, then

$$\varphi = \sqrt{\frac{12}{5}} \cdot \frac{a\omega}{c}.$$

637. Let the dipole moment of the oscillator which leads in phase lie along the x -axis, and let the xy plane contain the dipole moments of both oscillators. The magnetic field is then given by

$$\begin{aligned} \mathbf{H}(r, t) &= \mathbf{H}e^{-i\omega t'} = \frac{\omega^2 p}{c^2 r} \{ \mathbf{e}_\vartheta [\sin \alpha + l \sin(\alpha - \varphi)] + \\ &\quad + \mathbf{e}_\alpha [\cos \alpha + l \cos(\alpha - \varphi)] \cos \vartheta \} e^{-i\omega t'}, \quad (1) \\ \frac{dI}{d\Omega} &= \frac{p^2 \omega^4}{8\pi c^3} \{ 2 - [\cos^2 \alpha + \cos^2(\alpha - \varphi)] \sin^2 \vartheta \}, \\ \bar{I} &= \frac{2p^2 \omega^4}{3c^3}. \end{aligned}$$

where the unit vector $\mathbf{n}$ in the direction of propagation of the wave is defined by the polar angles $\vartheta = 0, \pi$.

The angular distribution is a maximum in the directions $\vartheta = 0$ and $\vartheta = \pi$, which are perpendicular to the dipole moments. The

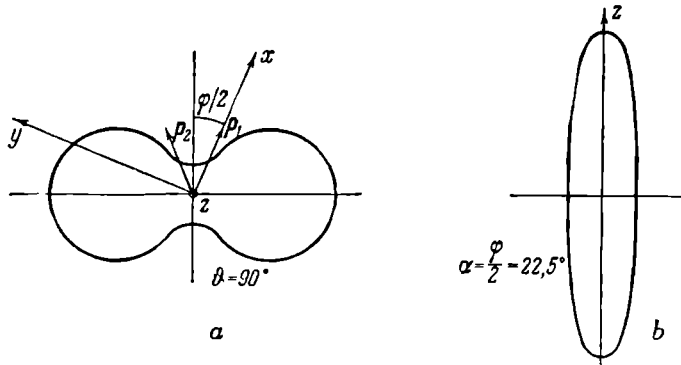


Fig. 89

distribution is illustrated in Fig. 89 for $\varphi = 45^\circ$. Fig. 89a shows the angular distribution in the plane $\vartheta = 90^\circ$, and Fig. 89b the angular distribution in the plane $\alpha = \frac{1}{2} \varphi = 22.5^\circ$.

When the origin of the phase is displaced by γ , the new amplitude becomes $\mathbf{H} \exp(-i\gamma) = \mathbf{H}_1 - i\mathbf{H}_2$. Since $\mathbf{H}_1 \cdot \mathbf{H}_2 = 0$, we have

$$\tan 2\gamma = 2 \frac{\sin \alpha \sin(\alpha - \varphi) + \cos \alpha \cos(\alpha - \varphi) \cos^2 \vartheta}{\sin^2 \alpha - \sin^2(\alpha - \varphi) + [\cos^2 \alpha - \cos^2(\alpha - \varphi)] \cos^2 \vartheta}. \quad (2)$$

$\cos \gamma$ and $\sin \gamma$ can now be determined from (2), and hence $\mathbf{H}_1$ and $\mathbf{H}_2$ may be found as functions of ϑ , α , and φ .

When $\vartheta = 90^\circ$, the polarisation is linear and the plane of polarisation is perpendicular to the xy plane. When $\vartheta = 0, \pi$, the polarisation is elliptical and the ratio of semi-axes of the ellipse is equal to $\tan \frac{1}{2} \varphi$. In particular, the polarisation is circular when $\varphi = \frac{1}{2} \pi$ and $\vartheta = 0, \pi$. The values $\alpha = \frac{1}{2} \varphi$, $\frac{1}{2}(\varphi \pm \pi)$, and

$\frac{1}{2}\varphi + \pi$ are equally simple to discuss. In all these cases the polarisation is in general elliptical. When $\alpha = \frac{1}{2}\varphi$, $\frac{1}{2}\varphi + \pi$, the polarisation is circular in the directions defined by $\tan\frac{1}{2}\varphi = |\cos\vartheta|$. When $\alpha = \frac{1}{2}(\varphi \pm \pi)$, the directions in which the polarisation is circular are defined by $\cotan\frac{1}{2}\varphi = |\cos\vartheta|$.

639.

$$\bar{\gamma} = \frac{e^2 a^2 \omega^4}{8\pi c^3 r^2} (1 + \cos^2 \vartheta) \mathbf{e}_r + \frac{e^2 a^2 \omega^3}{4\pi c^2 r^3} \sin \vartheta \mathbf{e}_a,$$

$$\mathbf{N} = \frac{2}{3} \frac{e^2 a^2 \omega^3}{c^3} \mathbf{e}_z.$$

The latter result may be obtained by recalling that the rate of loss of angular momentum $d\mathbf{K}/dt = -(2/3c^3)\dot{\mathbf{p}} \times \mathbf{p}$ (cf. Problem 632) is equal to the couple $\mathbf{N}$ on the screen. Alternatively, the couple may be computed directly from the formula

$$\mathbf{N} = \frac{1}{c} \int_{r \gg a} \mathbf{r} \times \bar{\gamma} r^2 d\Omega.$$

640.

$$\mathbf{H} = \frac{m\omega^2 \sin \varphi}{c^2 r} (\mathbf{e}_\vartheta \cos \vartheta + i\mathbf{e}_a) e^{i(kr - \omega t + \alpha)},$$

$$\mathbf{E} = \frac{m\omega^2 \sin \varphi}{c^2 r} (-\mathbf{e}_a \cos \vartheta + i\mathbf{e}_\vartheta) e^{i(kr - \omega t + \alpha)},$$

where

$$m = \frac{4\pi}{3} a^3 M,$$

$$\frac{d\bar{I}}{d\Omega} = \frac{m^2 \omega^4 \sin^2 \varphi}{8\pi c^3} (1 + \cos^2 \vartheta),$$

$$\bar{I} = \frac{2m^2 \omega^4 \sin^2 \varphi}{3c^3}.$$

641.

$$\frac{d\bar{I}}{d\Omega} = \frac{9}{800\pi} \cdot \frac{\omega^6 q^2 R_0^4 a^2}{c^5} \sin^2 \vartheta \cos^2 \vartheta,$$

$$\bar{I} = \frac{3}{500} \cdot \frac{\omega^6 q^2 R_0^4 a^2}{c^5}.$$

$$642. \quad \mathbf{E} = \frac{q\mathbf{r}}{r^3}, \quad \mathbf{H} = 0.$$

643. The Hertz vector $\mathbf{Z}(\mathbf{r}, t)$ may be resolved into monochromatic components. Using the expansion given by Eqn. (20) of Appendix III, we have

$$\mathbf{Z}_p(\mathbf{r}, t) = \frac{\mathbf{p}(t')}{r}, \quad (1)$$

where $t' = t - (r/c)$ and

$$\mathbf{Z}_Q(\mathbf{r}, t) = \frac{1}{2r^2} \mathbf{Q}(t') + \frac{1}{2rc} \dot{\mathbf{Q}}(t'), \quad (2)$$

$$\mathbf{Z}_m(\mathbf{r}, t) = \frac{\mathbf{m}(t') \times \mathbf{n}}{r} + \frac{c}{r^2} \left[\int \mathbf{m}(t') dt' \right] \times \mathbf{n}. \quad (3)$$

These formulae will hold for $r \gg a$, where a is a linear dimension of the system. The arbitrary constant which arises as a result of the integration in (3) will have no effect on the magnitude of the field strength.

644. The field of the magnetic dipole is

$$\mathbf{E}_m(\mathbf{r}, t) = -\frac{1}{c} \dot{\mathbf{A}}_m = \frac{\mathbf{n} \times \ddot{\mathbf{m}}(t')}{c^2 r} + \frac{\mathbf{n} \times \dot{\mathbf{m}}(t')}{cr^2},$$

$$\mathbf{H}_m(\mathbf{r}, t) = \text{curl } \mathbf{A}_m = \frac{3\mathbf{n}(\mathbf{m} \cdot \mathbf{n}) - \mathbf{m}}{r^3} + \frac{3\mathbf{n}(\dot{\mathbf{m}} \cdot \mathbf{n}) - \dot{\mathbf{m}}}{cr^2} + \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{m}})}{c^2 r}.$$

The field of the electric dipole may be obtained from that due to the magnetic dipole by substituting

$$\mathbf{m} \rightarrow \mathbf{p}, \quad \mathbf{H}_m \rightarrow \mathbf{E}_e, \quad \mathbf{E}_m \rightarrow -\mathbf{H}_e.$$

645. The dipole moments of the system are zero, and the electric quadrupole moment has the single non-zero component Q_{zz} (if the z -axis is parallel to $\mathbf{p}_0$).

Hence, the vector $\mathbf{Q}$ is parallel to the z -axis and is equal to $\mathbf{Q}(t') = Q_0 \cos \vartheta \cos \omega t' \mathbf{e}_z$ (when the origin of time is suitably chosen), where $Q_0 = 2p_0 a$.

It is convenient to carry out the calculation in the complex form using expression (2) of the solution of Problem 643, and taking the components of $\mathbf{Z}$ along the axes of the spherical system of coordinates. Separating out the real part, we have the following final result:

$$\begin{aligned} H_\alpha &= \frac{Q_0 \sin 2\vartheta}{4} \left[\left(\frac{k^3}{r} - \frac{3k}{r^3} \right) \sin(\omega t - kr) - \frac{3k^2}{r^2} \cos(\omega t - kr) \right], \\ E_r &= \frac{Q_0 (3 \cos^2 \vartheta - 1)}{2} \left[\left(\frac{3}{r^4} - \frac{k^2}{r^2} \right) \cos(\omega t - kr) - \frac{3k}{r^3} \sin(\omega t - kr) \right], \\ E_\vartheta &= \frac{Q_0 \sin 2\vartheta}{4} \left[\left(\frac{6}{r^4} - \frac{3k^2}{r^2} \right) \cos(\omega t - kr) + \left(\frac{k^3}{r} - \frac{6k}{r^3} \right) \sin(\omega t - kr) \right], \\ \frac{\overline{dI}}{d\Omega} &= \frac{Q_0^2 \omega^6}{32\pi c^5} \sin^2 \vartheta \cos^2 \vartheta, \quad \overline{I} = \frac{Q_0^2 \omega^6}{60c^5}, \end{aligned}$$

where $Q_0 = 2p_0 a$.

646. Consider the coordinate system shown in Fig. 90. The current distribution in the antenna is given by

$$\mathcal{J} = \mathcal{J}_0 \sin k \left(\xi + \frac{l}{2} \right) e^{-i\omega t},$$

where

$$k = \frac{\omega}{c} = \frac{m\pi}{l}.$$

According to (XII.9) the electric dipole moment per unit length of the antenna is $P = (i/\omega)\mathcal{J}$. The element $d\xi$ of the antenna may be looked upon as an electric dipole oscillator with a moment $dp = P d\xi$. Since $d\xi \ll \lambda$, it follows that the magnetic field at the point A due to this element may be calculated from (XII.17) and (XII.20):

$$dH_0(\mathbf{r}_0, t) = -\frac{\omega^2}{c^2 r} \mathbf{e}_a \sin \vartheta P \left(t - \frac{r}{c} \right) d\xi,$$

where

$$r = r_0 - \xi \cos \vartheta.$$

Since we are only interested in the field in the wave zone, the quantity $\sin \vartheta/r$, which is a slowly varying function in the region $r \gg l$, may be taken out from under the integral sign. Thus,

$$H_r = H_\vartheta = 0,$$

$$H_\alpha = -\frac{i\omega \sin \vartheta}{c^2 r_0} \mathcal{J}_0 e^{i(kr_0 - \omega t)} \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ik\xi \cos \vartheta} \sin m\pi \left(\frac{\xi}{l} + \frac{1}{2} \right) d\xi.$$

The angular distribution may be found by evaluating the above integral, since $dI/d\Omega = (c/4\pi) \overline{H_\alpha^2} r_0^2$:

$$\frac{dI}{d\Omega} = \begin{cases} \frac{\mathcal{J}_0^2}{2\pi c} \cdot \frac{\cos^2 \left(\frac{m\pi}{2} \cos \vartheta \right)}{\sin^2 \vartheta} & m \text{ odd,} \\ \frac{\mathcal{J}_0^2}{2\pi c} \cdot \frac{\sin^2 \left(\frac{m\pi}{2} \cos \vartheta \right)}{\sin^2 \vartheta} & m \text{ even.} \end{cases}$$

The angular distribution is illustrated by the polar diagrams in Fig. 91. The dashed curves show the current distribution along

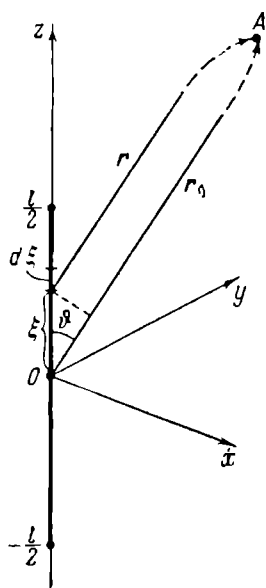


Fig. 90

the antenna; the continuous curves represent the angular distribution of the emitted radiation.

647.

$$\bar{I} = \frac{\mathcal{I}_0^2}{2c} [\ln(2\pi m) + C - Cl(2\pi m)],$$

$$R = 2 \frac{\bar{I}}{\mathcal{I}_0^2} = \frac{1}{c} [\ln(2\pi m) + C - Cl(2\pi m)].$$

648.

$$\frac{d\bar{I}}{d\Omega} = \frac{\mathcal{I}_0^2}{2\pi c} \frac{\sin^2 \vartheta \sin^2 \left[\frac{kl}{2} (1 - \cos \vartheta) \right]}{(1 - \cos \vartheta)^2},$$

$$\bar{I} = \frac{\mathcal{I}_0^2}{c} \left[C - 1 + \ln \frac{4\pi l}{\lambda} - Cl \left(\frac{4\pi l}{\lambda} \right) + \frac{\sin(4\pi l/\lambda)}{4\pi l/\lambda} \right],$$

where $\lambda = 2\pi/k$ is the wavelength of the emitted wave, and ϑ is the polar angle measured from the ξ -axis.

It is easy to show that a travelling wave is emitted more intensely than a standing wave with the same values of l , λ , and $\mathcal{I}_0$.

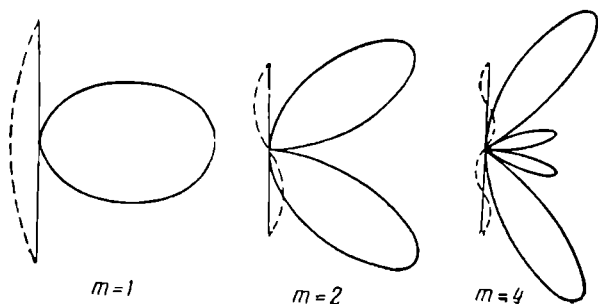


Fig. 91

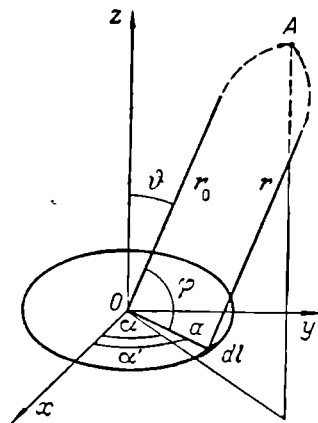


Fig. 92

649. If the distance r of the point of observation $A(r_0, \vartheta, \alpha)$ (Fig. 92) from the loop is large ($r \gg a$), then the position vectors $\mathbf{r}$ of all the elements of the ring, $d\mathbf{l}$, may be regarded as parallel and $r = r_0 - a \cos \varphi = r_0 - a \sin \vartheta \cos(\alpha' - \alpha)$ (cf. Problem 1). An element $d\mathbf{l}$ has an electric dipole moment $d\mathbf{p} = P d\mathbf{l} = (i/\omega) \mathcal{I} d\mathbf{l}$, where P is the electric dipole moment per unit length of the wire

which gives rise to a magnetic field at A of magnitude [cf. (XII.20)]

$$\begin{aligned} d\mathbf{H}(\mathbf{r}_0, t) &= -\frac{\omega^2}{c^2} \frac{d\mathbf{p}(t') \times \mathbf{n}}{r} = \\ &= -i \frac{\omega a}{c^2} \cdot \frac{\mathcal{I}_0}{r_0} e^{-i\omega t + ikr_0 - iak \sin \vartheta \cos(\alpha' - \alpha)} \times \\ &\quad \times \sin n\alpha' [\cos(\alpha' - \alpha) \mathbf{e}_\vartheta + \cos \vartheta \sin(\alpha' - \alpha) \mathbf{e}_\alpha] d\alpha'. \end{aligned}$$

In the denominator of the latter expression quantities of the order of a have been neglected in comparison with r_0 . This cannot be done in the exponential, since ak is not small in general and has an important effect on the phase.

The field may be found by integration:

$$H_\vartheta = -\frac{i\omega a}{c^2} \frac{\mathcal{I}_0}{r_0} e^{i(kr_0 - \omega t)} \int_{-\pi}^{\pi} \cos(\alpha' - \alpha) \sin n\alpha' e^{-ika \sin \vartheta \cos(\alpha' - \alpha)} d\alpha'.$$

The expression for H_α may be obtained from that for H_ϑ by replacing $\cos(\alpha' - \alpha)$ by $\sin(\alpha' - \alpha)$ in the pre-exponential factor in the integration.

Substituting $\beta = \alpha' - \alpha$ we have

$$\begin{aligned} H_\vartheta = -\frac{i\omega a}{c^2} \cdot \frac{\mathcal{I}_0}{r_0} e^{i(kr_0 - \omega t)} &\left(\cos n\alpha \int_{-\pi}^{\pi} \cos \beta \sin n\beta e^{-ika \sin \vartheta \cos \beta} d\beta + \right. \\ &\left. + \sin n\alpha \int_{-\pi}^{\pi} \cos \beta \cos n\beta e^{-ika \sin \vartheta \cos \beta} d\beta \right). \end{aligned}$$

The first of the integrals in the brackets will vanish because the integrand is odd. The second integral has an even integrand so that its limits may be transformed to $0, \pi$ and the integral itself may be expressed in terms of the derivative of a Bessel function (cf. Appendix III). Thus,

$$H_\vartheta(r_0, t) = -E_\alpha = \frac{2\pi\omega a}{c^2} \cdot \frac{\mathcal{I}_0}{r_0} e^{i(kr_0 - \omega t - n\frac{\pi}{2})} \sin n\alpha J'_n(ka \sin \vartheta).$$

Since $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$, similar calculations will yield

$$H_\alpha(r_0, t) = E_\vartheta = \frac{2\pi\omega a n \mathcal{I}_0 e^{i(kr_0 - \omega t - n\frac{\pi}{2})}}{c^2 r_0} \cos n\alpha \frac{J_n(ka \sin \vartheta)}{ka \tan \vartheta}.$$

650.

$$\frac{d\bar{I}}{d\vartheta} = \frac{\omega^4 p_0^2}{8\pi c^3} (1 - \sin^2 \vartheta \cos^2 \alpha) \frac{\sin^2\left(\pi \cos^2 \frac{\vartheta}{2}\right)}{\sin^2\left(\frac{\pi}{2} \cos^2 \frac{\vartheta}{2}\right)},$$

where ϑ, α are the polar angles characterising the direction of emission (*cf.* Fig. 93). The oscillator which leads in phase is in the upper position along the z -axis.

651. Since $\mathbf{j} = \rho \mathbf{v} = \rho d\mathbf{r}/dt$, it follows that $(j_x, j_y, j_z) \rightarrow (-j_x, -j_y, +j_z)$ where the reflected currents are calculated at the reflected points: $j_x(\mathbf{r}) = -j'_x(\mathbf{r}')$, and so on.

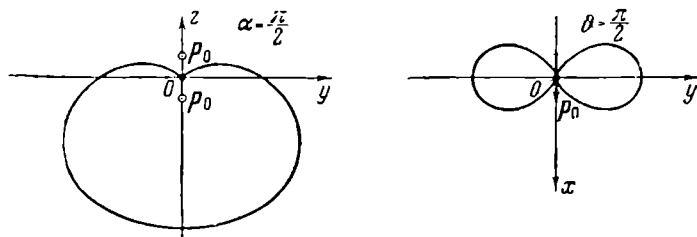


Fig. 93

Similarly, from usual definitions and Eqns. (XII.1) and (XII.2) in the Cartesian form, we have

$$\begin{aligned} (p_x, p_y, p_z) &\rightarrow (-p_x, -p_y, p_z), & (Q_x, Q_y, Q_z) &\rightarrow (-Q_x, -Q_y, -Q_z), \\ (m_x, m_y, m_z) &\rightarrow (m_x, m_y, -m_z), & (E_x, E_y, E_z) &\rightarrow (-E_x, -E_y, E_z), \\ (H_x, H_y, H_z) &\rightarrow (H_x, H_y, -H_z). \end{aligned}$$

652. The boundary conditions $H_n = 0$ and $E_\tau = 0$ are satisfied at the surface ($z = 0$) of the conductor: this is a direct consequence of the results of Problem 651. In the special case of the electric dipole oscillator, the electromagnetic field in the half-space $z > 0$ is the same as the field due to an electric dipole oscillator of moment $\mathbf{p} = 2\mathbf{e}_z f(t) \sin \varphi_0$. The field vanishes when $\varphi_0 = 0$ (dipole parallel to the plane) and reaches a maximum when $\varphi_0 = \frac{1}{2}\pi$ (dipole perpendicular to the plane). The total energy emitted into the half-space $z > 0$ for $\varphi_0 = \frac{1}{2}\pi$ is greater by a factor of 4 than the energy emitted by this oscillator when it is at a large distance from the conducting plane.

653.

$$E_\vartheta = H_\alpha = \frac{\omega^3 p_0 a}{2c^3 r} \cos 2\vartheta \cos \alpha \cos \omega t',$$

$$E_\alpha = -H_\vartheta = \frac{\omega^3 p_0 a}{2c^3 r} \cos \vartheta \sin \alpha \cos \omega t',$$

$$\frac{dI}{d\Omega} = \frac{p_0^2 a^2 \omega^6}{32\pi c^6} (\cos^2 2\vartheta \cos^2 \alpha + \cos^2 \vartheta \sin^2 \alpha).$$

654. b) $H_r = 0, H_\vartheta = -\frac{ik}{\sin \vartheta} \cdot \frac{\partial u}{\partial z}, H_\alpha = ik \frac{\partial u}{\partial \vartheta},$

$$E_r = k^2 u r + \frac{\partial^2 (ur)}{\partial r^2}, E_\vartheta = \frac{1}{r} \frac{\partial^2 (ru)}{\partial r \partial \vartheta}, E_\alpha = \frac{1}{r \sin \vartheta} \frac{\partial^2 (ru)}{\partial r \partial \alpha}.$$

656.

$$u = \frac{\rho_0}{b} \cdot \frac{e^{ikR}}{R} - ik\rho_0 \sum_{l=0}^{\infty} (2l+1) \frac{h_l^{(1)}(kb)}{b} \times \\ \times \left. \frac{\frac{d[rj_l(kr)]}{dr}}{\frac{d[rh_l^{(1)}(kr)]}{dr}} \right|_{r=a} \cdot h_l^{(1)}(kr) P_l(\cos \vartheta).$$

The fields $\mathbf{E}$ and $\mathbf{H}$ are given by the formulae obtained in the solution of Problem 654. In order to determine the angular distribution, it is necessary to use the asymptotic expressions for the spherical Hankel functions [Eqn. (19), Appendix III]. This gives

$$E_a = H_\vartheta = 0, \\ H_a|_{r=\infty} = ik \frac{\partial u}{\partial \vartheta} \Big|_{r=\infty} = F(\vartheta) \frac{e^{ikr}}{r} = E_\vartheta \Big|_{r=\infty},$$

where

$$F(\vartheta) = \frac{\rho_0 k^2}{b} \sum_{l=0}^{\infty} \frac{2l+1}{i^{l-1}} \frac{j_l(kb) \frac{d[rh_l^{(1)}(kr)]}{dr} - h_l^{(1)}(kb) \frac{d[rj_l(kr)]}{dr}}{\frac{d[rh_l^{(1)}(kr)]}{dr}} \Big|_{r=a} \times \\ \times \frac{dP_l(\cos \vartheta)}{d\vartheta}, \\ \frac{dI}{d\Omega} = \frac{c}{8\pi} |H_a|^2 r^2 = \frac{c}{8\pi} |F(\vartheta)|^2.$$

2. The electromagnetic field of a moving point charge

657. The potential φ for the field of the particle is given by

$$\varphi(\mathbf{r}, t) = \int \frac{\varrho(\mathbf{r}', t - \frac{R}{c})}{R} dV' = e \int \frac{\delta[\mathbf{r}' - \mathbf{r}_0(t - \frac{R}{c})]}{R} dV', \quad (1)$$

where $R = |\mathbf{r} - \mathbf{r}'|$. This integral may be evaluated with the aid of the formula $\int f(\mathbf{R}_1) \delta(\mathbf{R}_1) (d\mathbf{R}_1) = f(0)$ (cf. Appendix I). Let $\mathbf{R}_1 = \mathbf{r}' - \mathbf{r}_0(t - R/c)$. The Jacobian of the transformation is

$$\left| \frac{\partial \mathbf{R}_1}{\partial \mathbf{r}'} \right| = 1 - \frac{\mathbf{v} \cdot \mathbf{R}}{cR}. \quad (2)$$

In terms of the new variables, the integral in (1) becomes

$$\varphi(\mathbf{r}, t) = e \int \frac{\delta(\mathbf{R}_1) (d\mathbf{R}_1)}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}} = \frac{e}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}} \bigg|_{\mathbf{R}_1=0}.$$

The condition $\mathbf{R}_1 = 0$ means that on the right-hand side of this formula all the quantities are referred to the point $\mathbf{r}' = \mathbf{r}_0(t - R/c)$ at which the charge was located at the retarded time $t' = t - R/c$. The vector potential may be calculated in a similar way.

658.

$$\begin{aligned} \varphi(\mathbf{r}, t) &= \int \frac{\rho(\mathbf{r}', t - \frac{R}{c})}{R} dV' = \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n n!} \frac{\partial^n}{\partial t^n} \int R^{n-1} \rho(\mathbf{r}', t) dV' = \\ &= e \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n n!} \frac{d^n R_0^{n-1}}{dt^n}, \end{aligned}$$

where $R_0 = |\mathbf{r} - \mathbf{r}_0(t)|$ and

$$\mathbf{A}(\mathbf{r}, t) = e \sum_{n=0}^{\infty} \frac{(-1)^n}{c^{n+1} n!} \frac{d^n (\mathbf{v}(t) R_0^{n-1})}{dt^n}.$$

All the quantities on the right- and left-hand sides of these equations are to be taken at the same instants of time. The retarded interaction is thus formally reduced to the instantaneous interaction. The above expansions may be used for a sufficiently slow ($v \ll c$) and continuous motion (finite acceleration and all its derivatives), provided R is not too large.

661. For small v/c , Eqn. (XII.25) becomes

$$\begin{aligned} \mathbf{E} &= \frac{e\mathbf{r}}{r^3} + 3 \frac{e\mathbf{r}(\mathbf{r} \cdot \mathbf{v})}{cr^4} - \frac{e\mathbf{v}}{cr^2} + \frac{e\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{v}})}{c^2 r^3} \bigg|_{t'=t-\frac{r}{c}}, \\ \mathbf{H} &= \frac{e\mathbf{v} \times \mathbf{r}}{cr^3} + \frac{e\dot{\mathbf{v}} \times \mathbf{r}}{c^2 r^2} \bigg|_{t'=t-\frac{r}{c}}, \end{aligned}$$

where $\mathbf{r}$ is the distance between the point of observation and any point in the region in which the motion of the charge takes place.

The first three terms in the expression for $\mathbf{E}$ and the first term in the expression for $\mathbf{H}$ are proportional to r^{-2} and are the leading terms at relatively small distances from the charge (quasi-stationary zone). The electric field in this zone is essentially the Coulomb field $\mathbf{E} = e\mathbf{r}/r^3$, while the magnetic field is given by the Biot-Savart law $\mathbf{H} = e\mathbf{v} \times \mathbf{r}/cr^3$. At large distances from the charge (in the wave zone), the leading terms are the last terms, which are proportional to r^{-1} . These terms represent

the radiation field and are of the form

$$\mathbf{E} = \frac{e\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{v}})}{c^2 r},$$

$$\mathbf{H} = \frac{e\dot{\mathbf{v}} \times \mathbf{n}}{c^2 r},$$

where $\mathbf{n} = \mathbf{r}/r$. The position of the boundary between the quasi-stationary and the wave zones is defined by the condition

$$\frac{e/r_b^2}{e|\dot{\mathbf{v}}|/c^2 r_b} \sim 1,$$

and hence

$$r_b \simeq a \left(\frac{c^2}{v^2} \right),$$

since $|\dot{\mathbf{v}}| \sim v^2/a$, where a is of the same order of magnitude as the linear dimensions of the region in which the motion of the charge takes place.

662.

$$\frac{dI}{d\Omega} = \frac{e^2}{4\pi c^3} (\dot{\mathbf{v}} \times \mathbf{n})^2,$$

$$I = \frac{2e^2}{3c^3} \dot{\mathbf{v}}^2,$$

where $\mathbf{n} = \mathbf{r}/r$.

663. (a) The portion of energy $-d\mathcal{E}$ emitted by the particle into a solid angle $d\Omega$ in a time dt' will travel past the point of observation in a time dt . Hence, $-d\mathcal{E}/dt'd\Omega = (dI/d\Omega)(dt/dt')$. Since $t = t' + R/c$ and $\partial R/\partial t' = -\mathbf{n} \cdot \mathbf{v}$ where $\mathbf{n} = \mathbf{R}/R$, we have

$$dt = dt' \left(1 - \frac{\mathbf{n} \cdot \mathbf{v}}{c} \right),$$

and hence, finally,

$$-\frac{d\mathcal{E}}{dt'd\Omega} = \left(1 - \frac{\mathbf{n} \cdot \mathbf{v}}{c} \right) \frac{dI}{d\Omega}.$$

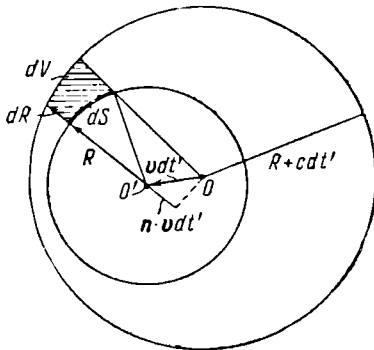


Fig. 94.

(b) The energy emitted by the charge in a time interval dt' is localised between two spheres. The first sphere has its centre at the point O at which the charge was located at time t' , while the second sphere has its centre at the point O' where the charge was located at time $t' + dt'$ (Fig. 94). The radius of the first sphere is R and the radius of the second sphere is $R + c dt'$.

Consider the volume element $dV = dSdR = R^2 d\Omega (c - \mathbf{n} \cdot \mathbf{v}) dt'$. The electromagnetic energy residing in this volume is $dW = (E^2/4\pi) dV = (cE^2/4\pi)(1 - \mathbf{n} \cdot \mathbf{v}/c) R^2 d\Omega dt'$. Hence, the rate of loss of energy $-d\mathcal{E}/dt' d\Omega = dW/dt' d\Omega$ is given by the expression quoted above.

$$666. \quad (a) \quad -\frac{d\mathcal{E}}{dt'} = \frac{2e^2}{3c^3} \dot{\gamma}^6 \left[\dot{\mathbf{v}}^2 - \left(\dot{\mathbf{v}} \times \frac{\mathbf{v}}{c} \right)^2 \right];$$

$$(b) \quad -\frac{d\mathcal{E}}{dt'} = \frac{2e^4}{3m^2 c^3} \frac{\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{H}}{c} \right)^2 - \frac{(\mathbf{E} \cdot \mathbf{v})^2}{c^2}}{1 - \frac{v^2}{c^2}}.$$

667.

$$-\frac{d\mathbf{p}}{dt'} = -\frac{\mathbf{v}}{c^2} \cdot \frac{d\mathcal{E}}{dt'},$$

where $\mathbf{v}$ is the velocity of the particle at time t' .

668. Consider the rate of loss of energy by the particle in two reference frames, namely, the frame S_0 in which the particle is instantaneously at rest, and the laboratory frame S in which the particle has a velocity $\mathbf{v}$. In S_0 the emission is of the electric dipole type, and hence there is no loss of momentum in S_0 . This follows from the central symmetry of the angular distribution of the emitted radiation in this reference frame (or from the result of Problem 667).

Consider now the amount of energy $-d\mathcal{E}_0$ emitted by the particle in a time $dt'_0 = d\tau$ in the frame S_0 . The energy loss observed in S in a time interval $dt' = d\tau/(1 - v^2/c^2)^{1/2}$ is then $-d\mathcal{E} = -d\mathcal{E}_0/(1 - v^2/c^2)^{1/2}$. Hence, the rate of loss of energy is given by

$$-\frac{d\mathcal{E}}{dt'} = -\frac{d\mathcal{E}_0/\sqrt{1 - \frac{v^2}{c^2}}}{d\tau/\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{d\mathcal{E}_0}{d\tau}.$$

This result is independent of v , which means that the rate of loss of energy integrated over all directions is the same in all reference frames.

The total radiated intensity at a time t is given by the integral of the normal component of the Poynting vector over a sphere of radius R and centred at the point occupied by the particle at the retarded time $t' = t - R/c$. In contrast to the invariant quantity $-d\mathcal{E}/dt'$, the total radiated intensity does not exhibit simple relativistic transformation properties.

$$669. \quad \frac{dI(t)}{d\Omega} = \frac{c}{4\pi} E^2 R^2 = \frac{e^2 \dot{v}^2 \sin^2 \vartheta}{4\pi c^3 (1 - \beta \cos \vartheta)^5},$$

where ϑ is the angle between the velocity $\mathbf{v}$ and the direction of emission $\mathbf{n}$, and $\beta = v/c$. The polar diagram of the radiation is

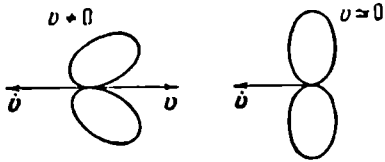


Fig. 95.

shown in Fig. 95. At low velocities v , the intensity emitted in the forward and backward directions is the same. Emission in the forward direction predominates when v becomes comparable with c . Maximum emission is observed at an angle ϑ_0 which

is given by

$$\cos \vartheta_0 = \frac{1}{4\beta} (\sqrt{1 + 12\beta^2} - 1).$$

When $\beta \rightarrow 0$, $\vartheta_0 \rightarrow \pi/2$; when $\beta \rightarrow 1$, $\vartheta_0 \rightarrow 0$. The total intensity is

$$I = \int \frac{dI}{d\Omega} d\Omega = \frac{2e^2 \dot{v}^2}{3c^3} \frac{1 + \beta^2/5}{(1 - \beta^2)^4}.$$

The total rate of loss of energy is

$$-\frac{d\mathcal{E}}{dt'} = \frac{2e^2}{3c^3} \cdot \frac{\dot{v}^2}{(1 - \beta^2)^3}.$$

670. The total radiation emitted is given by

$$\begin{aligned} \frac{d\Delta W}{d\Omega} &= \int \frac{dI}{d\Omega} dt = \int \left(-\frac{d\mathcal{E}}{d\Omega dt'} \right) dt' = \\ &= \frac{e^2 v_0^2}{16\pi c^3 \tau} \cdot \frac{\sin^2 \vartheta}{\cos \vartheta} \left[\frac{1}{\left(1 - \frac{v_0}{c} \cos \vartheta\right)^4} - 1 \right], \end{aligned}$$

where ϑ is the angle between the direction of motion of the particle and the direction of emission of the radiation $\mathbf{n}$.

The observed duration of the pulse depends on the angle ϑ between the velocity of the particle and the direction of emission:

$$\Delta t = \tau \left[1 - \frac{v_0}{2c} \cos \vartheta \right].$$

671.

$$-\frac{d\mathcal{E}}{dt'} = \frac{2e^4 H^2 p^2}{3m^4 c^5}.$$

$$672. \quad \frac{dI}{d\Omega} = \frac{e^2 |\dot{\mathbf{v}}|^2}{4\pi c^3} \cdot \frac{(1 - \beta \cos \vartheta)^2 - (1 - \beta^2) \sin^2 \vartheta \cos^2 \alpha}{(1 - \beta \cos \vartheta)^6},$$

where $\beta = v/c$.

Suppose that the polar axis is in the direction of the velocity and the azimuth angle α is measured from the direction of the acceleration. The angular distribution of the emitted radiation is shown in Fig. 96. No radiation is emitted in directions defined

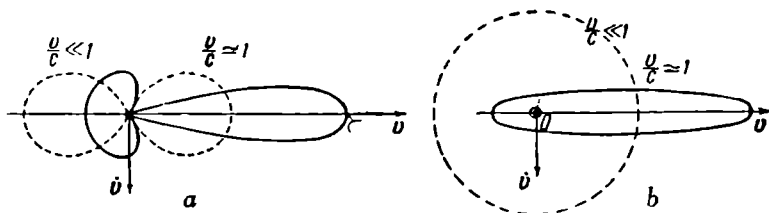


Fig. 96.

by $\gamma[1 - (v/c) \cos \vartheta] = \sin \vartheta |\cos \alpha|$. In particular, when $\alpha = 0, \pi$ (Fig. 96a) no radiation is emitted at $\vartheta = \cos^{-1}(v/c)$. When $\alpha = \pi/2, 3\pi/2$ (Fig. 96b) the intensity is finite for all ϑ .

$$\begin{aligned} 673. \quad \frac{\bar{dI}}{d\Omega} &= -\frac{\bar{d\mathcal{E}}}{d\Omega dt'} = \frac{e^4 H^2 \beta^2}{8\pi^2 m^2 c^3} (1 - \beta^2) \times \\ &\times \int_0^{2\pi} \frac{(1 - \beta^2) \cos^2 \vartheta + (\beta - \sin \vartheta \cos \alpha)^2}{(1 - \beta \sin \vartheta \cos \alpha)^5} d\alpha = \\ &= \frac{e^4 H^2 \beta^2 (1 - \beta^2)}{8\pi m^2 c^3} \cdot \frac{1 + \cos^2 \vartheta - \frac{1}{4} \beta^2 (1 + 3\beta^2) \sin^4 \vartheta}{(1 - \beta^2 \sin^2 \vartheta)^{\frac{7}{2}}}, \end{aligned}$$

where $\beta = v/c$.

The origin of the azimuth angle is chosen so that the direction of the vector $\mathbf{n}$ is characterised by the polar angles $\vartheta, \pi/2$. In the ultra-relativistic case, the emission of radiation is confined to the neighbourhood of the plane of the orbit in the angular range $\Delta\vartheta \simeq (1 - \beta^2)^{1/2}$.

$$674. \quad \left. \begin{aligned} A_{n\vartheta} &= \frac{e\beta e^{ikR_0}}{2\pi R_0} \cos \vartheta \int_0^{2\pi} \cos \alpha' e^{i(n\alpha' - n\beta \sin \vartheta \sin \alpha')} d\alpha', \\ A_{n\alpha} &= \frac{e\beta e^{ikR_0}}{2\pi R_0} \int_0^{2\pi} \sin \alpha' e^{i(n\alpha' - n\beta \sin \vartheta \sin \alpha')} d\alpha', \end{aligned} \right\} \quad (1)$$

where $\mathbf{k} = (\omega/c)\mathbf{n}$, the origin of coordinates is at the centre of the

orbit, the z -axis is perpendicular to the plane of the orbit, the direction of $\mathbf{k}$ is characterised by the polar angles ϑ , $\pi/2$, and R_0 is the distance from the centre of the orbit to the point of observation. Hence

$$\left. \begin{aligned} H_{n\alpha} &= i \frac{\omega}{c} n A_{n\vartheta} \simeq i \frac{\beta e n e^{ikR_0}}{a R_0} \cotan \vartheta J_n(n\beta \sin \vartheta), \\ H_{n\vartheta} &= -i \frac{\omega}{c} n A_{n\alpha} = \frac{e \beta^2 n e^{ikR_0}}{a R_0} J'_n(n\beta \sin \vartheta). \end{aligned} \right\} \quad (2)$$

The polarisation of the emitted radiation is, in general, elliptical. The ratio of the semi-axes of the ellipse is equal to $\beta \tan \vartheta [J'_n(n\beta \sin \vartheta)/J_n(n\beta \sin \vartheta)]$. The sense in which the ellipse is traversed is determined by the sign of this ratio. When $\vartheta = 0$ and $\vartheta = \pi/2$, the polarisation is circular and linear respectively. For sufficiently large n and β , the polarisation is linear in those directions for which J'_n/J_n is zero or has a pole.

675. The presence of higher harmonics in the spectrum of the radiated field is explained by the fact that the time taken by the field between equivalent points on the orbit is finite, and is, in general, comparable with the orbital period (provided the velocity of the charge is comparable with the velocity of light, c). Hence, the time taken by the field emitted during one-half of an orbital period to travel past the point of observation, when the particle approaches this point, is smaller than the corresponding time during the next half-period. Since the coordinates of the particle are not simple harmonic functions of time, it follows that the field is a complicated periodic function of time which can be represented as a superposition of Fourier components.

It is to be expected that when $\beta \rightarrow 0$ the higher harmonics will vanish. In fact, when $x \simeq 0$, $n > 0$ we have (*cf.* Appendix III)

$$J_n(x) \simeq \frac{x^n}{2^n n!}, \quad J'_n(x) \simeq \frac{x^{n-1}}{2^n (n-1)!}.$$

It is clear from these equations that when $\beta \rightarrow 0$, the important harmonics are those with $|n| = 1$. Thus (*cf.* solution of Problem 634),

$$H_\alpha = H_{1\alpha} + H_{-1\alpha} = -\frac{e\beta^2}{a} \cdot \frac{\cos \vartheta \sin kR_0}{R_0},$$

$$H_\vartheta = H_{1\vartheta} + H_{-1\vartheta} = \frac{e\beta^2}{a} \cdot \frac{\cos kR_0}{R_0}.$$

$$\begin{aligned}
 676. \quad \frac{dI_n}{d\Omega} &= \frac{c}{2\pi} |\mathbf{H}_n|^2 R_0^2 = \\
 &= \frac{cn^2 e^2 \beta^2}{2\pi a^2} [\cotan^2 \vartheta J_n^2(n\beta \sin \vartheta) + \beta^2 J_n'^2(n\beta \sin \vartheta)].
 \end{aligned}$$

If the circular motion occurs under the action of a constant uniform magnetic field H , then

$$a = \frac{mc^2 \beta}{eH \sqrt{1 - \beta^2}}.$$

677. The n -th harmonic of the field emitted by a single charge was obtained in the solution of Problem 674 [Eqn. (2)]. The expressions for the harmonics due to a number of charges will clearly differ from each other only in the initial phases. Let ψ_l be the phase difference between the field due to the l -th electron relative to the electron which we shall agree to call electron number 1, so that the final field (in real form) is

$$H_{n\vartheta} = \frac{e\beta^2 n}{aR_0} J_n'(n\beta \sin \vartheta) \sum_{l=1}^N \cos n \left(\omega t - \frac{\omega R_0}{c} + \psi_l \right).$$

There is an analogous expression for $H_{n\alpha}$. The average intensity over the period $T = 2\pi/\omega$ is

$$dI_{nN} = \frac{c}{4\pi} \cdot \frac{1}{T} \int_0^T (H_{n\vartheta}^2 + H_{n\alpha}^2) dt R_0^2 d\Omega = S_N dI_n,$$

where dI_n is the intensity due to a single electron (*cf.* preceding problem) and S_N is a factor which represents interference effects between the fields emitted by the electrons (the so-called coherence factor):

$$S_N = N + \sum_{\substack{l, l'=1 \\ (l \neq l')}}^N \cos n(\psi_l - \psi_{l'}).$$

Consider three special cases:

(a) electrons distributed at random on the orbit:

$$\sum \cos n(\psi_l - \psi_{l'}) = 0;$$

(b) uniform distribution of electrons on the orbit

$$\psi_l = \frac{2\pi}{N}(l-1)$$

and

$$S_N = N \sum_{l=2}^N \cos 2\pi(l-1) \frac{n}{N} = \frac{N}{2} \left[\sum_{l=1}^N e^{2\pi(l-1) \frac{n}{N}} + \sum_{l=1}^N e^{-2\pi(l-1) \frac{n}{N}} \right] = N(-1)^n \frac{\sin n\pi}{\tan \frac{n\pi}{N}} = \begin{cases} 0, & \frac{n}{N} \text{ not an integer,} \\ N^2, & \frac{n}{N} \text{ an integer.} \end{cases}$$

(c) if the electrons form a bunch, then all the differences $\psi_l - \psi_{l'}$ are small, provided n is not too large, so that the linear dimensions of the bunch are small compared with the corresponding wavelength. All the $\cos n(\psi_l - \psi_{l'})$ in the expression for S_N can then be replaced by 1. Hence, $S_N = N^2$. The factor S_N decreases with increasing n . The actual value of S_N depends on the precise distribution of the electrons in the bunch, and cannot be given in a general form.

678. Let the origin be at the centre of inertia of the system of charges. The electric dipole moment of the system is then given by

$$\mathbf{p} = e_1 \mathbf{r}_1 + e_2 \mathbf{r}_2 = \mu \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right) \mathbf{r}, \quad (1)$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mu = m_1 m_2 / (m_1 + m_2)$.

Since the ratios e/m are different, it follows that $\mathbf{p} \neq 0$ and the system will radiate as an electric dipole ($v/c \ll 1$). The instantaneous intensity is

$$I(t) = \frac{2\ddot{\mathbf{p}}^2}{3c^3} = \frac{2\mu^2}{3c^3} \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 \ddot{\mathbf{r}}^2(t').$$

Since the equation of motion of the charges is $\mu \ddot{\mathbf{r}} = e_1 e_2 \mathbf{r} / r^3$, it follows that $I = (2e_1^2 e_2^2 / 3c^3) (e_1 / m_1 - e_2 / m_2)^2 / r^4$. To calculate

the time average of the intensity, $\bar{I} = T^{-1} \int_0^T I dt'$, we can replace

integration with respect to t' by integration with respect to the angle α by substituting $dt' = (\mu r^2 / K) d\alpha$ (K is the angular momentum) and using the equation of the trajectory. The result is

$$\bar{I} = \frac{2^{\frac{3}{2}}}{3c^3} \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 \frac{\mu^{\frac{5}{2}} |e_1 e_2|^3 |\mathfrak{G}|^{\frac{3}{2}}}{K^5} \left(3 - \frac{2|\mathfrak{G}|K^2}{\mu e_1^2 e_2^2} \right).$$

679.

$$\frac{d\mathbf{K}}{dt} = - \frac{2^{\frac{7}{2}} \mu^{\frac{3}{2}} |\mathfrak{G}|^{\frac{3}{2}}}{3c^3} \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 \frac{K}{K^3}.$$

680. Proceeding as in the solution of Problem 678, we can write the second derivative of the dipole moment as

$$\ddot{\mathbf{p}} = \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right) \frac{e_1 e_2 \mathbf{r}}{r^3}. \quad (1)$$

The evaluation of A presents no difficulties. To determine B it is necessary to know $\ddot{p}_z$, i. e. the component of $\ddot{\mathbf{p}}$ in the direction of the initial motion of the scattered particles as a function of the coordinates r, α (the polar coordinates in the plane of the relative motion of the particles). It must be remembered that in the equation for the trajectory of relative motion $[-1 + \epsilon \cos \alpha = a(\epsilon^2 - 1)/r]$, the angle α is measured from the axis of symmetry (z' -axis) of the trajectory. Thus, $y' = r \sin \alpha$, $z' = r \cos \alpha$. The angle between the z - and z' -axes is $\pi - \alpha_0$ ($\cos \alpha_0 = 1/\epsilon$), and hence $z = -z' \cos \alpha_0 - y' \sin \alpha_0 = -(r/\epsilon)[\cos \alpha + (\epsilon^2 - 1)^{1/2} \sin \alpha]$. Using Eqn. (1), and recalling that $\sin \alpha$ is an odd function, we have

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{+\infty} \ddot{p}_z dt ds &= \\ &= e_1^2 e_2^2 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 \int_0^\infty \int_{-\infty}^{+\infty} \frac{\cos^2 \alpha + (\epsilon^2 - 1) \sin^2 \alpha}{\epsilon^2 r^4} dt ds. \end{aligned}$$

The equation for the trajectory may be used to express $\cos^2 \alpha$ and $\sin^2 \alpha$ in terms of r and ϵ . Substituting $\epsilon^2 = u$, $s ds = \frac{1}{2} a^2 du$ we find that the above integral can be transformed to

$$\begin{aligned} \frac{a}{v_0} \int_{2a}^\infty \frac{dr}{r^3} \int_1^{\left(\frac{r}{a}-1\right)^2} &\left[-\frac{a^2}{r^2} u + \left(4 \frac{a^2}{r^2} - 2 \frac{a}{r} + 1 \right) + \right. \\ &\left. + \left(-5 \frac{a^2}{r^2} + 6 \frac{a}{r} - 2 \right) \frac{1}{u} + 2 \left(\frac{a}{r} - 1 \right)^2 \frac{1}{u^2} \right] \frac{du}{\sqrt{\left(\frac{r}{a}-1\right)^2 - u}}. \end{aligned}$$

The evaluation of the integral with respect to u leads to a logarithmic term which may be transformed by integration by parts. To evaluate the integral with respect to r it is convenient to substitute $x = 2a/r$ which converts this integral into a sum of a number of B-functions which are defined by

$$B(k, l) = \int_0^1 x^{k-1} (1-x)^{l-1} dx = \frac{\Gamma(k) \Gamma(l)}{\Gamma(k+l)}.$$

The final result is

$$A = \frac{8\pi}{9} e_1 e_2 \left(\frac{e_1}{m_1} - \frac{e_2}{m_2} \right)^2 \mu v_0,$$

$$B = 0.$$

681. On the present approximation, $v = \text{const}$ and the trajectory of the particle is a straight line. Suppose that the motion of the particle takes place in the xz plane in the direction parallel to the z -axis. In terms of these coordinates,

$$\mathbf{n} = (n_x, n_y, n_z), \text{ where } n_x = \sin \vartheta \cos \alpha, \quad n_y = \sin \vartheta \sin \alpha,$$

$$n_z = \cos \vartheta, \quad \mathbf{r} = (s, 0, vt'), \quad r = \sqrt{s^2 + v^2 t'^2},$$

$$\mathbf{v} = (0, 0, v).$$

Since $\mathbf{v} = c^2 \dot{\mathbf{p}}/\mathcal{E}$ where $\mathcal{E} = mc^2/(1 - \beta^2)^{1/2}$ and $\beta = v/c$, we have $\dot{\mathbf{v}} = c^2 \ddot{\mathbf{p}}/\mathcal{E} - c^2 \dot{\mathbf{p}} \dot{\mathcal{E}}/\mathcal{E}^2$. Moreover, from the equation of motion of the particle we have $\dot{\mathbf{p}} = e_1 e_2 \mathbf{r}/r^3$, and the law of conservation of energy requires that $\mathcal{E} + e_1 e_2/r = \text{const}$. When the last equation is differentiated with respect to t' we have

$$\dot{\mathcal{E}} = \frac{e_1 e_2 \dot{r}}{r^2} = \frac{e_1 e_2 \mathbf{r} \cdot \mathbf{v}}{r^3},$$

so that

$$\dot{\mathbf{v}} = \frac{e_1 e_2 c^2}{\mathcal{E}} \left[\mathbf{r} - \frac{\mathbf{p}(\mathbf{r} \cdot \mathbf{v})}{\mathcal{E}} \right] = \frac{e_1 e_2 c^2}{\mathcal{E} r^3} [s \mathbf{e}_x + vt' (1 - \beta^2) \mathbf{e}_z].$$

Substituting these expressions into (XII.26) we have

$$\frac{d \Delta W_{\mathbf{n}}}{d\Omega} = \frac{e_1^4 e_2^2 c^4}{4\pi c^3 \mathcal{E}^2 (1 - \beta n_z)^5} \left\{ s^2 [(1 - \beta n_z)^2 - n_x^2 (1 - \beta^2)] \times \right.$$

$$\left. \times \int_{-\infty}^{+\infty} \frac{dt'}{(s^2 + v^2 t'^2)} + c^2 \beta^2 (1 - \beta^2)^2 (1 - n_z)^2 \int_{-\infty}^{\infty} \frac{t'^2 dt'}{(s^2 + v^2 t'^2)^3} \right\}.$$

Integration yields

$$\frac{d \Delta W_{\mathbf{n}}}{d\Omega} = \frac{e_1^4 e_2^2 (1 - \beta^2)}{32 m^2 c^3 s^3 v (1 - \beta n_z)^5} [4 - 3n_x^2 - n_z^2 - 6\beta n_z +$$

$$+ \beta^2 (-2 + 3n_x^2 + 5n_z^2) + \beta^4 (1 - n_z^2)]. \quad (1)$$

In the non-relativistic limit $\beta \rightarrow 0$ and

$$\frac{d \Delta W_{\mathbf{n}}}{d\Omega} = \frac{e_1^4 e_2^2}{32 m^2 c^3 s^3 v} (4 - 3n_x^2 - n_z^2).$$

In the ultra-relativistic limit $\beta \simeq 1$ and

$$\frac{d\Delta W_n}{d\Omega} = \frac{3e_1^4 e_2^2 (1-\beta)}{2^9 m^2 c^4 s^3 \sin^4 \frac{\vartheta}{2}}.$$

When $\vartheta \leq (1-\beta)^{1/2}$, the latter formula becomes incorrect, and the exact expression (1) must be employed.

$$682. \quad \Delta W = \frac{\pi e_1^4 e_2^2}{12 m^2 c^3 s^3 v} \cdot \frac{4-\beta^2}{1-\beta^2}, \quad \Delta p = \frac{v \Delta W}{c^2}.$$

$$683. \quad \frac{d\Delta W_\omega}{d\omega} = \frac{8e_1^4 e_2^2 \omega^2 c}{3\pi v^4} \left[K_1^2\left(\frac{\omega s}{v}\right) + K_0^2\left(\frac{\omega s}{v}\right) \right].$$

684. Eqn. (XII.33) will hold for all frequencies ω since the collision time $\tau = 0$. For scattering by a perfectly hard sphere, the angle of incidence is equal to the angle of scattering, and hence $|\mathbf{v}_2 - \mathbf{v}_1|^2 = 2v \sin^2 \frac{1}{2} \vartheta$, where ϑ is the angle of scattering which is related to the impact parameter s by $s = a \sin^2 \frac{1}{2} \vartheta$ when $s \leq a$. When $s > a$, the particle is not scattered. Hence,

$$d\kappa_\omega = \frac{2e^2}{3\pi c^3} 4v^2 \int_0^a \sin^2 \frac{\vartheta}{2} 2\pi s ds d\omega = \frac{4e^2 a^2 v^2}{3c^3} d\omega.$$

The above differential effective emission is independent of frequency. Hence, the total effective emission is

$$\kappa = \int_0^\infty d\kappa_\omega = \infty.$$

This is due to the fact that the sphere was assumed to be perfectly hard. In fact, perfectly hard spheres do not exist, $\tau \neq 0$, and for large values of ω the above expression for $d\kappa_\omega$ is invalid.

685. Eqn. (XII.30) for the effective differential emission may be written in the form

$$\frac{d\kappa_n}{d\Omega} = 2\pi \int_0^\infty \int_{-\infty}^\infty \frac{\overline{dI}}{d\Omega} dt s ds. \quad (1)$$

The intensity is given by $dI/d\Omega = (c/4\pi) H^2 r^2$, where $\mathbf{H} = \mathbf{A} \times \mathbf{n}/c$. In (1), the averaging of the radiated intensity should be carried out over all directions in the plane perpendicular to the direction of the incident beam. It is convenient to write the vector product in the expression for $\mathbf{H}$ in the form $H_\alpha = e_{\alpha\beta\gamma} \dot{A}_\beta n_\gamma / c$ where $e_{\alpha\beta\gamma}$ is the skew-symmetric unit pseudotensor (cf. Problems 24 and 26), and the summation is carried out over the repeated indices. The components of the vector potential A_β may be expressed in

terms of the components of the quadrupole moment $Q_{\beta\epsilon}$ which are given by Eqn. (XII.19):

$$A_{\beta} = \frac{1}{2c^2 r} \ddot{Q}_{\beta\epsilon} n_{\epsilon}.$$

Thus,

$$H_{\alpha} = \frac{1}{2c^3 r} e_{\alpha\beta\gamma} \ddot{Q}_{\beta\epsilon} n_{\gamma} n_{\epsilon}$$

and

$$\frac{dI}{d\Omega} = \frac{1}{16\pi c^5} \ddot{Q}_{\beta\epsilon} \ddot{Q}_{\beta'\epsilon'} e_{\alpha\beta\gamma} e_{\alpha\beta'\gamma'} \overline{n_{\gamma} n_{\epsilon} n_{\gamma'} n_{\epsilon'}}.$$

Consider a polar system of coordinates with the polar axis in the direction of the incident beam, and the particle e_2 , m_2 at the origin. The average should be taken for a fixed value of the component $n_z \equiv n_3 = \cos \vartheta$, where ϑ represents the direction in which the radiation is emitted. It is easy to show that

$$\left. \begin{aligned} \overline{n_i n_k} &= \frac{1}{2} \delta_{ik} (1 - n_3^2), \\ \overline{n_i n_k n_l n_m} &= \frac{1}{8} (\delta_{ik} \delta_{lm} + \delta_{il} \delta_{km} + \delta_{im} \delta_{kl}) (1 - n_3^2)^2, \\ \overline{n_i} &= \overline{n_i n_k n_l} = 0, \end{aligned} \right\} \quad (2)$$

where the subscripts i, k, l assume the values 1, 2.

Using (2) and the equation

$$e_{\alpha\beta\gamma} e_{\alpha\beta'\gamma'} = \delta_{\beta\beta'} \delta_{\gamma\gamma'} - \delta_{\beta\gamma'} \delta_{\gamma\beta'},$$

we have

$$\begin{aligned} \frac{dI}{d\Omega} &= \frac{1}{16\pi c^5} \left\{ (\ddot{Q}_{33}^2 - \ddot{Q}_{33}^2) \cos^4 \vartheta + \right. \\ &\quad + \frac{1}{2} (\ddot{Q}_{33}^2 - 3\ddot{Q}_{33}^2 + 6\ddot{Q}_{33}^2 - 2\ddot{Q}_{33}\ddot{Q}_{33}) \sin^2 \vartheta \cos^2 \vartheta + \\ &\quad \left. + \frac{1}{8} [2\ddot{Q}_{33}^2 - (\ddot{Q}_{33})^2 - 3\ddot{Q}_{33}^2 + 2\ddot{Q}_{33}\ddot{Q}_{33}] \sin^4 \vartheta \right\}. \end{aligned} \quad (3)$$

Substituting (3) into (1) we have finally

$$\frac{d\tau_n}{d\Omega} = A + BP_2(\cos \vartheta) + CP_4(\cos \vartheta), \quad (4)$$

where P_2 and P_4 are the Legendre polynomials (cf. Appendix II)

and

$$\left. \begin{aligned} A &= \frac{1}{120c^5} \int_{-\infty}^{\infty} \int_0^{\infty} [3\ddot{Q}_{\beta\beta}^2 - (\ddot{Q}_{\beta\beta})^2] s ds dt, \\ B &= \frac{1}{168c^5} \int_{-\infty}^{\infty} \int_0^{\infty} [-3\ddot{Q}_{\beta\beta}^2 + 2(\ddot{Q}_{\beta\beta})^2 + 9\ddot{Q}_{\beta 3}^2 - 6\ddot{Q}_{33}\ddot{Q}_{\beta\beta}] s ds dt, \\ C &= \frac{1}{280c^5} \int_{-\infty}^{\infty} \int_0^{\infty} [-2\ddot{Q}_{\beta\beta}^2 + 2\ddot{Q}_{\beta 3}^2 - (\ddot{Q}_{\beta\beta})^2 - \\ &\quad - 35\ddot{Q}_{33}^2 + 10\ddot{Q}_{33}\ddot{Q}_{\beta\beta}] s ds dt. \end{aligned} \right\} \quad (5)$$

686. The total effective emission is

$$\kappa = \int \frac{dx_n}{d\Omega} d\Omega.$$

Using Eqns. (4) and (5) of the preceding solution, we have (*cf.* Appendix II):

$$\kappa = 4\pi A = \frac{\pi}{30c^5} \int_{-\infty}^{\infty} \int_0^{\infty} [3\ddot{Q}_{\alpha\beta}^2 - \ddot{Q}_{\beta\beta}^2] s ds dt. \quad (1)$$

Let x_α be the cartesian components of the relative position vector of the particles and $v_\alpha = \dot{x}_\alpha$ the cartesian components of the corresponding velocities. Hence, bearing in mind the equation of relative motion of the particles, we have

$$\ddot{x}_\alpha = \frac{2e^2 x_\alpha}{mr^3}, \quad \ddot{x}_\alpha = \frac{2e^2}{m} \cdot \frac{rx_\alpha - 3x_\alpha v_r}{r^6},$$

where $v_r = \dot{r}$. Substituting these expressions into (1) we have

$$\kappa = \frac{4\pi e^6}{15m^2c^5} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{v^2 + 11v_\alpha^2}{r^4} s ds dt, \quad (2)$$

where $v^2 = v_\alpha^2 + v_r^2$.

In view of the conservation of energy and angular momentum, $v^2 = v_0^2 - 4e^2/mr$ and $v_\alpha = v_0 s/r$. In order to evaluate the integral in (2) it is convenient to substitute $dt = dr/v_r = dr/(v^2 - v_\alpha^2)^{1/2}$. The final result is

$$\kappa = \frac{4\pi}{9} \cdot \frac{e^4 v_0^3}{mc^5}.$$

3. Interaction of charged particles with radiation

687. The momentum of the field is

$$\mathbf{G} = \int \mathbf{g} dV,$$

where $\mathbf{g} = \mathbf{E} \times \mathbf{H}/4\pi c$ and the integral is evaluated over all space. The magnetic field of the moving particle is $\mathbf{H} = \mathbf{v} \times \mathbf{E}/c$, since in the rest system of the particle (S') the magnetic field is zero. Hence,

$$\mathbf{g} = \frac{1}{4\pi c^2} [\mathbf{v} E^2 - \mathbf{E}(\mathbf{v} \cdot \mathbf{E})].$$

Using Eqn. (X.25) we have

$$E_x = E'_x, \quad E_y = \frac{E'_y}{\sqrt{1-\beta^2}}, \quad E_z = \frac{E'_z}{\sqrt{1-\beta^2}},$$

where the x -axis is parallel to $\mathbf{v}$. Moreover, in view of the Lorentz contraction, the relation between the volume elements is $dV = dV'(1 - \beta^2)^{1/2}$ and hence,

$$\mathbf{G} = \frac{\mathbf{v}}{4\pi c^2 \sqrt{1-\beta^2}} \int (E_y'^2 + E_z'^2) dV' = \frac{\mathbf{v}}{4\pi c^2 \sqrt{1-\beta^2}} \cdot \frac{2}{3} \int E'^2 dV'. \quad (1)$$

The last transformation is a consequence of the spherical symmetry of the field in the S' system.

If it is assumed that the rest mass of the particle is of purely electromagnetic origin, *i.e.* it is the mass of its electric field and is given by Einstein's relation $W' = m_0 c^2$, then

$$m_0 = \frac{1}{c^2} \cdot \frac{1}{8\pi} \int E'^2 dV'. \quad (2)$$

The momentum of the field should then be equal to $m_0 \mathbf{v}/(1 - \beta^2)^{1/2}$, although it follows from Eqn. (1) that this cannot be so†. The momentum of the field depends on the velocity $\mathbf{v}$ just as for a particle, *i.e.*

$$\mathbf{G} = \frac{m'_0 \mathbf{v}}{\sqrt{1-\beta^2}}. \quad (3)$$

However, the "mass" is $m'_0 = \frac{4}{3} m_0 \neq m_0$, *i.e.* it is not equal to

† On this assumption the energy of the field should be equal to $m_0 c^2 / (1 - \beta^2)^{1/2}$; it will be shown in the next problem that this is not correct.

the rest mass of the particle m_0 as given by Eqn. (2).

The presence of the coefficient $4/3$ in the expression for $\mathbf{G}$ shows that the energy and momentum of the electromagnetic field of the particle do not form a 4-vector and cannot be identified with the energy and momentum of the particle.

It should be noted that the electromagnetic mass defined by Eqn. (2) is infinite for a point particle.

$$688. \quad W_m = \frac{1}{8\pi} \int H^2 dV = \frac{1}{2} \cdot \frac{m'_0 v^2}{\sqrt{1-\beta^2}},$$

where m'_0 is defined in the preceding solution.

The total electromagnetic energy of the particle is

$$W = \frac{1}{8\pi} \int (E^2 + H^2) dV = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - \frac{1}{4} \sqrt{1-\beta^2} \right)$$

which is different from the velocity dependence $m_0 c^2 / (1 - \beta^2)^{1/2}$ of the energy of a particle (cf. Problem 687).

689. We shall neglect terms of the order of v/c and higher, and consider the effect of an element de_1 on another element de_2 . The Coulomb part of the electric field is spherically symmetric and does not contribute to the self-force. The quasi-stationary magnetic field cannot contribute to it either. Thus, it is sufficient to consider only that part of the electric field strength $d\mathbf{E}$ of the element de_1 which depends on the acceleration. The force acting on the element de_2 is

$$d\mathbf{F} = -de_2 d\mathbf{E} = \frac{de_1 de_2}{c^2 r} [\dot{\mathbf{v}} - \mathbf{r}_0 (\mathbf{r}_0 \cdot \dot{\mathbf{v}})],$$

where $\mathbf{r}_0 = \mathbf{r}/r$, and $\mathbf{r}$ is the position vector of de_2 relative to de_1 . The total force acting on the particle is

$$\mathbf{F} = \int d\mathbf{F} = -\frac{4}{3} \cdot \frac{W_0}{c^2} \dot{\mathbf{v}},$$

where $W_0 = \frac{1}{2} \int de_1 de_2 / r$ is the energy of the electromagnetic field of the particle at rest. The factor $4/3$ is obtained as a result of integration over the directions $\mathbf{r}_0$. If the rest mass of the particle is defined by $m'_0 = 4W_0/3c^2$ (cf. Problem 687) then the total self-force is given by

$$\mathbf{F} = -m'_0 \dot{\mathbf{v}}.$$

Thus, when retarded effects are neglected, this force becomes identical with the inertial force.

690. The force acting on an element of charge de_2 due to an element de_1 is determined by the acceleration $\dot{\mathbf{v}}$ of the latter at the time t' :

$$dF(t) = -\frac{de_1 de_2}{c^2 r} [\dot{\mathbf{v}} - \mathbf{r}_0 (\mathbf{r}_0 \cdot \dot{\mathbf{v}})] \Big|_{t'=t-\frac{r}{c}}.$$

Expanding the acceleration $\dot{\mathbf{v}}$ in powers of $t' - t = -r/c$ we have

$$\dot{\mathbf{v}}(t') = \dot{\mathbf{v}}(t) + (t' - t) \ddot{\mathbf{v}}(t) = \dot{\mathbf{v}}(t) - \frac{r}{c} \ddot{\mathbf{v}}(t).$$

The required force is found by integrating with respect to e_1 and e_2 (cf. preceding solution):

$$\mathbf{F} = -m_0' \dot{\mathbf{v}} + \frac{2}{3} \frac{e^2}{c^3} \ddot{\mathbf{v}}.$$

The second term on the right-hand side is the force of radiative friction. It is independent of the structure of the particle and retains its form as the size of the particle tends to zero. The proper energy W_0 , and therefore the electromagnetic mass m_0 , become infinite in the limit of a point particle. The terms which have been neglected are of the order of $(t' - t)^n$, where $n \geq 2$, and are proportional to r_0^{n-1} and tend to zero in the limit of a point particle (r_0 is the radius of the particle).

691. $T = m^2 c^3 a_0^3 / 4 e^4 = 3.2 \times 10^{-13}$ sec. The approximations made in the solution will be valid if the energy loss per period is small in comparison with the total energy of the electron, i.e. $\tau |d\mathcal{E}/dt| \ll |\mathcal{E}|$, and hence $ac/v \gg r_0 = e^2/mc^2$, where r_0 is the classical radius of the electron. This condition ceases to be valid at distances of the order of 10^{-13} cm, when classical electrodynamics breaks down.

The very short lifetime obtained from the above calculations clearly shows that classical ideas about the motion of an electron in an atom (definite orbits and so on) are incorrect. These (and other) fundamental difficulties of classical physics have been removed by quantum mechanics.

$$692. \quad \mathcal{E}(t) = mc^2 \coth \left[\frac{2e^4 H^2}{3m^3 c^5} t + \frac{1}{2} \ln \frac{\mathcal{E}_0 + mc^2}{\mathcal{E}_0 - mc^2} \right].$$

When $t \rightarrow \infty$ we have $\mathcal{E}(t) \rightarrow mc^2$, i.e. the particle comes to rest.

The radius of the orbit can be expressed in terms of $\mathcal{E}(t)$:

$$r(t) = \frac{cp}{eH} = \frac{1}{eH} \sqrt{\mathcal{E}^2(t) - m^2 c^4}.$$

When $t \rightarrow \infty$ we have $r(t) \rightarrow 0$, *i.e.* the particle moves on a closing spiral.

693. $\mathcal{E}_{cr} = mc^2[(3a^2\omega/2cr_0)^{1/3}]$ where $r_0 = e^2/mc^2$.

694. The equation of motion (including the radiative friction term) is

$$\ddot{\mathbf{r}} + \omega_0^2 \mathbf{r} = \frac{2}{3} \cdot \frac{e^2}{mc^3} \ddot{\ddot{\mathbf{r}}}. \quad (1)$$

The characteristic equation corresponding to (1) is

$$k^2 + \omega_0^2 = \frac{2}{3} \cdot \frac{e^2}{mc^3} k^3. \quad (2)$$

Since the radiative friction force is small in comparison with the quasi-elastic force, Eqn. (2) can be solved by successive approximations. On the zero-order approximation the right-hand side is neglected and $k \simeq k_0 = \pm i\omega_0$. On the first-order approximation k_0 may be substituted for k on the right-hand side of (2) so that $k \simeq k_1 = \pm i\omega_0 - \frac{1}{2}\gamma$ where

$$\gamma = \frac{2}{3} \cdot \frac{e^2 \omega_0^2}{mc^3}. \quad (3)$$

Consider the solution corresponding to the negative sign:

$$\mathbf{r} = \mathbf{r}_0 e^{-\frac{\gamma t}{2}} \cdot e^{-i\omega_0 t} \quad (t > 0). \quad (4)$$

This solution holds when $\gamma \ll \omega_0$ and represents damped oscillations.

The energy of the oscillator falls off as the square of the modulus of its amplitude:

$$W = W_0 e^{-\gamma t}. \quad (5)$$

The quantity $1/\gamma$ can therefore be called the lifetime of the excited state of the oscillator.

The electric field strength of the radiation is proportional to $\mathbf{r}$ so that

$$\mathbf{E} = \int_{-\infty}^{+\infty} \mathbf{E}_\omega e^{-i\omega t} d\omega = \begin{cases} \mathbf{E}_0 e^{-i\omega_0 t} e^{-\frac{\gamma}{2} t} & t > 0, \\ 0 & t < 0. \end{cases}$$

and

$$\mathbf{E}_\omega = \frac{\mathbf{E}}{2\pi} \int_0^\infty e^{-\left(\frac{\gamma}{2} + i\omega_0\right)t + i\omega t} dt = \frac{\mathbf{E}_0}{2\pi \left[i(\omega - \omega_0) - \frac{\gamma}{2} \right]}.$$

Hence, the spectral distribution of the emitted intensity is

$$\frac{dI_\omega}{d\omega} = \frac{I_0\gamma}{2\pi} \cdot \frac{1}{(\omega - \omega_0)^2 + \frac{\gamma^2}{4}}, \quad (6)$$

where $I_0 = \int_{-\infty}^{+\infty} dI_\omega$ is the total intensity. The spectral distribution (6) is of the characteristic resonance form (Fig. 97).

The width of the spectral line is characterised by the quantity $\Delta\omega = \gamma$. The natural width is very small [on the wavelength graph it is $\Delta\lambda = \Delta(2\pi c/\omega) = 4\pi r_0/3 = 1.17 \times 10^{-12}$ cm].

If it is assumed that the radiation is emitted in discrete portions (this assumption does, of course, take us outside the framework of classical electrodynamics), then the uncertainty in the

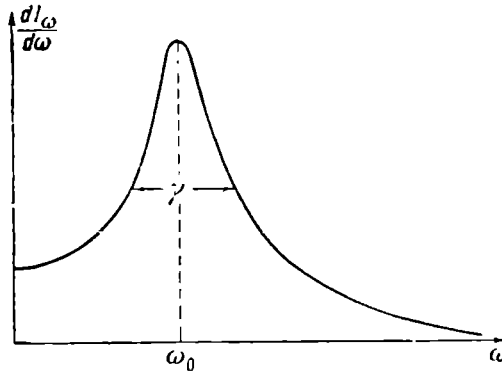


Fig. 97.

energy of the photons, $\Delta\mathcal{E} = \hbar\Delta\omega = \hbar\gamma$, is related to the lifetime of the excited state, $\tau = 1/\gamma$, by

$$\Delta\mathcal{E} \cdot \tau = \hbar. \quad (7)$$

This is a special case of the very general quantum mechanical uncertainty relation for energy and time.

$$695. \quad \frac{dI_\omega}{d\omega} = I_0 e^{-\left(\frac{\omega - \omega_0}{\gamma_D}\right)^2},$$

where $\gamma_D = (2kT\omega_0^2/mc^2)^{1/2}$ is the Doppler line width and I_0 is the intensity at $\omega = \omega_0$. The line width is a function of temperature and may be used as a measure of the temperature of a gas.

696.

$$\frac{dI_{\omega}}{d\omega} = \frac{I\Gamma}{2\pi} \cdot \frac{1}{(\omega - \omega_0)^2 + \frac{\Gamma^2}{4}},$$

where $I = \int_{-\infty}^{+\infty} dI_{\omega}$.

697. If the wave is polarised along the x -axis then

$$x_{\omega} = \frac{eE_{x\omega}}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}, \quad (1)$$

where $\gamma = \frac{2}{3}(e^2 \omega_0^2 / mc^3)$.

The energy absorbed by the oscillating electron is

$$\Delta W = \int_{-\infty}^{+\infty} eE_x(t) \dot{x}(t) dt = \frac{2\pi e^2}{m} \int_{-\infty}^{+\infty} |E_{x\omega}|^2 \frac{2\omega^2\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} d\omega$$

since $(\dot{x})_{\omega} = -i\omega x_{\omega}^{\dagger}$. The integrand in the latter expression represents the spectral distribution of absorption. It follows that γ is a measure of the width of the absorption line, just as in the case of emission. Since, by definition, the width of the spectral distribution is large compared with the natural line width γ , it follows that

$$\Delta W = \frac{2\pi e^2}{m} |E_{x\omega_0}|^2 2\omega_0^2\gamma \int_{-\omega_0}^{\infty} \frac{d\xi}{(2\omega_0\xi)^2 + \omega_0^2\gamma^2},$$

where $\xi = \omega - \omega_0$. The lower limit may be replaced by $-\infty$ since $\gamma \ll \omega_0$. The result of the integration is

$$\Delta W = \frac{2\pi^2 e^2}{m} |E_{x\omega_0}|^2 = 2\pi^2 r_0 c S_{\omega_0},$$

where $r_0 = e^2/mc^2$ is the classical radius of the electron. As can be seen, the result is independent of γ . The dependence on frequency is indirect: the quantity ΔW is proportional to the spectral density S_{ω_0} at the resonant frequency ω_0 of the oscillator. It is clear from the above derivation that the same result would be obtained for an unpolarised and curved group of waves incident on an anisotropic oscillator. The density S_{ω} would then be the

† It is easy to show that

$$\int_{-\infty}^{+\infty} A(t) \cdot B(t) dt = 2\pi \int_0^{\infty} (A_{\omega} B_{\omega}^* + A_{\omega}^* B_{\omega}) d\omega.$$

sum of the intensities of all the polarised waves of frequency ω entering into this group.

698. a) $\Delta W = 2\pi^2 r_0 c S_{\omega_0} \cos^2 \vartheta$;

b) $\Delta W = \pi^2 r_0 c S_{\omega_0} \sin^2 \vartheta$;

c) $\Delta W = \frac{2}{3} \pi^2 r_0 c S_{\omega_0}$.

699. The equation of motion of the harmonic oscillator is

$$\ddot{\mathbf{r}} + \omega_0^2 \mathbf{r} = \frac{2e^2}{3mc^3} \ddot{\mathbf{r}} + \frac{e}{m} \mathbf{E}_0 e^{-i\omega t}, \quad (1)$$

where it is assumed that the electric field is uniform in the region occupied by the oscillator, and the effects of the magnetic force, *i.e.* effects of the order of v/c , can be neglected.

The solution of (1) which corresponds to forced oscillations is

$$\mathbf{r} = \frac{e}{m} \cdot \frac{\mathbf{E}}{\omega_0^2 - \omega^2 - i\omega\gamma}.$$

Hence the time average of the intensity scattered in a given direction is given by

$$\frac{\overline{dI}}{d\Omega} = \frac{1}{4\pi c^3} |\overline{e\ddot{\mathbf{r}} \times \mathbf{n}}|^2 = \frac{cE_0^2 r_0^2}{8\pi} \cdot \frac{\omega^4 \sin^2 \vartheta}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2},$$

where ϑ is the angle between the direction of propagation of the scattered radiation, $\mathbf{n}$, and the direction of polarisation of the incident wave. The time average of the energy flux density in the incident wave is $\overline{\gamma}_0 = cE_0^2/8\pi$. The differential scattering cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{\overline{\gamma}_0} \frac{\overline{dI}}{d\Omega} = r_0^2 \frac{\omega^4 \sin^2 \vartheta}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}.$$

The total scattering cross-section is therefore given by

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} r_0^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}.$$

For a strongly bound electron, when $\omega_0 \gg \omega$,

$$\sigma = \frac{8\pi}{3} \cdot \frac{r_0^2 \omega^4}{\omega_0^4}.$$

A characteristic feature of this relation is the dependence on the 4th power of the frequency ω .

For a weakly bound electron and small radiative friction $\gamma \simeq 0$, $\omega \simeq 0$, and $\sigma = 8\pi r_0^2/3$.

$$700. \quad \mathbf{H} = -\frac{Ae^2}{mc^2r}(\mathbf{e}_\alpha \cos \vartheta - i\mathbf{e}_\vartheta) e^{-i(\omega t' - \alpha)},$$

where ϑ , α are the polar angles of the direction of propagation of the scattered wave (the incident wave is assumed to propagate in the direction of the z -axis) and A is the amplitude of the incident wave.

It is clear from the expression for $\mathbf{H}$ that the scattered wave will, in general, be elliptically polarised. The waves scattered in the forward and backward directions will be circularly polarised, while the waves scattered into the xy plane will be linearly polarised. The differential and total scattering cross-sections are given by

$$\frac{d\sigma}{d\Omega} = r_0^2 \frac{(1 + \cos^2 \vartheta)}{2}, \quad \sigma = \frac{8\pi}{3} r_0^2.$$

$$701. \quad \rho = \cos^2 \vartheta.$$

702. For a linearly polarised wave

$$d\sigma_{\text{pol}} = r_0^2 \frac{(1 - \beta^2)(1 - \beta)^2}{(1 - \beta \cos \vartheta)^6} [(1 - \beta \cos \vartheta)^2 - (1 - \beta^2) \sin^2 \vartheta \cos^2 \alpha],$$

where ϑ , α are the polar angles of the direction of propagation of the scattered wave, the z -axis is parallel to the velocity $\mathbf{v}$ of the charge, $\beta = v/c$, and the azimuth angle α is measured from the direction of the vector $\mathbf{E}$ in the incident wave.

For an unpolarised wave

$$d\sigma_{\text{unpol}} = r_0^2 \frac{(1 - \beta^2)(1 - \beta)^2}{(1 - \beta \cos \vartheta)^6} \left[\frac{1 + \beta^2}{2} (1 + \cos^2 \vartheta) - 2\beta \cos \vartheta \right].$$

703. Consider the motion of the oscillator in a magnetic field $\mathbf{H} \parallel z$ and suppose that

$$\omega_0 \gg \frac{eH}{2mc} = \omega_L.$$

Following the method employed in the solution of Problem 597, we have

$$\mathbf{r} = A_1(\mathbf{e}_x + i\mathbf{e}_y)e^{-i(\omega_0 - \omega_L)t} + A_2(\mathbf{e}_x - i\mathbf{e}_y)e^{-i(\omega_0 + \omega_L)t} + A_3\mathbf{e}_ze^{-i\omega_L t},$$

where A_1 , A_2 , and A_3 are the constants of integration which can be determined from the initial conditions.

It is clear from the expression for $\mathbf{r}$ that when the oscillator is placed in the magnetic field it becomes anisotropic and the frequency of its oscillations splits into three frequencies ω_0 and $\omega_0 \pm \omega_L$. When the radiation is observed in an arbitrary direction, the polarisation of each of the monochromatic components is in general found to be elliptical. In particular, two spectral lines are observed along the z -axis (along the field $\mathbf{H}$). These lines are circularly polarised in opposite directions. All three monochromatic components are observed in the direction perpendicular to the field and are linearly polarised. In this situation, the electric vector of the undisplaced spectral line is parallel to the magnetic field, while the electric vectors of the two displaced lines are perpendicular to this direction.

4. Expansion of an electromagnetic field in terms of plane waves

$$705. \quad \mathbf{E}_\omega(\mathbf{r}) = -\text{grad } \varphi_\omega(\mathbf{r}) + i \frac{\omega}{c} \mathbf{A}_\omega(\mathbf{r}),$$

$$\mathbf{H}_\omega(\mathbf{r}) = \text{curl } \mathbf{A}_\omega(\mathbf{r}),$$

$$\mathbf{E}_\mathbf{k}(t) = -ik\varphi_\mathbf{k}(t) - \frac{1}{c} \dot{\mathbf{A}}_\mathbf{k},$$

$$\mathbf{H}_\mathbf{k}(t) = ik \times \mathbf{A}_\mathbf{k}(t),$$

$$\mathbf{E}_{\mathbf{k}\omega} = -ik\varphi_{\mathbf{k}\omega} + i \frac{\omega}{c} \mathbf{A}_{\mathbf{k}\omega},$$

$$\mathbf{H}_{\mathbf{k}\omega} = ik \times \mathbf{A}_{\mathbf{k}\omega}.$$

$$706. \quad \text{a) } \text{curl } \mathbf{E}_\omega = \frac{i\omega\mu}{c} \mathbf{H}_\omega, \quad \text{div } \epsilon \mathbf{E}_\omega = 4\pi\rho_\omega,$$

$$\text{curl } \mathbf{H}_\omega = -\frac{i\omega\epsilon}{c} \mathbf{E}_\omega + \frac{4\pi}{c} \mathbf{j}_\omega, \quad \text{div } \mu \mathbf{H}_\omega = 0;$$

$$\text{b) } ik \times \mathbf{E}_\mathbf{k} = -\frac{1}{c} \dot{\mathbf{B}}_\mathbf{k}, \quad ik \cdot \mathbf{D}_\mathbf{k} = 4\pi\rho_\mathbf{k},$$

$$ik \times \mathbf{H}_\mathbf{k} = \frac{1}{c} \dot{\mathbf{D}}_\mathbf{k} + \frac{4\pi}{c} \mathbf{j}_\mathbf{k}, \quad \mathbf{k} \cdot \mathbf{B}_\mathbf{k} = 0;$$

$$\text{c) } \mathbf{k} \times \mathbf{E}_{\mathbf{k}\omega} = \frac{\omega\mu}{c} \mathbf{H}_{\mathbf{k}\omega}, \quad ik \cdot \mathbf{E}_{\mathbf{k}\omega} = \frac{4\pi\rho_{\mathbf{k}\omega}}{\epsilon},$$

$$ik \times \mathbf{H}_{\mathbf{k}\omega} = -\frac{i\omega\epsilon}{c} \mathbf{E}_{\mathbf{k}\omega} + \frac{4\pi}{c} \mathbf{j}_{\mathbf{k}\omega}, \quad \mathbf{k} \cdot \mathbf{H}_{\mathbf{k}\omega} = 0.$$

$$707. \quad \text{a) } \Delta\varphi_\omega + \frac{\epsilon\mu\omega^2}{c^2} \varphi_\omega = -\frac{4\pi\rho_\omega}{\epsilon}, \quad \Delta\mathbf{A}_\omega + \frac{\epsilon\mu\omega^2}{c^2} \mathbf{A}_\omega = -\frac{4\pi\mu\mathbf{j}_\omega}{c},$$

$$\text{div } \mathbf{A}_\omega - \frac{i\omega\epsilon\mu}{c} \varphi_\omega = 0;$$

$$\text{b) } \epsilon\mu\ddot{\varphi}_\mathbf{k} + k^2c^2\varphi_\mathbf{k} = \frac{4\pi c^2\rho_\mathbf{k}}{\epsilon},$$

$$\epsilon\mu\ddot{\mathbf{A}}_\mathbf{k} + k^2c^2\mathbf{A}_\mathbf{k} = 4\pi c\mu\mathbf{j}_\omega, \quad -ick \cdot \mathbf{A}_\mathbf{k} + \epsilon\mu\dot{\varphi}_\mathbf{k} = 0;$$

$$c) \left(k^2 - \frac{\epsilon_1 \mu \omega^2}{c^2} \right) \zeta_{\mathbf{k}\omega} = \frac{4\pi}{\epsilon} \rho_{\mathbf{k}\omega}, \quad \left(k^2 - \frac{\epsilon_1 \mu \omega^2}{c^2} \right) \mathbf{A}_{\mathbf{k}\omega} = \frac{4\pi\mu}{c} \mathbf{j}_{\mathbf{k}\omega},$$

$$\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}\omega} - \frac{\omega \epsilon_1 \mu}{c} \zeta_{\mathbf{k}\omega} = 0.$$

708. Consider Eqn. (XII.40'). Integration with respect to the angles yields

$$\varphi_{\mathbf{k}} = \frac{1}{(2\pi)^3} \int \varphi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} (d\mathbf{r}) = \frac{e}{2\pi^2 k} \int_0^\infty \sin kr \, dr.$$

The last integral does not, in general, have a definite value, since the quantity

$$I_N = \int_0^N \sin kr \, dr = \frac{1 - \cos kN}{k}$$

does not tend to any particular limit as $N \rightarrow \infty$. It is easy to see, however, that the indeterminate term containing $\cos kN$ does not contribute to the potential $\varphi(\mathbf{r})$ when I_N is substituted into (XII.40) with $N \rightarrow \infty$.

This follows from the fact that as $N \rightarrow \infty$, owing to rapid oscillations,

$$\int_0^\infty \frac{\cos kN}{k} e^{i\mathbf{k} \cdot \mathbf{r}} (d\mathbf{k}) \rightarrow 0.$$

Thus, in effect,

$$\zeta_{\mathbf{k}} = \frac{e}{2\pi^2 k^2}.$$

We note that the result $I = \lim_{N \rightarrow \infty} I_N = 1/k$ may be obtained, for example, if I is defined as the limit of $\int_0^\infty \exp(-br) \sin kr \, dr$ as $b \rightarrow 0$.

The same result can also be obtained in another way. The application of the Laplace operator Δ to both sides of the equation

$$\varphi(\mathbf{r}) = \int \varphi_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r}) d\mathbf{k} \text{ yields}$$

$$(\Delta \varphi)_{\mathbf{k}} = -k^2 \varphi_{\mathbf{k}}.$$

On the other hand, the expression for the Fourier component $(\Delta \varphi)_{\mathbf{k}} = -e/2\pi^2$ may be obtained from a Fourier expansion of both sides of Poisson's equation $\Delta \varphi = -4\pi e \delta(\mathbf{r})$. The required result

is obtained by equating these two expressions for $(\Delta\varphi)_{\mathbf{k}}$.

$$709. \quad \mathbf{E}_{\mathbf{k}} = -i\mathbf{k}\varphi_{\mathbf{k}} = -\frac{ie\mathbf{k}}{2\pi^2k^2}.$$

710. Since the volume density is $\rho(\mathbf{r}, t) = e\delta(\mathbf{r} - \mathbf{v}t)$ it follows that

$$\begin{aligned} \rho_{\mathbf{k}\omega} &= \frac{e}{(2\pi)^4} \int \int \delta(\mathbf{r} - \mathbf{v}t) e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} (d\mathbf{r}) dt = \\ &= \frac{e}{(2\pi)^4} \int_{-\infty}^{+\infty} e^{i(\omega - \mathbf{k}\cdot\mathbf{v})t} dt = \frac{e}{8\pi^3} \delta(\mathbf{k} \cdot \mathbf{v} - \omega). \end{aligned}$$

Hence, using the results of the preceding solution, we have

$$\varphi_{\mathbf{k}\omega} = \frac{e}{2\pi^2} \cdot \frac{\delta(\mathbf{k} \cdot \mathbf{v} - \omega)}{k^2 - \frac{\omega^2}{c^2}},$$

and similarly

$$\mathbf{A}_{\mathbf{k}\omega} = \frac{e\mathbf{v}}{2\pi^2c} \cdot \frac{\delta(\mathbf{k} \cdot \mathbf{v} - \omega)}{k^2 - \frac{\omega^2}{c^2}}.$$

Using the expressions for the component of the field strength (cf. solution of Problem 705) we have

$$\begin{aligned} \mathbf{E}_{\mathbf{k}\omega} &= i \frac{e}{2\pi^2} \cdot \frac{\delta(\mathbf{k} \cdot \mathbf{v} - \omega)}{k^2 - \frac{\omega^2}{c^2}} \left(-\mathbf{k} + \frac{\mathbf{v}\omega}{c^2} \right), \\ \mathbf{H}_{\mathbf{k}\omega} &= i\mathbf{k} \times \mathbf{A}_{\mathbf{k}\omega} = i \frac{e}{2\pi^2c} (\mathbf{k} \times \mathbf{v}) \frac{\delta(\mathbf{k} \cdot \mathbf{v} - \omega)}{k^2 - \frac{\omega^2}{c^2}}. \end{aligned}$$

All the field components contain the factor $\delta(\mathbf{k}\cdot\mathbf{v} - \omega)$ which is due to the dispersion relation $\omega = \mathbf{k}\cdot\mathbf{v}$. Hence, all the Fourier expansions of the electromagnetic field in this case are, in fact, three- rather than four-dimensional. For example, the potential is given by

$$\varphi = \int_{(\mathbf{k})} \int_{-\infty}^{\infty} \frac{e}{2\pi^2} \cdot \frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})}{k^2 - \frac{\omega^2}{c^2}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} (d\mathbf{k}) d\omega = \int \varphi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{r}} (d\mathbf{k}),$$

where

$$\varphi_{\mathbf{k}}(t) = \frac{e}{2\pi^2} \cdot \frac{e^{-i(\mathbf{k}\cdot\mathbf{v})t}}{k^2 - \frac{\omega^2}{c^2}}.$$

712. Consider the scalar potential. According to equations (c) of the solution of Problem 707 ($\epsilon = \mu = 1$), $\varphi_{\mathbf{k}\omega} = 4\pi\rho_{\mathbf{k}\omega}/(k^2 - \omega^2/c^2)$. The Fourier component of the charge

density is given by

$$\begin{aligned}\rho_{\mathbf{k}\omega} &= -\frac{1}{(2\pi)^4} \int [\mathbf{p} \cdot \text{grad } \delta(\mathbf{r} - \mathbf{v}t)] e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} (d\mathbf{r}) dt = \\ &= \frac{1}{(2\pi)^4} \int [\mathbf{p} \cdot \text{grad } e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}] \delta(\mathbf{r} - \mathbf{v}t) (d\mathbf{r}) dt = -i \frac{\mathbf{p} \cdot \mathbf{k}}{(2\pi)^3} \delta(\omega - \mathbf{k} \cdot \mathbf{v}).\end{aligned}$$

The dispersion relation $\omega = \mathbf{k} \cdot \mathbf{v}$ is identical with that for the field of a uniformly moving point charge (*cf.* Problem 710). Using the method given in the solution of Problem 711 we have

$$\varphi(\mathbf{r}, t) = -\mathbf{p} \cdot \text{grad } \frac{1}{r^*} = \frac{\mathbf{p} \cdot \mathbf{r}_0}{r^{*3}}, \quad (1)$$

where

$$\mathbf{r}_0 = \left(x - vt, \frac{y}{\gamma}, \frac{z}{\gamma} \right), \quad r^* = \sqrt{(x - vt)^2 + \frac{1}{\gamma^2}(y^2 + z^2)}.$$

A similar calculation of the vector potential yields

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{m} \times \mathbf{r}^*}{r^{*3}} + \frac{\mathbf{v}(\mathbf{p} \cdot \mathbf{r}_0)}{cr^{*3}}. \quad (2)$$

713.

$$\begin{aligned}\text{a) } \mathbf{A} &= \frac{\mathbf{m}_0 \times \mathbf{r}^*}{\gamma r^{*3}}, \quad \varphi = 0; \\ \text{b) } \mathbf{A} &= \frac{\mathbf{m}_0 \times \mathbf{r}^*}{r^{*3}}, \quad \varphi = \frac{\mathbf{v} \cdot \mathbf{A}}{c}.\end{aligned}$$

716. Let us expand all the vectors in Maxwell's equations into irrotational and solenoidal parts[†] (or longitudinal and transverse; *cf.* Problem 715):

$$\left. \begin{aligned} \mathbf{E} &= \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}, & \mathbf{j} &= \mathbf{j}_{\parallel} + \mathbf{j}_{\perp}, \\ \mathbf{H} &= \mathbf{H}_{\perp}, & \mathbf{H}_{\parallel} &= 0. \end{aligned} \right\} \quad (1)$$

Equating the transverse parts of the vectors, we have from Maxwell's equations,

$$\left. \begin{aligned} \text{curl } \mathbf{E}_{\perp} &= -\frac{1}{c} \dot{\mathbf{H}}, & \text{curl } \mathbf{H} &= \frac{1}{c} \dot{\mathbf{E}}_{\perp} + \frac{4\pi}{c} \mathbf{j}_{\perp}, \\ \text{div } \mathbf{E}_{\perp} &= 0, & \text{div } \mathbf{H} &= 0. \end{aligned} \right\} \quad (2)$$

[†] The expansion of the electromagnetic field into longitudinal and transverse parts is used in one of the variants of quantum electrodynamics. In this expansion the transverse part of the field is quantised while the longitudinal part is not quantised; photons correspond to the former part.

The longitudinal (irrotational) part of the electric field is given by

$$\left. \begin{aligned} \operatorname{div} \mathbf{E}_{\parallel}(\mathbf{r}, t) &= 4\pi\rho(\mathbf{r}, t), \\ \operatorname{curl} \mathbf{E}_{\parallel}(\mathbf{r}, t) &= 0, \end{aligned} \right\} \quad (3)$$

which resemble the equations of electrostatics. The time enters into them as a parameter. It follows that $\mathbf{E}_{\parallel}$ is the Coulomb field.

717. According to the solution of 707(b),

$$\ddot{q}_{\mathbf{k}\lambda} + \omega_{\mathbf{k}}^2 q_{\mathbf{k}\lambda} = 0, \quad (1)$$

where $\omega_{\mathbf{k}} = kc$.

This is the equation of a linear harmonic oscillator. Its general solution is

$$q_{\mathbf{k}\lambda}(t) = a_{\mathbf{k}\lambda} e^{-i\omega t} + b_{\mathbf{k}\lambda} e^{i\omega t}.$$

The coefficients $a_{\mathbf{k}\lambda}$ and $b_{\mathbf{k}\lambda}$ are related by

$$\mathbf{e}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} = \mathbf{e}_{-\mathbf{k}\lambda}^* b_{-\mathbf{k}\lambda}^*, \quad \mathbf{e}_{\mathbf{k}\lambda} b_{\mathbf{k}\lambda} = \mathbf{e}_{-\mathbf{k}\lambda}^* a_{-\mathbf{k}\lambda}^*,$$

which follow from the fact that $\mathbf{A}(\mathbf{r}, t) = \mathbf{A}^*(\mathbf{r}, t)$.

If the unit vectors which describe the main states of polarisation of waves with opposite wave vectors, $\mathbf{k}$ and $-\mathbf{k}$, are such that

$$\mathbf{e}_{\mathbf{k}\lambda} = \mathbf{e}_{-\mathbf{k}\lambda}^*, \quad (2)$$

then

$$\left. \begin{aligned} a_{\mathbf{k}\lambda} &= b_{-\mathbf{k}\lambda}^*, \quad b_{\mathbf{k}\lambda} = a_{-\mathbf{k}\lambda}^* \\ q_{\mathbf{k}\lambda}(t) &= a_{\mathbf{k}\lambda} e^{-i\omega t} + a_{-\mathbf{k}\lambda}^* e^{i\omega t}. \end{aligned} \right\} \quad (3)$$

The fields can be expressed in terms of the coordinates $q_{\mathbf{k}\lambda}(t)$:

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{\pi\sqrt{2}} \int \mathbf{e}_{\mathbf{k}\lambda} q_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} (d\mathbf{k}), \quad (4)$$

$$\mathbf{H} = \operatorname{curl} \mathbf{A} = \frac{ic}{\pi\sqrt{2}} \int \mathbf{k} \times \mathbf{e}_{\mathbf{k}\lambda} q_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} (d\mathbf{k}). \quad (5)$$

Consider the energy of the electromagnetic field

$$W = \frac{1}{8\pi} \int (E^2 + H^2) (d\mathbf{r}).$$

Since $\mathbf{E}$, $\mathbf{H}$ are real, we have

$$\begin{aligned} \frac{1}{8\pi} \int E^2(d\mathbf{r}) &= \frac{1}{8\pi} \int \mathbf{E} \cdot \mathbf{E}^*(d\mathbf{r}) = \\ &= \frac{1}{2\pi^2} \frac{1}{8\pi} \int \int \int \sum_{\lambda\lambda'} \mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{e}_{\mathbf{k}'\lambda'} \dot{q}_{\mathbf{k}\lambda} \dot{q}_{\mathbf{k}'\lambda'}^* e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}(d\mathbf{r})(d\mathbf{k})(d\mathbf{k}') = \\ &= \frac{1}{2} \int \sum_{\lambda} \dot{q}_{\mathbf{k}\lambda} \dot{q}_{\mathbf{k}\lambda}^*(d\mathbf{k}), \end{aligned}$$

where we have used Eqn. (15) of Appendix I and the orthogonality of the polarisation unit vectors belonging to the same $\mathbf{k}$ but different λ ($\mathbf{e}_{\mathbf{k}1} \cdot \mathbf{e}_{\mathbf{k}2}^* = 0$). The energy of the magnetic field may be calculated in a similar way. The total energy of the electromagnetic field is thus given by

$$W = \frac{1}{2} \int \sum_{\lambda} (\dot{q}_{\mathbf{k}\lambda} \dot{q}_{\mathbf{k}\lambda}^* + \omega_{\mathbf{k}}^2 q_{\mathbf{k}\lambda} q_{\mathbf{k}\lambda}^*)(d\mathbf{k}). \quad (6)$$

It consists of the energies of the separate field oscillators

$$W_{\mathbf{k}\lambda} = \frac{1}{2} (\dot{q}_{\mathbf{k}\lambda} \dot{q}_{\mathbf{k}\lambda}^* + \omega_{\mathbf{k}}^2 q_{\mathbf{k}\lambda} q_{\mathbf{k}\lambda}^*). \quad (7)$$

The field energy (6) may be expressed directly in terms of the coefficients $a_{\mathbf{k}\lambda}$ with the aid of Eqn. (3):

$$W = 2 \int \sum_{\lambda} \omega_{\mathbf{k}}^2 a_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^*(d\mathbf{k}). \quad (8)$$

Similarly, the momentum of the field may be shown to be

$$\begin{aligned} \mathbf{G} &= \frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{H}(d\mathbf{r}) = \frac{1}{8\pi c} \int (\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H})(d\mathbf{r}) = \\ &= \frac{i}{2} \int \sum_{\lambda} \mathbf{k} (\dot{q}_{\mathbf{k}\lambda} q_{\mathbf{k}\lambda}^* - \dot{q}_{\mathbf{k}\lambda}^* q_{\mathbf{k}\lambda})(d\mathbf{k}). \end{aligned} \quad (9)$$

The oscillator coordinates $q_{\mathbf{k}\lambda}$ are analogous to the coordinates describing the normal oscillations of a mechanical system. The main difference is that the field is a system with an infinite number of degrees of freedom. This analogy may be employed in the application of formal methods of quantum mechanics to the solution of problems in quantum electrodynamics.

718.

$$\mathbf{A}(\mathbf{r}, t) = \frac{c}{\pi \sqrt{2}} \int \sum_{\lambda} \mathbf{e}_{\mathbf{k}\lambda} \left[Q_{\mathbf{k}\lambda}(t) \cos \mathbf{k} \cdot \mathbf{r} - \frac{1}{\omega} \dot{Q}_{\mathbf{k}\lambda}(t) \sin \mathbf{k} \cdot \mathbf{r} \right] (d\mathbf{k}),$$

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{\pi \sqrt{2}} \int \sum_{\lambda} \mathbf{e}_{\mathbf{k}\lambda} [\dot{Q}_{\mathbf{k}\lambda} \cos \mathbf{k} \cdot \mathbf{r} + \omega Q_{\mathbf{k}\lambda} \sin \mathbf{k} \cdot \mathbf{r}] (d\mathbf{k}),$$

$$\mathbf{H}(\mathbf{r}, t) = -\frac{1}{\pi \sqrt{2}} \int \sum_{\lambda} (\mathbf{k} \times \mathbf{e}_{\mathbf{k}\lambda}) \left[Q_{\mathbf{k}\lambda} \sin \mathbf{k} \cdot \mathbf{r} + \frac{1}{\omega} \dot{Q}_{\mathbf{k}\lambda} \cos \mathbf{k} \cdot \mathbf{r} \right] (d\mathbf{k}).$$

In deriving the expression for $\mathbf{E}(\mathbf{r}, t)$ we have made use of the fact that

$$\ddot{Q}_{\mathbf{k}\lambda} + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}\lambda} = 0.$$

The expression for the field energy can be derived most simply from Eqn. (8) of the preceding solution by expressing the coefficients $a_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}^*$ in terms of $Q_{\mathbf{k}\lambda}$ and $\dot{Q}_{\mathbf{k}\lambda}$:

$$a_{\mathbf{k}\lambda} = \frac{1}{2} Q_{\mathbf{k}\lambda} e^{i\omega t} + \frac{i}{2\omega} \dot{Q}_{\mathbf{k}\lambda} e^{i\omega t},$$

$$a_{\mathbf{k}\lambda}^* = \frac{1}{2} Q_{\mathbf{k}\lambda} e^{-i\omega t} - \frac{i}{2\omega} \dot{Q}_{\mathbf{k}\lambda} e^{-i\omega t}.$$

Hence,

$$W = \frac{1}{2} \sum_{\lambda} \int (\dot{Q}_{\mathbf{k}\lambda}^2 + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}\lambda}^2) (d\mathbf{k}).$$

It is evident from the latter equation that the energy of a free electromagnetic field may be written as a sum of the energies of field oscillators, which is similar in form to the corresponding expression for a mechanical oscillating system:

$$W = \int \sum_{\lambda} W_{\mathbf{k}\lambda} (d\mathbf{k}), \quad (9)$$

where

$$W_{\mathbf{k}\lambda} = \frac{1}{2} (\dot{Q}_{\mathbf{k}\lambda}^2 + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}\lambda}^2).$$

Evaluation of the field momentum $\mathbf{G}$ yields

$$\mathbf{G} = \int \sum_{\lambda} W_{\mathbf{k}\lambda} \frac{\mathbf{k}}{kc} (d\mathbf{k}) = \frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{H} (d\mathbf{r}).$$

The relation between the momentum of a single oscillator, $\mathbf{G}_{\mathbf{k}\lambda}$, and its energy is

$$\mathbf{G}_{\mathbf{k}\lambda} = \frac{\mathbf{k} W_{\mathbf{k}\lambda}}{kc}. \quad (10)$$

A similar equation relates the energy and the momentum of particles moving with the velocity of light in the direction $\mathbf{k}$ (photons!).

719. If the equations given in the solution of Problem 707(b) are multiplied by $\mathbf{e}_{\mathbf{k}\lambda}^*$ then it is found that the transverse component of the potential $\mathbf{A}_{\mathbf{k}}(t)$ is given by

$$\ddot{q}_{\mathbf{k}\lambda}(t) + \omega_{\mathbf{k}}^2 q_{\mathbf{k}\lambda}(t) = F_{\mathbf{k}\lambda}(t), \quad (1)$$

where

$$F_{\mathbf{k}\lambda}(t) = \frac{ec}{2\pi^2} [\mathbf{e}_{\mathbf{k}\lambda}^* \cdot \mathbf{v}(t)] e^{-i\mathbf{k} \cdot \mathbf{r}_0(t)}, \quad (2)$$

and $\mathbf{r}_0(t)$ is the position vector of the particle and $\mathbf{v}$ is its velocity at time t . In the non-relativistic limit

$$m\ddot{\mathbf{r}}_0 = \mathbf{F} + e\mathbf{E}(\mathbf{r}_0), \quad (3)$$

where m is the mass of the particle, $\mathbf{F}$ is that part of the force on the particle which is of non-electromagnetic origin, and

$$\mathbf{E}(\mathbf{r}_0) = -\frac{1}{\pi\sqrt{2}} \int \mathbf{e}_{\mathbf{k}\lambda} \dot{q}_{\mathbf{k}\lambda} e^{i\mathbf{k} \cdot \mathbf{r}_0} (d\mathbf{k}) \quad (4)$$

is the field strength at the point at which the particle is located. The force on the particle which is due to the magnetic field is neglected, since it is assumed that $v \ll c$. Eqn. (1) is, in fact, the equation of forced oscillations of the oscillator under the action of the external force $F_{\mathbf{k}\lambda}(t)$. The motion of the particle and of the electromagnetic field which interact with each other is described by the system of equations (1), (3).

720. The change in the energy of one oscillator is

$$\frac{dW_{\mathbf{k}\lambda}}{dt} = \frac{1}{2} (F_{\mathbf{k}\lambda} \dot{q}_{\mathbf{k}\lambda}^* + F_{\mathbf{k}\lambda}^* \dot{q}_{\mathbf{k}\lambda}).$$

The rate of change of the field energy is

$$\frac{dW}{dt} = \frac{1}{2} \int \sum_{\lambda} (F_{\mathbf{k}\lambda} \dot{q}_{\mathbf{k}\lambda}^* + F_{\mathbf{k}\lambda}^* \dot{q}_{\mathbf{k}\lambda}) (d\mathbf{k}).$$

721. The force $F_{\mathbf{k}\lambda}(t)$ is given by

$$F_{\mathbf{k}\lambda}(t) = b_{\mathbf{k}\lambda} \cos \omega_0 t,$$

where

$$b_{\mathbf{k}\lambda} = \frac{e}{\pi\sqrt{2}} (\mathbf{v}_0 \cdot \mathbf{e}_{\mathbf{k}\lambda}), \quad \mathbf{v}_0 = \omega_0 \mathbf{r}_0$$

(for the sake of simplicity the discussion is confined to linearly polarised field oscillators, so that the unit vectors $\mathbf{e}_{\mathbf{k}\lambda}$ are real). Integration of Eqn. (1) of Problem 719 yields

$$q_{\mathbf{k}\lambda} = \frac{b_{\mathbf{k}\lambda}}{\omega_{\mathbf{k}}^2 - \omega_0^2} (\cos \omega_0 t - \cos \omega_{\mathbf{k}} t),$$

if the oscillators are not excited at $t = 0$. Substituting this expression for $q_{\mathbf{k}\lambda}$ into the expression for the rate of change of the field energy $dW_{\mathbf{k}\lambda}/dt$ found in the solution of Problem 720, we have

$$\frac{dW_{\mathbf{k}\lambda}}{dt} = \frac{b_{\mathbf{k}\lambda}^2}{\omega_{\mathbf{k}}^2 - \omega_0^2} (\omega_{\mathbf{k}} \cos \omega_0 t \sin \omega_{\mathbf{k}} t - \omega_0 \cos \omega_0 t \sin \omega_{\mathbf{k}} t).$$

Integration of this expression with respect to t between 0 and t yields the amount of energy transmitted by the particle to the field oscillator $(\mathbf{k}, \lambda)$ in a time t :

$$W_{\mathbf{k}\lambda} = \int_0^t \frac{dW_{\mathbf{k}\lambda}}{dt} dt = \frac{b_{\mathbf{k}\lambda}^2}{\omega_{\mathbf{k}}^2 - \omega_0^2} \left[\frac{\omega_{\mathbf{k}}}{2} \cdot \frac{1 - \cos(\omega_{\mathbf{k}} + \omega_0)t}{\omega_{\mathbf{k}} + \omega_0} + \right. \\ \left. + \frac{\omega_{\mathbf{k}}}{2} \cdot \frac{1 - \cos(\omega_{\mathbf{k}} - \omega_0)t}{\omega_{\mathbf{k}} - \omega_0} - \frac{\omega_0}{4} \cdot \frac{1 - \cos 2\omega_0 t}{\omega_0} \right].$$

When $\omega_{\mathbf{k}} = \omega_0$ and $t \rightarrow \infty$ the second term in brackets is very large in comparison with the first and third terms. The excitation of the oscillators is therefore found to exhibit a resonance property: the oscillators whose frequency is close to the frequency of the exciting force $F_{\mathbf{k}\lambda}$ are excited first. We shall therefore retain only the resonance term, and sum the energies received by the field oscillators whose frequencies are near ω_0 , with $\mathbf{k}$ lying inside a solid angle $d\Omega$ and the polarisation unit vector $\mathbf{e}_{\mathbf{k}1}(\mathbf{e}_{\mathbf{k}2})$ fixed in direction:

$$dW = \sum_{\mathbf{k}, \lambda} W_{\mathbf{k}\lambda} = \frac{d\Omega}{2c^3} \int_{\omega_0 - \delta}^{\omega_0 + \delta} \sum_{\lambda} \frac{\omega_{\mathbf{k}}^3 b_{\mathbf{k}\lambda}^2}{\omega_{\mathbf{k}} + \omega_0} \cdot \frac{1 - \cos(\omega_{\mathbf{k}} - \omega_0)t}{(\omega_{\mathbf{k}} - \omega_0)^2} d\omega_{\mathbf{k}}.$$

The integrand in the latter expression has a sharp maximum at $\omega_{\mathbf{k}} = \omega_0$. The maximum is particularly narrow for large t . For sufficiently large t the slowly varying factor $\sum_{\lambda} \omega_{\mathbf{k}}^3 b_{\mathbf{k}\lambda}^2 / (\omega_{\mathbf{k}} + \omega_0)$

can be taken outside the integral sign and $\omega_{\mathbf{k}}$ in it may be replaced by ω_0 . In the remaining integral δ may be allowed to tend to infinity so that (*cf.* Appendix I)

$$\int_{-\infty}^{\infty} \frac{1 - \cos at}{a^2} da = \pi t, \quad t \rightarrow \infty.$$

The final result is therefore

$$\frac{dW}{d\Omega} = \frac{\pi(b_{\mathbf{k}1}^2 + b_{\mathbf{k}2}^2)\omega_0^2}{2c^3} t.$$

Hence, the intensity emitted in a given direction is given by the following well-known formula

$$\frac{\overline{dI}}{d\Omega} = \frac{1}{t} \frac{dW}{d\Omega} = \frac{e^2 \omega_0^2 \overline{v^2} \sin^2 \vartheta}{4\pi c^3},$$

where $\overline{v^2} = \frac{1}{2} v_0^2$ is the mean square velocity of the oscillating particle and ϑ is the angle between $\mathbf{v}_0$ and $\mathbf{k}$. In deriving the latter formula we have used the following relation, which can easily be proved:

$$(\mathbf{v}_0 \cdot \mathbf{e}_{k1})^2 + (\mathbf{v}_0 \cdot \mathbf{e}_{k2})^2 = v_0^2 \sin^2 \vartheta.$$

Integration with respect to the angles yields the total intensity:

$$\overline{I} = \frac{2e^2 \omega_0^2 \overline{v^2}}{3}.$$

723. Consider the approximate solution of Eqns. (1) and (3) in the solution of Problem 719. Neglect the reaction of the radiation, and substitute the field strength $\mathbf{E} = \mathbf{E}_0 \cos \omega t$ of the incident wave into Eqn. (3). The solution corresponding to forced oscillations is

$$\mathbf{r}(t) = \frac{e}{m} \mathbf{E}_0 \frac{\cos \omega t}{\omega_0^2 - \omega^2}. \quad (1)$$

The motion of the particle under the action of the incident wave will excite radiation field oscillators in accordance with Eqn. (1) of Problem 719, in which the force $F_{k\lambda}$ should be expressed in terms of $\mathbf{r}(t)$ so that

$$F_{k\lambda} = \frac{e^2 \omega}{m\pi \sqrt{2}} \cdot \frac{\mathbf{e}_{k\lambda} \cdot \mathbf{E}_0}{\omega^2 - \omega_0^2} \sin \omega t.$$

The polarisation unit vectors are chosen to be real. The solution of Eqn. (1) of Problem 719, subject to the initial condition $q_{k\lambda}(0) = 0$, is

$$q_{k\lambda}(t) = \frac{e^2}{m\pi \sqrt{2}} \cdot \frac{\omega (\mathbf{E}_0 \cdot \mathbf{e}_{k\lambda})}{(\omega_k^2 - \omega^2)(\omega^2 - \omega_0^2)} (\sin \omega t - \sin \omega_k t).$$

Proceeding as in the solution of Problem 721, we obtain the following expression for the intensity in the direction $\mathbf{k}$, corresponding to polarisation characterised by the unit vector $\mathbf{e}_{k\lambda}$:

$$\frac{dI_{k\lambda}}{d\Omega} = \frac{1}{t} \frac{dW_{k\lambda}}{d\Omega} = \frac{e^4}{8\pi m^2 c^3} \cdot \frac{\omega^4 (\mathbf{E}_0 \cdot \mathbf{e}_{k\lambda})^2}{(\omega^2 - \omega_0^2)^2}. \quad (2)$$

It follows from (2) that the scattered radiation is linearly polarised in the plane containing $\mathbf{E}_0$ and $\mathbf{k}$. If ϑ is the angle between $\mathbf{E}_0$ and $\mathbf{k}$, then

$$\frac{d\sigma}{d\Omega} = \frac{8\pi}{cE_0^2} \frac{dI}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2} \sin^2 \vartheta,$$

which is in agreement with the result of Problem 699. Integration with respect to the angles yields the following expression for the total scattering cross-section:

$$\sigma = \frac{8\pi}{3} \left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}.$$

Chapter

13

THE RADIATION EMITTED DURING THE INTERACTION OF CHARGED PARTICLES WITH MATTER

726. By expressing the field vectors in the form of the Fourier integral over the space and time coordinates

$$E(\mathbf{R}, t) = \int \mathbf{E}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{R} - \omega t)} d\mathbf{k} d\omega \dots,$$

we obtain the following system of algebraic equations for the Fourier amplitudes from Maxwell's equations:

$$\left. \begin{aligned} \kappa \mathbf{n} \times \mathbf{E}(\mathbf{k}, \omega) &= \mathcal{H}(\mathbf{k}, \omega), \\ \kappa \mathbf{n} \times \mathcal{H}(\mathbf{k}, \omega) &= -\varepsilon(\omega) \mathbf{E}(\mathbf{k}, \omega) - i \frac{e\mathbf{v}}{2\pi^2\omega^2} \delta\left(\frac{\kappa}{c} \mathbf{n} \cdot \mathbf{v} - 1\right), \\ \kappa \varepsilon(\omega) \mathbf{n} \cdot \mathbf{E}(\mathbf{k}, \omega) &= -i \frac{e\mathbf{v}}{2\pi^2\omega^2} \delta\left(\frac{\kappa}{c} \mathbf{n} \cdot \mathbf{v} - 1\right), \\ \kappa \mathbf{n} \cdot \mathcal{H}(\mathbf{k}, \omega) &= 0, \end{aligned} \right\} \quad (1)$$

where $\mathcal{H}(\mathbf{k}, \omega)$ is the Fourier amplitude of the magnetic field, $\mathbf{k} = (\omega/c)\kappa\mathbf{n}$, κ is a parameter which can be expressed in terms of ω and $\mathbf{k}$, and $\mathbf{n}$ is a unit vector. In deriving (1) it must be remembered that the Fourier amplitude of the function $\delta(\mathbf{R} - \mathbf{v}t)$ is equal to $\delta(\mathbf{k} \cdot \mathbf{v} - \omega)/8\pi^3$ and that $\delta(\alpha x) = \delta(x)/|\alpha|$. The system (1) yields

$$\left. \begin{aligned} \mathbf{E}(\mathbf{k}, \omega) &= -\frac{iec}{2\pi^2\omega^2} \cdot \frac{\kappa\mathbf{n} - (\mathbf{v}/c)\varepsilon}{\varepsilon(\kappa^2 - \varepsilon)} \delta\left(\frac{\kappa}{c} \mathbf{n} \cdot \mathbf{v} - 1\right), \\ \mathcal{H}(\mathbf{k}, \omega) &= \frac{ie\kappa}{2\pi^2\omega^2} \cdot \frac{\mathbf{n} \times \mathbf{v}}{\kappa^2 - \varepsilon} \delta\left(\frac{\kappa}{c} \mathbf{n} \cdot \mathbf{v} - 1\right). \end{aligned} \right\} \quad (2)$$

The fields may be determined by the inverse Fourier transformation. Consider $E_z(\mathbf{R}, t)$ first. It follows from (2) that

$$\mathcal{E}_z(\mathbf{k}, \omega) = -\frac{iec}{2\pi^2\omega^2} \cdot \frac{\kappa \cos \theta - \beta\varepsilon}{\varepsilon(\kappa^2 - \varepsilon)} \delta(\beta\kappa \cos \theta - 1),$$

and hence,

$$\begin{aligned} E_z(\mathbf{R}, t) &= -\frac{ie}{2\pi^2c^2} \int_{-\infty}^{\infty} \omega d\omega e^{-i\omega t} \int_0^{\omega} \kappa^2 d\kappa \int \frac{\kappa \cos \theta - \beta\varepsilon}{\varepsilon(\kappa^2 - \varepsilon)} \times \\ &\times \exp\left\{i \frac{\omega}{c} z [r \sin \theta \cos(\Phi - \varphi) - z \cos \theta]\right\} \delta(\beta\kappa \cos \theta - 1) \sin \theta d\theta d\Phi, \end{aligned} \quad (3)$$

where $\mathbf{r}$ is the component of $\mathbf{R}$ in the xy plane, φ is the angle

between $\mathbf{r}$ and the x -axis, $\beta = v/c$, and θ and Φ are the polar angles of $\mathbf{n}$.

The integral with respect to Φ may be expressed in terms of the Bessel function $J_0[(\omega/c)\kappa r \sin \theta]$ [cf. Eqn. (11) of Appendix III].

The integral with respect to θ takes the form

$$\int_0^\pi f(\theta) \delta(\beta \kappa \cos \theta - 1) \sin \theta d\theta = \frac{1}{\beta \kappa} \int_{-\beta \kappa}^{\beta \kappa} \varphi(y) \delta(y - 1) dy. \quad (4)$$

This integral has a non-zero value only when $\beta \kappa \geq 1$ and hence the lower limit of κ is equal to $1/\beta$. In (3) this is automatically allowed for because of the presence of the δ -function. However, after integration with respect to y , the δ -function will vanish, and it will be necessary to take into account explicitly the lower limit of integration. Integration of (4) with respect to y yields

$$\frac{1}{\beta \kappa} \varphi(1) = \frac{1}{\beta \kappa} f(\theta) \Big|_{\cos \theta = \frac{1}{\beta \kappa}}. \quad (5)$$

Now substitute (5) into (3), and change the variable so that $x = (\kappa^2 - \beta^{-2})^{1/2}$. Since κ varies between $1/\beta$ and infinity, the limit of x will be 0 and infinity. $E_z(\mathbf{R}, t)$ will then be of the form

$$E_z(\mathbf{R}, t) = \frac{ie}{\pi c^2} \int_{-\infty}^{\infty} \omega d\omega e^{i\omega(\frac{z}{v} - t)} \left(1 - \frac{1}{\beta^2 \varepsilon}\right) \int_0^\infty \frac{J_0\left(\frac{\omega}{c} r x\right) x dx}{x^2 + 1/\beta^2 - \varepsilon}.$$

Eqn. (16) of Appendix III may be used to complete the integration with respect to x :

$$E_z(\mathbf{R}, t) = \frac{ie}{\pi c^2} \int_{-\infty}^{\infty} \left(1 - \frac{1}{\beta^2 \varepsilon}\right) K_0(sr) e^{i\omega(\frac{z}{v} - t)} \omega d\omega, \quad (6)$$

where $s^2 = (\omega^2/v^2) - (\omega^2/c^2)\varepsilon(\omega)$. The sign of s should be chosen so that $\text{Re } s > 0$ since otherwise the integral with respect to ω will diverge. The integration with respect to ω in (6) can only be completed by specifying the precise form of the function $\varepsilon(\omega)$.

In order to evaluate $E_x(\mathbf{R}, t)$, let us again begin with integration with respect to Φ .

Integration with respect to θ can be carried out with the aid of the δ -function, and the subsequent integration with respect to $x = (\kappa^2 - \beta^{-2})^{1/2}$ may be completed with the aid of the formula

$$\int_0^\infty \frac{J_1(xr) x^2 dx}{x^2 + k^2} = k K_1(kr),$$

which is obtained in Appendix III [Eqn. (16)] after differentiation

with respect to r ($J'_0 = -J_1$, $K'_0 = -K_1$).

The result is

$$E_x(\mathbf{R}, t) = \cos \varphi \frac{e}{\pi v} \int_{-\infty}^{\infty} \frac{s}{\epsilon} K_1(sr) e^{i\omega \left(\frac{z}{v} - t\right)} d\omega.$$

$E_y(\mathbf{R}, t)$ and $\mathbf{H}(\mathbf{R}, t)$ may be determined in a similar way. E_y differs from E_x in that $\cos \varphi$ is replaced by $\sin \varphi$. Hence, in cylindrical coordinates

$$E_r(\mathbf{R}, t) = \frac{e}{\pi v} \int_{-\infty}^{\infty} \frac{s}{\epsilon} K_1(sr) e^{i\omega \left(\frac{z}{v} - t\right)} d\omega, \quad E_\varphi = 0. \quad (7)$$

The result for $\mathbf{H}$ is

$$H_\varphi(\mathbf{R}, t) = \frac{e}{\pi c} \int_{-\infty}^{\infty} s K_1(sr) e^{i\omega \left(\frac{z}{v} - t\right)} d\omega, \quad H_z = H_r = 0. \quad (8)$$

It follows from Eqns. (6)–(8) that the electromagnetic field is axially symmetric.

The above formulae will hold only in the region $r \gg a$, where a is of the order of the interatomic distance. When $r \leq a$, it is necessary to take into account the spatial dispersion of the permittivity.

727. It follows from Eqns. (6)–(8) of the preceding solution that the monochromatic components of the fields, $\mathbf{E}_\omega(\mathbf{R}, t)$ and $\mathbf{H}_\omega(\mathbf{R}, t)$, are

$$E_{\omega z}(\mathbf{R}, t) = \frac{ie\omega}{\pi c^2} \left(1 - \frac{1}{\beta^2 \epsilon}\right) K_0(sr) e^{i\omega \left(\frac{z}{v} - t\right)} \dots, \quad (1)$$

where

$$s^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} \epsilon(\omega), \quad \text{Re } s > 0, \quad (2)$$

and K_n are the modified Bessel functions.

In the wave zone $|sr| \gg 1$ and hence the asymptotic expression given by Eqn. (8) of Appendix III may be used:

$$K_n(sr) = \sqrt{\frac{\pi}{2sr}} e^{-sr}. \quad (3)$$

It follows from (2) that when $\epsilon(\omega)$ is real s will also be real provided $\beta^{-2} > \epsilon(\omega)$ or $\beta n(\omega) < 1$ [$n(\omega)$ is the refractive index at frequency ω]. When $\beta n(\omega) > 1$, the quantity s will be purely imaginary.

For real s [$s > 0$ in view of (2)], the field will decay exponentially in the wave zone, and there will be no emission of radiation. For a purely imaginary s , the field amplitude in the wave zone will vary as $r^{-1/2}$, which corresponds to cylindrical waves. We shall now show that these waves will diverge, *i.e.* radiation will, in fact, be emitted.

Suppose that

$$s = \pm \frac{\omega}{c} \sqrt{\frac{1}{\beta^2} - \epsilon(\omega)} = \pm i \frac{\omega}{c} n \sqrt{\beta^2 n^2 - 1} \quad (4)$$

and consider the sign in front of the square root. The lossless dielectric under consideration is the limiting case of a weakly absorbing dielectric with a complex refractive index $n = n' + in''$. In order that the imaginary part of the refractive index, n'' , should, in fact, represent the absorption of energy (so that the amplitude of the corresponding wave will decrease rather than increase), it is necessary to satisfy the condition $n'' > 0$ when $\omega > 0$ and $n'' < 0$ when $\omega < 0$. Assuming that n'' is very small, we may write

$$\sqrt{\beta^2 (n' + in'')^2 - 1} \simeq \sqrt{\beta^2 n'^2 - 1} \left(1 + i \frac{\beta n' n''}{\beta^2 n'^2 - 1} \right).$$

It follows that the condition $\text{Re } s > 0$ will be satisfied provided the negative sign is taken in (4). Letting $n'' \rightarrow 0$, we have

$$s = -i \frac{\omega n}{c} \sqrt{\beta^2 n^2 - 1}. \quad (5)$$

The negative sign corresponds to diverging waves, since the exponential factor in (1) will be of the form

$$\exp i(\mathbf{k} \cdot \mathbf{R} - \omega t) = \exp i[k(z \cos \theta + r \sin \theta) - \omega t], \quad (6)$$

where $k = \omega n/c$, $\cos \theta = (\beta n)^{-1}$, $\sin \theta = [1 - (\beta n)^{-2}]^{1/2}$, $k \cos \theta = k_z = k_{\parallel}$, and $k \sin \theta = k_{\perp}$ (the components of the wave vector). Thus, when the condition $\beta n(\omega) > 1$ is satisfied, a particle moving in the dielectric with a constant velocity $v = \beta c$ will emit electromagnetic waves of frequency ω (Vavilov-Cherenkov radiation).

The condition $\beta n > 1$ means that the velocity of the particle should exceed the phase velocity of the wave of frequency ω in the given medium. It follows from the expression for the wave vector $\mathbf{k}$ that the radiation will be emitted at an angle θ to the

velocity of the particle, where

$$\cos \theta = \frac{1}{\beta n(\omega)}. \quad (7)$$

This characteristic directional property of the radiation is a consequence of the coherence of the waves emitted by the particle at the various points along its trajectory (*cf.* Problem 729).

The phase velocity of Vavilov-Cherenkov waves

$$v_{\varphi} = \frac{\omega}{c} = \frac{c}{n}$$

is the same as for all the transverse electromagnetic waves. The polarisation of the radiation can easily be determined from (1). Thus, the vector $\mathbf{H}$ is perpendicular to the plane containing the trajectory of the particle and the wave vector $\mathbf{k}$, while the vector $\mathbf{E}$ lies in this plane and is perpendicular to $\mathbf{k}$ in the wave zone. The fact that $\mathbf{k}$ and $\mathbf{E}$ are perpendicular may be verified by evaluating the scalar product $\mathbf{k} \cdot \mathbf{E}_{\omega}$. The total energy per unit path length of the Vavilov-Cherenkov radiation, w_{V-Ch} , is equal to the time integral of the flux of the Poynting vector through an infinitely distant cylindrical surface of unit length, surrounding the trajectory of the particle:

$$w_{V-Ch} = 2\pi r \int_{-\infty}^{\infty} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H})_r dt = -\frac{cr}{2} \int_{-\infty}^{\infty} H_{\varphi} E_z dt. \quad (8)$$

This equation may be rewritten with the aid of the formula given in the solution to Problem 697:

$$w_{V-Ch} = -2\pi cr \operatorname{Re} \int_{\beta n(\omega) > 1} H_{\omega\varphi}^* E_{\omega z} d\omega, \quad (9)$$

where the monochromatic components $H_{\omega\varphi}$, $E_{\omega z}$ should be taken in the wave zone, and the integration should be carried out over the frequency region in which the radiation condition $\beta n(\omega) > 1$ is satisfied. From (1)–(3) we have, finally,

$$w_{V-Ch} = \frac{e^2}{c^2} \int_{\beta n(\omega) > 1} \left(1 - \frac{c^2}{v^2 n^2}\right) \omega d\omega. \quad (10)$$

$$728. \quad w_{V-Ch} = \frac{e^2 \omega_0^2}{2v^2} (\beta^2 - 1) + \frac{e^2 \omega_0^2}{2v^2} (\epsilon_0 - 1) \ln \frac{\epsilon_0}{\epsilon_0 - 1}.$$

Under the conditions given in the problem, $w_{V-Ch} \approx 5000$ eV/cm.

The radiation is largely concentrated in the angular range $\theta_0 \leq \theta \leq \pi/2$ where $\beta^2 \epsilon_0 \cos^2 \theta_0 = 1$.

729. Each point of the trajectory may be regarded as a source of elementary excitation propagating in the form of a spherical wave with a velocity $v_\varphi = c/n$ (Fig. 98). The resultant wave front will be the envelope of the elementary spherical waves. The normal to the wave front will be at an angle θ to the trajectory. It follows from the figure that $\cos \theta = 1/\beta n$.

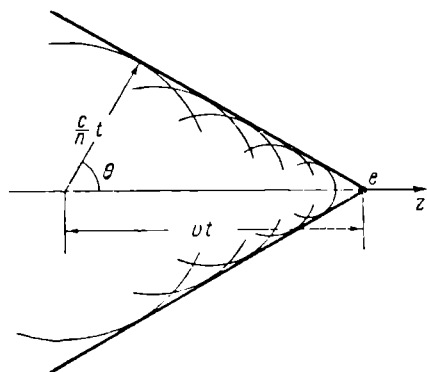


Fig. 98.

730. The field due to a uniformly moving charged particle is a result of the superposition of plane waves with frequencies $\omega = \mathbf{k} \cdot \mathbf{v}$, where $\mathbf{v}$ is the velocity

of the particle and $\mathbf{k}$ is the wave vector (*cf.* Problem 710). In an infinite dielectric, the possible oscillations have frequencies $\omega = kc/n$, where n is the refractive index of the medium. It follows from the resonance condition

$$\frac{kc}{n} = \mathbf{k} \cdot \mathbf{v} = kv \cos \theta$$

that $\cos \theta = c/vn$. Since $\cos \theta \leq 1$, it follows that $vn/c \geq 1$, which is the condition for the existence of the Vavilov-Cherenkov radiation.

732. When $\beta n < 1$, *i.e.* when $v < v_\varphi$,

$$\varphi = \frac{e}{\epsilon \sqrt{(z-vt)^2 + r^2 (1 - \beta^2 n^2)}}. \quad (1)$$

This expression may be obtained by the method used in the solution of Problem 711.

When $\beta n > 1$, this method cannot be used, since the integrand will then have a pole at $k^2 = \epsilon \mu (\mathbf{k} \cdot \mathbf{v})^2 / c^2$.

In terms of cylindrical coordinates in $\mathbf{k}$ -space,

$$\varphi(\mathbf{R}, t) = \frac{e}{2\pi^2 \epsilon} \int \frac{e^{ik_z(z-vt) + ik_\perp r \cos \alpha}}{k_\perp^2 - k_z^2 (\beta^2 n^2 - 1)} k_\perp dk_\perp dk_z d\alpha.$$

The integral with respect to k_z may be evaluated by the method of residues. The denominator will have zeros at $k_z = \pm k_\perp / (\beta^2 n^2 - 1)^{1/2}$.

Suppose that n has a small imaginary part $n'' > 0$ when $k_z > 0$ and $n'' < 0$ when $k_z < 0$ (cf. analogous analysis in Problem 727); here, the sign of ω is the same as the sign of k_z , since $\omega = \mathbf{k} \cdot \mathbf{v}$. Hence, both zeros will be displaced into the lower half-plane of the complex variable k_z . When $z > vt$, the contour of integration may be completed by an arc of infinite radius in the upper half-plane (on which the integrand will be zero). Since the denominator has no zeros in the upper half-plane, the integral with respect to k_z will then vanish. When $z < vt$, the contour of integration will be closed in the lower half-plane. Both poles will then contribute to the integral and the result will be

$$\int_{-\infty}^{\infty} \frac{e^{ik_z(z-vt)}}{k_{\perp}^2 - k_z^2(\beta^2 n^2 - 1)} dk_z = -\frac{2\pi}{k_{\perp} \sqrt{\beta^2 n^2 - 1}} \sin \frac{k_{\perp}(z-vt)}{\sqrt{\beta^2 n^2 - 1}}.$$

The integral with respect to α may be expressed in terms of the Bessel function $J_0(k_{\perp} r)$ [cf. Appendix III, Eqn. (11)]. Thus, when $\beta n > 1$

$$\varphi(R, t) = \begin{cases} \frac{2e}{\epsilon \sqrt{(z-vt)^2 - r^2(\beta^2 n^2 - 1)}} & \text{when } z < vt - r\sqrt{\beta^2 n^2 - 1}, \\ 0 & \text{in the remaining space.} \end{cases} \quad (2)$$

The vector potential $\mathbf{A}$ may be obtained by multiplying φ by $\epsilon \mu \mathbf{v}/c$.

Eqn. (2) shows that when the Vavilov-Cherenkov condition $\beta n > 1$ is satisfied, the field is discontinuous. It exists only within a cone whose surface is described by the equation

$$z - vt + r\sqrt{\beta^2 n^2 - 1} = 0. \quad (3)$$

The normal to the surface of the cone is at an angle $\theta = \cos^{-1}(1/\beta n)$ to the direction of motion of the particle. It follows from (3) that the conical wave propagates along the z -axis with the velocity of the particle.

Electromagnetic waves are not the only waves which may have the above structure. For example, discontinuous acoustic waves of the above type may be excited by a projectile moving in air with a velocity greater than the velocity of sound (ballistic shock wave). Similar waves may be produced on the surface of water by a fast moving vessel.

733. The Vavilov-Cherenkov radiation will be emitted provided $\beta n > 1$ where $n(\omega) = [\epsilon(\omega)\mu(\omega)]^{1/2}$; the vector potential is

of the form

$$A_x = \frac{i\mathcal{J}}{c} \int \frac{\exp i \frac{\omega}{v} (y - vt + \sqrt{\beta^2 n^2 - 1} |z|)}{\sqrt{\beta^2 n^2 - 1}} \cdot \frac{\mu(\omega) d\omega}{\omega};$$

$$w_{V-Ch} = \frac{2\mathcal{J}^2}{c^2 v} \int_{\beta n > 1} \frac{\mu(\omega) d\omega}{\sqrt{\beta^2 n^2 - 1}}.$$

The retarding force may be calculated from the formula $\mathbf{f} = (\mathbf{j} \times \mathbf{B})/c$ where $\mathbf{B}$ should be taken at the point $z = 0$, $y = vt$. The force acts in the negative direction of the y -axis, and its absolute magnitude is equal to the energy loss per unit path length: $F_y = -w_{V-Ch}$. This result is a direct consequence of the law of conservation of energy.

$$734. \quad w_{V-Ch} = \frac{2e^2}{c^2} \int_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) \left(1 \pm \cos \frac{\omega l}{v}\right) \omega d\omega.$$

The positive and negative signs correspond respectively to the two cases quoted in the problem. The spectral density of the radiation emitted by two identical charges differs from the spectral density of the radiation emitted by a single charge by a factor of $2[1 + \cos(\omega l/v)]$. Hence, the intensity of the harmonics with frequencies $\omega = (2\pi v/l)n$ ($n = 0, 1, 2, \dots$) will increase by a factor of 4, while harmonics with frequencies $\omega = (\pi v/l)(2n + 1)$ will vanish. The reverse situation will occur when the charges differ in sign.

In order to obtain the energy loss for a point dipole lying along the direction of motion, the quantity $1 - \cos(\omega l/v)$ should be expanded into a series. This yields

$$w_{V-Ch} = \frac{p^2}{c^2 v^2} \int_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) \omega^3 d\omega,$$

where p is the electric moment of the dipole in the laboratory system.

$$735. \quad w_{V-Ch} = \frac{p^2}{c^2 v^2} \int_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) \left[\cos^2 \alpha + \frac{1}{2} \sin^2 \alpha (\beta^2 n^2 - 1)\right] \omega^3 d\omega,$$

where $n = \epsilon^{1/2}$ and p is the electric dipole moment in the laboratory system.

$$736. \quad w_{V-Ch} = \frac{m^2}{c^2 v^2} \int_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) n^2 \omega^3 d\omega.$$

737. The energy loss per unit path length is given by the time integral of the energy flux through a cylindrical surface of

unit length and radius a , which surrounds the trajectory of the particle. The losses may be calculated with the aid of Eqn. (9) in the solution of Problem 727, in which the fields should be taken at $r = a$ and the integration should be carried out over all frequencies between 0 and infinity. Using the expressions for the field components obtained in the solution of Problem 726, we have

$$-\frac{d\mathcal{E}}{dt} = \frac{2e^2\omega_0^2}{\pi v^2} \operatorname{Re} i \int_0^\infty \left(\frac{1-x^2}{\epsilon_0-x^2} - \beta^2 \right) s^* a K_1(s^* a) K_0(sa) x dx, \quad (1)$$

where $x = \omega/\omega_0$, $\epsilon(0) = \epsilon_0 = 1 + \omega_p^2/\omega_0^2$ is the static permittivity, and

$$s^2 = \frac{\omega_0^2}{c^2} \left(\frac{1}{\beta^2} - 1 \right) \frac{b-x^2}{1-x^2} x^2, \quad b = \frac{c^2 - \epsilon_0 v^2}{c^2 - v^2}. \quad (2)$$

It follows from (1) that the imaginary part of the integral is the only one to contribute to the losses. The functions K_0 and K_1 are real for a real argument, and hence the imaginary part of the integral will be determined by the range of values of x for which s is complex. As can be seen from (2), this range will depend on the sign and magnitude of the parameter b . When $b > 0$ (this means that $v < c/\epsilon_0^{1/2}$), s will be purely imaginary for x in the range $(b^{1/2}, 1)$, and real outside this interval. When $b < 0$ (this corresponds to $v > c/\epsilon_0^{1/2}$), s will be imaginary for $0 \leq x \leq 1$ and real for $x > 1$.

In addition to the above intervals of x , certain individual points at which the denominator $\epsilon_0 - x^2$ is zero will contribute to the imaginary part of the integral. At these points $x = \pm\epsilon_0^{1/2}$, and since the integration in (1) is carried out for $x > 0$, it is sufficient to consider the single pole at $x = \epsilon_0^{1/2} > 1$. When losses are neglected this pole will lie on the real axis. It will be displaced into the lower half-plane of the complex variable, ω , when the losses are taken into account†. This can easily be seen from the explicit expression for $\epsilon(\omega)$ [cf. (VI.12)]. The integral may be evaluated either by introducing a damping parameter which tends to zero after the integration, or by slightly deforming the path of integration and by-passing the pole by a circle of infinitely small radius in the upper half-plane. Denoting integration over this

† This is in accordance with the general theorem according to which $\epsilon(\omega)$ has no zeros in the upper half-plane.

half-plane by $\cap$, we have

$$\int_{\cap} \frac{1-x^2}{\epsilon_0-x^2} s^* a K_1(s^* a) K_0(sa) x dx = \\ = i \frac{1-\epsilon_0}{2} \cdot \frac{\omega_0 a \sqrt{\epsilon_0}}{v} K_0\left(\frac{\omega_0 a \sqrt{\epsilon_0}}{v}\right) K_1\left(\frac{\omega_0 a \sqrt{\epsilon_0}}{v}\right). \quad (3)$$

Next, consider the integral over the region in which s is purely imaginary. We note that for purely imaginary argument the cylindrical functions K_0 and K_1 are related by

$$s^* a K_1(s^* a) K_0(sa) - sa K_1(sa) K_0(s^* a) = i \frac{\pi}{2},$$

which follows from the properties of the Wronskian of the system of solutions of the Bessel equation. Hence,

$$\operatorname{Re} i \int_{s^2 < 0} \left(\frac{1-x^2}{\epsilon_0-x^2} - \beta^2 \right) s^* a K_1(s^* a) K_0(sa) x dx = \\ = -\frac{\pi}{2} \int_{s^2 < 0} \left(\frac{1-x^2}{\epsilon_0-x^2} - \beta^2 \right) x dx. \quad (4)$$

The latter integral may be evaluated by elementary methods. The limits of integration are chosen as indicated above.

Substituting (3) and (4) into (1), we have for $v < c/\epsilon_0^{1/2}$,

$$-\frac{d\mathcal{E}}{dl} = \frac{2\pi e^4 N}{mv^2} \left[\frac{2a\omega_0 \sqrt{\epsilon_0}}{v} K_0\left(\frac{a\omega_0 \sqrt{\epsilon_0}}{v}\right) K_1\left(\frac{a\omega_0 \sqrt{\epsilon_0}}{v}\right) - \beta^2 - \ln(1-\beta^2) \right] \quad (5)$$

and for $v > c/\epsilon_0^{1/2}$,

$$-\frac{d\mathcal{E}}{dl} = \frac{2\pi e^4 N}{mv^2} \left[\frac{2a\omega_0 \sqrt{\epsilon_0}}{v} K_0\left(\frac{a\omega_0 \sqrt{\epsilon_0}}{v}\right) K_1\left(\frac{a\omega_0 \sqrt{\epsilon_0}}{v}\right) - \right. \\ \left. - \frac{1-\beta^2}{\epsilon_0-1} + \ln \frac{\epsilon_0}{\epsilon_0-1} \right]. \quad (6)$$

The part of total losses which does not vanish when $a \rightarrow \infty$ [terms which do not contain a in (5) and (6)] will represent the energy loss by the emission of transverse waves (Vavilov-Cherenkov effect):

$$-\left(\frac{d\mathcal{E}}{dl}\right)_{v < c_h} \equiv w_{v < c_h} = \frac{e^2 \omega_p^2}{2v^2} [-\beta^2 - \ln(1-\beta^2)] \quad v < \frac{c}{\sqrt{\epsilon_0}}, \quad (7a)$$

$$-\left(\frac{d\mathcal{E}}{dl}\right)_{v > c_h} \equiv w_{v > c_h} = \frac{e^2 \omega_p^2}{2v^2} \left(-\frac{1-\beta^2}{\epsilon_0-1} + \ln \frac{\epsilon_0}{\epsilon_0-1} \right) \quad v > \frac{c}{\sqrt{\epsilon_0}}. \quad (7b)$$

The latter expression was obtained in the solution of Problem 728.

The terms with K_0 , K_1 in (5) and (6), which depend on a , enter as a result of integration round the pole $\omega \equiv \Omega = (\omega_0^2 + \omega_p^2)^{1/2}$ at which ϵ becomes equal to zero. Longitudinal oscillations are excited at these frequencies (*cf.* Problem 456) and hence the expression

$$-\left(\frac{d\mathcal{E}}{dt}\right)_{\text{tot}} = \frac{e^2 \omega_p^2 \Omega a}{v^3} K_0\left(\frac{\Omega a}{v}\right) K_1\left(\frac{\Omega a}{v}\right) \quad (8)$$

describes the losses on excitation of longitudinal oscillations (polarisation losses). When $\Omega a/v \ll 1$, Eqn. (8) assumes the simple form [*cf.* Eqn. (6) of Appendix III]

$$-\left(\frac{d\mathcal{E}}{dt}\right)_{\text{tot}} = \frac{e^2 \omega_p^2}{v^2} \ln \frac{v}{\Omega a}. \quad (9)$$

When $\Omega a/v \gg 1$, the quantity $-(d\mathcal{E}/dt)_{\text{tot}}$ becomes very small [it is proportional to $\exp(-\Omega a/v)$]. This shows that the effect of the polarisation of the medium is small at low velocities.

The above macroscopic method of calculating the losses is due to Fermi (1940).

$$738. \quad -\frac{d\mathcal{E}}{dt} = \frac{e^2 \omega_p^3 a}{v^3} K_0\left(\frac{\omega_p a}{v}\right) K_1\left(\frac{\omega_p a}{v}\right). \quad (1)$$

When $\omega_p a/v \ll 1$, which occurs when the velocity of the particle is sufficiently large, one can use the approximate formula for K_n [Eqn. (6) of Appendix III]. Eqn. (1) will then become

$$-\frac{d\mathcal{E}}{dt} = \frac{e^2 \omega_p^2}{v^2} \ln \frac{2v}{\gamma \omega_p a}. \quad (2)$$

It follows from (1) and (2) that the losses experienced by the particle will be very dependent on ω_p [the frequency of longitudinal plasma oscillations (*cf.* Problems 456 and 739)].

Vavilov-Cherenkov radiation is not emitted by a plasma, since the condition $\beta^2 \epsilon \geq 1$ is not satisfied, in view of the fact that $\epsilon(\omega) < 1$ at all frequencies (however, Cherenkov emission will become possible if the plasma is placed in a magnetic field).

On the quantum mechanical interpretation, the excitation of plasma waves is equivalent to the appearance of certain discrete elementary excitations (these are the quasi-particles known as plasmons). The energy of each plasmon is equal to $\hbar \omega_p$ where $\hbar = 1.05 \times 10^{-27}$ erg sec (Planck's constant). For metals, the energy $\hbar \omega_p$ lies between 5 and 30 eV. Thus, when plasma

oscillations are excited, the particle will lose energy in discrete steps. Studies of these discrete energies yield valuable information about the properties of solids.

739. The electron density is given by

$$\rho(\mathbf{r}, t) = e \sum_l \delta[\mathbf{r} - \mathbf{r}_l(t)],$$

where $\mathbf{r}_l(t)$ is the position vector of an electron, and the summation is carried out over all the electrons. The Fourier components are of the form

$$\rho_{\mathbf{k}}(t) = \frac{1}{V} \int \rho(\mathbf{r}, t) e^{-i\mathbf{k} \cdot \mathbf{r}} (d\mathbf{r}) = \frac{1}{V} \sum_l e^{-i\mathbf{k} \cdot \mathbf{r}_l(t)},$$

where $k_x = (2\pi/a_1)n_1$, $k_y = (2\pi/a_2)n_2$, $k_z = (2\pi/a_3)n_3$, and n_1, n_2, n_3 are integers. The equation of motion of the l -th electron under the action of all the remaining electrons may be written in the form

$$\ddot{\mathbf{r}}_l = -l \frac{4\pi e^2}{m} \sum'_{n, \mathbf{k}} \left(\frac{\mathbf{k}}{k^2} \right) e^{i\mathbf{k} \cdot (\mathbf{r}_l - \mathbf{r}_n)}. \quad (1)$$

The summation on the right-hand side represents the Coulomb force acting on the l -th electron due to all the remaining electrons. The prime on the summation sign indicates that terms with $n = l$ and $\mathbf{k} = 0$ should be omitted (the term with $\mathbf{k} = 0$ is omitted since the component $\rho_{\mathbf{k}}$ of the electron charge for $\mathbf{k} = 0$ is compensated by the charge on a positive ion). When $\rho_{\mathbf{k}}(t)$ is differentiated twice with respect to time, we have from (1)

$$\begin{aligned} \frac{d^2 \rho_{\mathbf{k}}}{dt^2} = & -\frac{1}{V} \sum_l (\mathbf{k} \cdot \mathbf{v}_l) e^{-i\mathbf{k} \cdot \mathbf{r}_l} - \\ & - \frac{1}{V^2} \sum_{l \neq j, \mathbf{k}' \neq 0} \frac{4\pi e^2}{mk'^2} (\mathbf{k} \cdot \mathbf{k}') e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_l} \times e^{-i(\mathbf{k}' \cdot \mathbf{r}_j)}. \end{aligned}$$

Consider the sum $\sum_{\mathbf{k}'} (4\pi e^2/mk'^2) (\mathbf{k} \cdot \mathbf{k}') \sum_l \exp[i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_l]$. Since

the position vectors of the electrons, $\mathbf{r}_l$, vary randomly, it follows that the various terms in the sum will cancel out when $\mathbf{k}' \neq \mathbf{k}$. The values of $\sum_l \exp[i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}_l]$ for $\mathbf{k}' \neq \mathbf{k}$ will be much smaller

than its value for $\mathbf{k}' = \mathbf{k}$, which is equal to the total number of electrons, N . Hence, in the sum over $\mathbf{k}'$ it is possible to reject terms with $\mathbf{k}' \neq \mathbf{k}$. The final result is

$$\frac{d^2 \rho_{\mathbf{k}}}{dt^2} = -\omega_p^2 \rho_{\mathbf{k}} - \frac{1}{V} \sum_l (\mathbf{k} \cdot \mathbf{v}_l)^2 e^{-i(\mathbf{k} \cdot \mathbf{r}_l)}. \quad (2)$$

The first term on the right-hand side of (2) contains the quantity ω_p which characterises the collective properties of the plasma, since it represents the frequency of oscillations of the electron gas as a whole. The second term includes the quantities $(\mathbf{k} \cdot \mathbf{v}_l)$, which contain the velocities of the individual particles. Collective properties will predominate when ω_p^2 is much greater than the mean value of $(\mathbf{k} \cdot \mathbf{v})^2$. Under these conditions, the second term may be neglected, and the electron density will execute harmonic oscillations with the frequency ω_p .

Consider now the range of values of k for which the inequality $\omega_p^2 \gg \overline{(\mathbf{k} \cdot \mathbf{v})^2}$ will be satisfied. Since $\overline{(\mathbf{k} \cdot \mathbf{v})^2} = \sum_{m,n} k_m k_n \overline{v_m v_n} = \frac{1}{3} k^2 v^2$ we have $k^2 \ll 12\pi e^2 N / m \overline{v^2}$. If the electrons follow the Maxwell distribution then $\frac{1}{2} m \overline{v^2} = \frac{3}{2} k_0 T$ and thus $k \ll \kappa$, where $1/\kappa = k_0 T / 4\pi e^2 N$ is the Debye radius (*cf.* Problem 308), k_0 is Boltzmann's constant, and T is the temperature.

740. Consider the expansion of the current density (Fig. 99)

$$j = j_z = \begin{cases} -ev\delta(z - vt)\delta(x)\delta(y) & z \geq 0, \\ -ev\delta(z + vt)\delta(x)\delta(y) & z \leq 0 \end{cases} \quad (1)$$

into the Fourier integral

$$j = \int j_\omega e^{-i\omega t} dt, \quad j_\omega = \begin{cases} -\frac{e}{2\pi} e^{-i\frac{\omega}{v}z} \delta(x)\delta(y) & z \geq 0, \\ -\frac{e}{2\pi} e^{+i\frac{\omega}{v}z} \delta(x)\delta(y) & z \leq 0. \end{cases} \quad (2)$$

Next, introduce the polarisation vector (XII.9)

$$\mathbf{P}_\omega = -\frac{j_\omega}{i\omega}, \quad (3)$$

and suppose that $\mathbf{P}_\omega$ lies along the z -axis.

Eqs. (2) and (3) show that the charge and current densities due to the moving particle can be represented by a set of harmonic oscillators whose spatial distribution is of the form

$$\mathbf{P}_\omega = \begin{cases} -\frac{ie}{2\pi\omega} e^{-i\frac{\omega}{v}z} \delta(x)\delta(y) & z \geq 0, \\ -\frac{ie}{2\pi\omega} e^{+i\frac{\omega}{v}z} \delta(x)\delta(y) & z \leq 0. \end{cases} \quad (4)$$

The presence of $\delta(x)\delta(y)$ in (4) shows that the oscillators are, in fact, distributed along the line of motion of the charge.

The oscillators lying along the element dz will produce at a point M in the wave zone a magnetic field (cf. Fig. 99)

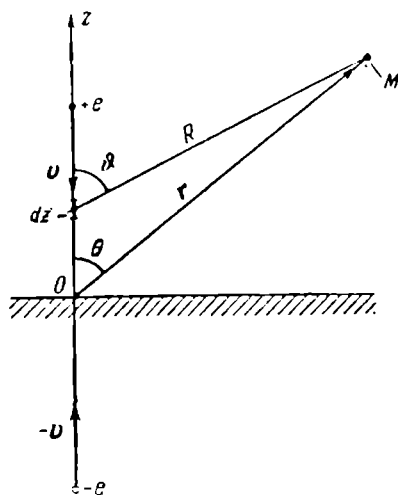


Fig. 99.

$$dH_{\omega} = - \frac{\omega^2 e^{ikR}}{c^2 R^2} (\mathbf{p}_{\omega} \times \mathbf{R}) dz = \\ = - \frac{\omega^2 e^{ikR}}{c^2 R} P_{\omega} \sin \vartheta \mathbf{e}_a dz. \quad (5)$$

After integrating (5) with respect to z the following expression for the total field is obtained:

$$H_{\omega a} = \frac{ie\omega}{2\pi c^2} \left[\int_{-\infty}^0 \frac{e^{i\left(\frac{\omega}{v}z + kR\right)} \sin \vartheta}{R} dz + \right. \\ \left. + \int_0^{\infty} \frac{e^{-i\left(\frac{\omega}{v}z - kR\right)} \sin \vartheta}{R} dz \right].$$

In the latter expression the integrands are the products of oscillating and decreasing functions, and therefore the main contribution to the integral will be that due to the region near $z = 0$. This is due to the fact that the emission takes place at the boundary between the vacuum and the metal. To evaluate the integrals approximately, let $R = r - z \cos \theta$ in the exponentials. Expressing $\sin \vartheta$ in terms of R we have

$$H_{\omega a} = \frac{ie\omega e^{ikr} \sin \theta}{2\pi c^2} \left[\int_{-\infty}^0 \frac{e^{i\frac{\omega}{v}(1-\beta \cos \theta)z}}{R^2} dz + \int_0^{\infty} \frac{e^{-i\frac{\omega}{v}(1+\beta \cos \theta)z}}{R^2} dz \right].$$

These integrals can be expanded into series in powers of R^{-1} after integrating by parts. Retaining only terms up to $1/R$, we have

$$H_a = E_0 = \frac{e\omega}{2\pi c^2} \left[\frac{1}{\frac{\omega}{v}(1-\beta \cos \theta)} + \frac{1}{\frac{\omega}{v}(1+\beta \cos \theta)} \right] \frac{\sin \theta e^{ikr}}{r}. \quad (6)$$

The second term in this expression represents the radiation field due to the sudden deceleration of the charge, while the first term represents the radiation due to the image.

The intensity of the radiation of frequency ω which is emitted into a solid angle $d\Omega$ is given by

$$dI(\omega, \theta) = c |E(\omega, \theta)|^2 r^2 d\Omega = \frac{e^2 v^2}{\pi^2 c^3} \cdot \frac{\sin^2 \theta d\Omega}{(1 - \beta^2 \cos^2 \theta)^2}. \quad (7)$$

In the non-relativistic limit ($\beta \ll 1$), Eqn. (7) gives the dipole

radiation

$$dI(\omega, \theta) = \frac{e^2 v^2}{\pi^2 c^3} \sin^2 \theta d\Omega, \quad (8)$$

whose intensity is proportional to the square of the velocity of the particle. We note that the intensity of the radiation is independent of the mass of the particle.

The integrals of (7) and (8) with respect to ω , which represent the angular distribution of the total radiation, will be found to diverge. This is due to the fact that the metal is regarded as a perfect conductor. In reality, the metal cannot be looked upon as a perfect conductor even in the infrared region, since at high frequencies both (7) and (8) will not hold.

The spectral distribution of the total radiation may be obtained by integrating (7) over the upper hemisphere:

$$I(\omega) = \frac{4e^2 v^2}{3\pi c^3} \left(\frac{3}{8} \cdot \frac{\beta^2 - 1}{\beta^3} \ln \frac{1 + \beta}{1 - \beta} - \frac{3}{4\beta^2} \right). \quad (9)$$

In the ultra-relativistic limit, when the total energy, $\mathcal{E}$, of the particle is much smaller than the rest energy mc^2 , Eqn. (9) may be simplified to read

$$I(\omega) = \frac{2e^2}{\pi c} \ln \frac{\mathcal{E}}{mc^2}.$$

The intensity of the radiation increases logarithmically with the energy.

In the non-relativistic limit the expression in brackets in Eqn. (9) becomes equal to unity and

$$I(\omega) = \frac{4e^2 v^2}{3\pi c^3}. \quad (10)$$

741. The Fourier component of the polarisation vector is

$$P_\omega = -\frac{ie}{2\pi\omega} e^{-i\frac{\omega}{v}z} \delta(x) \delta(y). \quad (1)$$

To begin with, consider the field at the point A due to oscillators lying in the region $z > 0$ (Fig. 100). It is sufficient to consider the oscillators lying near $z = 0$, since they will be largely responsible for the radiated field (*cf.* preceding problem).

Consider an oscillator $\mathbf{p}_B$ on the z -axis near $z = 0$ (point B) and an oscillator $\mathbf{p}_A$ at the point A at which the field is to be

calculated. Suppose they are equal in absolute magnitude and parallel to the z -axis, and that the distance between them is large compared with the wavelength. The oscillator $\mathbf{p}_B$ will produce a field at A whose amplitude $\mathbf{E}_+$ will be at an angle to the z -axis which is approximately equal to $\frac{1}{2}\pi - \theta$ (Fig. 100). The waves

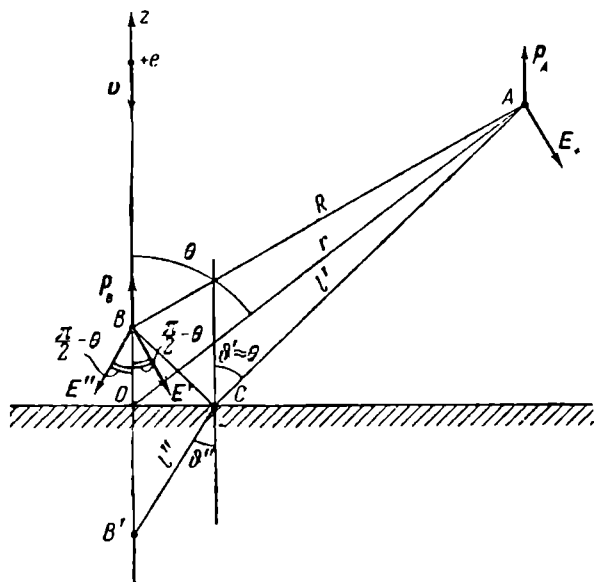


Fig. 100.

from A and B will arrive by two routes, namely, directly and after reflection from the boundary of the dielectric. In the figure, the corresponding amplitudes are denoted by $\mathbf{E}'$ and $\mathbf{E}''$. These two amplitudes are at angles $\frac{1}{2}\pi - \vartheta' \simeq \frac{1}{2}\pi - \theta$ to the z -axis. Hence, according to the reciprocity theorem, $\mathbf{E}_+ = \mathbf{E}' + \mathbf{E}''$ or, since in the wave zone $\mathbf{H} = \mathbf{n} \times \mathbf{E}$, we have $\mathbf{H}_+ = -\mathbf{H}' - \mathbf{H}''$ (all the three vectors $\mathbf{H}_+$, $\mathbf{H}'$, and $\mathbf{H}''$ are perpendicular to the Az plane).

The wave emitted by A and reaching B directly will give rise to a field

$$dH' = \frac{\omega^2 e^{ikR}}{c^2 R} P_\omega \sin \theta dz. \quad (2)$$

The amplitude of the reflected wave may be determined with the aid of the Fresnel formulæ. since the distance AC is large and the wave emitted at A may be looked upon as a plane wave near the point C . Using (VIII.20), and remembering that there is a phase change and that $\vartheta' \simeq \theta$, we have

$$dH'' = \frac{\omega^2 f e^{ikR'}}{c^2 R'} P_\omega \sin \theta dz, \quad (3)$$

where

$$f = \frac{\epsilon \cos \theta - \sqrt{\epsilon - \sin^2 \theta}}{\epsilon \cos \theta + \sqrt{\epsilon - \sin^2 \theta}}, \quad R' = ACB.$$

The field $\mathbf{H}_+$, which is produced at A by all the oscillators lying in the region $z > 0$, may be obtained by integrating the sum $-(dH' + dH'')$ with respect to z between 0 and ∞ . The integration may be carried out precisely as in the preceding problem. The result is

$$H_+ = \frac{ev}{2\pi c^2} \left(\frac{1}{1 + \beta \cos \theta} + \frac{f}{1 - \beta \cos \theta} \right) \frac{\sin \theta e^{ikr}}{r}. \quad (4)$$

This formula may easily be interpreted by comparison with the analogous formula (6) in the solution to the preceding problem. The first term represents the field of the particle moving in the vacuum and suddenly coming to rest at the point $z = 0$. The second term represents the field due to the image ($-ef$) which moves in the dielectric towards the particle and also comes to rest at $z = 0$. In contrast to a perfect conductor, the strength of the image is reduced by a factor f and its magnitude depends on the frequency ω of the particular harmonic [through $\epsilon(\omega)$] and on the position of the point of observation (through the angle θ).

The field $\mathbf{H}_-$ due to the dipoles at $z < 0$ can be determined in a similar way. The wave leaving A will arrive at B after refraction at the separation boundary. Using the Fresnel formulae again, we have

$$dH_- = -\frac{\omega^2}{\epsilon c^2 R''} (1 + f) P_\omega \sin \vartheta' e^{i\varphi} dz. \quad (5)$$

where $R'' = l' + l''$ is the length of the broken line ACB' (cf. Fig. 100). The phase φ is retarded, so that

$$\varphi = \frac{\omega}{c} l' + \frac{\omega}{c} \sqrt{\epsilon} l''.$$

When $|z| \ll r$ ($z < 0$), we have $l' = r + z \tan \vartheta'' \sin \theta$. $l'' = -z/\cos \vartheta''$. Since the law of refraction is $\sin \vartheta'' = \epsilon^{-1/2} \sin \vartheta'$ we have, on substituting θ for ϑ'

$$\varphi = \frac{\omega}{c} r - \frac{\omega}{c} z \sqrt{\epsilon - \sin^2 \theta}.$$

The field due to the dipoles at $z < 0$ may be obtained by integrating

(5) between $-\infty$ and 0, and the result is

$$H_- = -\frac{ev}{2\pi\epsilon c^2} (1+f) \frac{1}{1-\beta\sqrt{\epsilon-\sin^2\theta}} \cdot \frac{\sin\theta e^{ikr}}{r}. \quad (6)$$

The total field at A is equal to $H_+ + H_-$. The intensity of the radiation of frequency ω which is emitted into a solid angle $d\Omega$ is

$$dI(\omega, \theta) = \frac{e^2 v^2}{4\pi^2 c^3} A^2(\omega, \theta) \sin^2 \theta d\Omega,$$

where

$$A(\omega, \theta) = \frac{v^2 \cos \theta}{1-\beta \cos^2 \theta} + (1+f) \left[\frac{1}{\epsilon(1-\beta\sqrt{\epsilon-\sin^2\theta})} - \frac{1}{1-\beta\cos\theta} \right]. \quad (7)$$

The quantity A depends on the frequency through $\epsilon(\omega)$.

In the non-relativistic limit $\beta \ll 1$, and hence

$$dI(\omega, \theta) = \frac{e^2 v^2}{\pi^2 c^3} \cdot \frac{(\epsilon-1)^2 \sin^2 \theta \cos^2 \theta}{(\epsilon \cos \theta + \sqrt{\epsilon-\sin^2 \theta})^2} d\Omega. \quad (8)$$

APPENDIX I

THE δ -FUNCTION

Dirac's δ -function is defined by†

$$\delta(x-a) = \begin{cases} 0 & x \neq a, \\ \infty & x = a; \end{cases} \quad (1)$$

$$\int_{\Delta} \delta(x-a) dx = 1. \quad (2)$$

The integration in (2) is carried out over an arbitrary interval Δ which includes the point $x = a$.

The δ -function has the following properties

$$\delta(x) = \delta(-x), \quad (3)$$

$$\int_{\Delta} f(x) \delta(x-a) dx = f(a), \quad (4)$$

$$\delta(ax) = \frac{1}{|a|} \delta(x), \quad (5)$$

where $f(x)$ is a continuous function. The three-dimensional δ -function is defined in an analogous fashion:

$$\delta(\mathbf{r}-\mathbf{a}) = \delta(x-a_x) \delta(y-a_y) \delta(z-a_z) = \begin{cases} 0 & \mathbf{r} \neq \mathbf{a}, \\ \infty & \mathbf{r} = \mathbf{a}; \end{cases} \quad (6)$$

$$\int_V f(\mathbf{r}) \delta(\mathbf{r}-\mathbf{a}) dV = \begin{cases} f(\mathbf{a}), & \text{if } \mathbf{a} \text{ is inside the volume } V, \\ 0, & \text{if } \mathbf{a} \text{ is outside the volume } V, \end{cases} \quad (7)$$

where $f(\mathbf{r})$ is a continuous function.

The δ -function may be of use to describe the charge distribution of a point particle in space. The volume density of this distribution is given by

$$\rho(\mathbf{r}) = e\delta(\mathbf{r}-\mathbf{a}), \quad (8)$$

where e is the charge and $\mathbf{a}$ is the position vector of the particle.

† A mathematically correct definition of the δ -function requires a generalisation of the usual concept of a function.

It is also possible to define a derivative of the δ -function:

$$\int_a^\Delta f(x) \frac{\partial \delta(x-a)}{\partial x} dx = -\frac{\partial f(a)}{\partial a}, \quad (9)$$

where the integration is carried out by parts. Higher-order derivatives are defined in a similar fashion:

$$\int_a^\Delta f(x) \delta^{(n)}(x-a) dx = (-1)^n f^n(a). \quad (10)$$

The δ -function may also be looked upon as the derivative of a function which exhibits a finite discontinuous change b at the point $x = a$. Thus, if $f(a+0) - f(a-0) \equiv b$ then

$$\frac{\partial f}{\partial x} = b\delta(x-a) + \text{boundary function}. \quad (11)$$

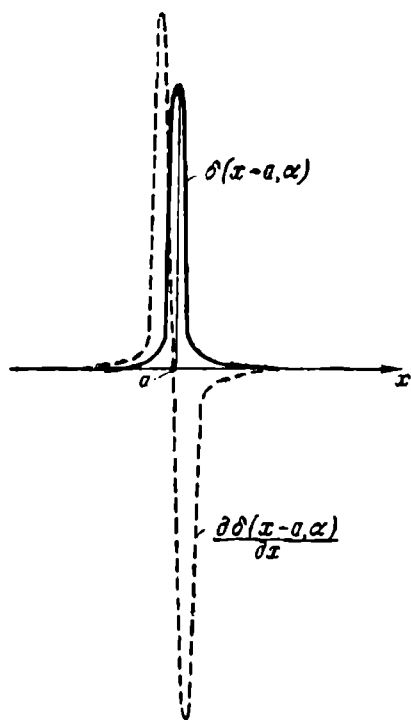


Fig. 101.

A convenient representation of the δ -function and its derivatives may be obtained by considering a graph of the continuous function $\delta(x-a, \alpha)$ which is illustrated in Fig. 101. The parameter α represents the width of the interval within which $\delta(x-a, \alpha)$ is different from zero. The

δ -function and its derivatives are then defined as the following limits:

$$\delta(x-a) = \lim_{\alpha \rightarrow 0} \delta(x-a, \alpha),$$

$$\frac{\partial \delta(x-a)}{\partial x} = \lim_{\alpha \rightarrow 0} \frac{\partial \delta(x-a, \alpha)}{\partial x}.$$

The following are some useful representations of the δ -function

$$\delta(x) = \lim_{\alpha \rightarrow 0} \frac{1}{\pi} \cdot \frac{\alpha}{\alpha^2 + x^2}, \quad (12)$$

$$\delta(x) = \lim_{k \rightarrow \infty} \frac{1}{\pi} \cdot \frac{\sin kx}{x}, \quad (13)$$

$$\delta(x) = \lim_{k \rightarrow \infty} \frac{1}{\pi} \cdot \frac{\sin^2 kx}{kx^2}. \quad (14)$$

These representations may be used to establish the properties of the δ -function which are given by (3)–(5). In evaluating the integral $\int f(x)\delta(x-a)dx$ with the aid of (12)–(14) it should be remembered that the limit corresponding to $\alpha \rightarrow 0$ should be taken after the integration:

$$\int f(x)\delta(x-a)dx = \lim_{\alpha \rightarrow 0} \int f(x)\delta(x-a, \alpha)dx.$$

A consideration of the Fourier integral leads to a further useful representation of the δ -function:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk = \frac{1}{\pi} \int_0^{\infty} \cos kx dk. \quad (15)$$

The two generalised functions $\delta_+(x)$ and $\delta_-(x)$ are similar to the δ -function. They are defined by a relation which is similar to that given by (15):

$$\delta_{\pm}(x) = \frac{1}{2\pi} \int_0^{\infty} e^{\pm ikx} dk. \quad (16)$$

The generalised functions are related to the δ -function:

$$\delta_{\pm}(x) = \frac{1}{2} \delta(x) \pm \frac{i}{2\pi} P \frac{1}{x}, \quad (17)$$

so that $\delta(x) = \delta_+(x) + \delta_-(x)$. The symbol P in (17) represents the principal value of the integral

$$\begin{aligned} \int_{a_1}^{a_2} f(x)\delta_{\pm}(x-a)dx &= \frac{1}{2} f(a) \pm \frac{i}{2\pi} P \int_{a_1}^{a_2} \frac{f(x)}{x-a} dx = \\ &= \frac{1}{2} f(a) \pm \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0} \left[\int_{a_1}^{a-\epsilon} \frac{f(x)}{x-a} dx + \int_{a+\epsilon}^{a_2} \frac{f(x)}{x-a} dx \right], \end{aligned}$$

where $a_1 < a < a_2$, $\epsilon > 0$. The symbol $\int_a^b$ is frequently used instead of $P \int_a^b$.

APPENDIX II

SPHERICAL LEGENDRE FUNCTIONS

The spherical function of order l, m is defined by

$$Y_{lm}(\vartheta, \alpha) = \delta_m \sqrt{\frac{(2l+1)}{4\pi} \cdot \frac{(l-|m|)!}{(l+|m|)!}} P_{lm}(\cos \vartheta) e^{im\alpha}, \quad (1)$$

where ϑ and α are the polar angles and the integers l, m are such that $l \geq 0$, $-l \leq m \leq l$ and $\delta_m = (-1)^m$ when $m \geq 0$ and $\delta_m = 1$ when $m < 0$. In the above definition $P_{lm}(\cos \vartheta)$ represents the associated Legendre polynomial

$$P_{lm}(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|} P_l(x)}{dx^{|m|}}, \quad (2)$$

where $P_l(x)$ is the ordinary Legendre polynomial which is identical with $P_{lm}(x)$ when $m = 0$ since

$$P_l(x) = \frac{1}{2^l l!} \cdot \frac{d^l (x^2-1)^l}{dx^l} = P_{l0}(x). \quad (3)$$

The associated Legendre polynomials are the solutions of the following differential equation:

$$\frac{d}{dx} \left[(x^2-1) \frac{dP_{lm}(x)}{dx} \right] + \left[l(l+1) - \frac{m^2}{x^2-1} \right] P_{lm}(x) = 0. \quad (4)$$

The following formulae are useful in practice:

$$\left. \begin{aligned} Y_{lm}(0, 0) &= \sqrt{\frac{2l+1}{4\pi}} \delta_{m0}, \quad Y_{lm}(\vartheta, \alpha) = (-1)^l Y_{lm}(\pi - \vartheta, \pi + \alpha), \\ P_l(1) &= 1, \quad P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \quad P_{2n+1}(0) = 0; \end{aligned} \right\} \quad (5)$$

$$\left. \begin{aligned} (l+1)P_{l+1}(x) &= (2l+1)xP_l(x) - lP_{l-1}(x), \\ (x^2-1)\frac{dP_l(x)}{dx} &= l[xP_l(x) - P_{l-1}(x)]. \end{aligned} \right\} \quad (6)$$

where the symbol $n!!$ represents the product of all the successive integers having the same parity as n between 2 and n when n is

even, and between 1 and n when n is odd. The spherical functions with $l = 0, 1, 2$ are of the form

$$\left. \begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}}, & Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos \vartheta, \\ Y_{1, \pm 1} &= \mp \sqrt{\frac{3}{8\pi}} \sin \vartheta e^{\pm i\alpha}, & Y_{20} &= \sqrt{\frac{5}{4\pi}} \cdot \frac{3 \cos^2 \vartheta - 1}{2}, \\ Y_{2, \pm 1} &= \mp \sqrt{\frac{15}{8\pi}} \sin \vartheta \cos \vartheta e^{\pm i\alpha}, & Y_{2, \pm 2} &= \sqrt{\frac{15}{32\pi}} \sin^2 \vartheta e^{\pm 2i\alpha}. \end{aligned} \right\} \quad (7)$$

The spherical functions form a complete orthonormal set of functions on the surface of a sphere. This means that

$$\int Y_{lm}^*(\vartheta, \alpha) Y_{l'm'}(\vartheta, \alpha) d\Omega = \delta_{ll'} \delta_{mm'}, \quad (8)$$

where $d\Omega = \sin \vartheta d\vartheta d\alpha$ is an element of solid angle. It also means that an arbitrary function of ϑ and α which has an integrable square may be expanded into a series in terms of the spherical functions, *i.e.*

$$f(\vartheta, \alpha) = \sum_{l,m} a_{lm} Y_{lm}(\vartheta, \alpha). \quad (9)$$

The coefficients a_{lm} are given by

$$a_{lm} = \int Y_{lm}^*(\vartheta, \alpha) f(\vartheta, \alpha) d\Omega. \quad (10)$$

Functions of the form $r^{-l-1} Y_{lm}(\vartheta, \alpha)$ and $r^l Y_{lm}(\vartheta, \alpha)$ are called spherical harmonics. It is easily shown with the aid of (4) that spherical harmonics are special solutions of the Laplace equation at all points other than $r = 0$:

$$\Delta \left[\frac{Y_{lm}(\vartheta, \alpha)}{r^{l+1}} \right] = 0, \quad \Delta [r^l Y_{lm}(\vartheta, \alpha)] = 0. \quad (11)$$

The superposition of spherical harmonics with arbitrary coefficients

$$\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^l (a_{lm} r^{-l-1} + b_{lm} r^l) Y_{lm}(\vartheta, \alpha), \quad (12)$$

is also a solution of the Laplace equation. If $\mathbf{r}(\vartheta, \alpha)$ and $\mathbf{r}'(\vartheta', \alpha')$ are the position vectors of any two points in space which are such that $r > r'$ (*cf.* Fig. 7) then

$$\frac{1}{R} = \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 - 2rr' \cos \gamma + r'^2}} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \gamma). \quad (13)$$

The function R^{-1} is referred to as the generating function for the Legendre polynomials. The following addition theorem holds for spherical functions:

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\vartheta, \alpha) Y_{lm}^*(\vartheta', \alpha'). \quad (14)$$

The angles ϑ, α and ϑ', α' enter symmetrically into (14). Substitution of (14) into (13) yields the following expansion:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{r'^l}{(2l+1) r^{l+1}} Y_{lm}(\vartheta, \alpha) Y_{lm}^*(\vartheta', \alpha'). \quad (15)$$

It follows from (13) that

$$\frac{1}{\sqrt{\cosh \xi - \cos \eta}} = \sqrt{2} \sum_{l=0}^{\infty} e^{-(l+\frac{1}{2})|\xi|} P_l(\cos \eta), \quad (16)$$

where $r'/r = \exp(-|\xi|)$.

APPENDIX III

CYLINDRICAL FUNCTIONS

The cylindrical functions $Z_p(kr)$ are the solutions of the Bessel equation

$$\frac{d^2 Z}{dr^2} + \frac{1}{r} \frac{dZ}{dr} + \left(k^2 - \frac{p^2}{r^2} \right) Z = 0. \quad (1)$$

The solution which is bounded at $r = 0$ for $p \geq 0$ is known as the cylindrical function of the first kind (or the Bessel function of the first kind). It is given by

$$J_p(x) = \frac{x^p}{2^p} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k} k! \Gamma(p+k+1)}. \quad (2)$$

Since the square of p enters into (1), it follows that J_{-p} is also a solution of the above equation. Any linear combination of J_p and J_{-p} will also be a solution. The cylindrical function of the second kind (Neuman's function) is defined by

$$N_p(x) = \frac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi}. \quad (3)$$

This function is occasionally called Weber's function and is denoted by $Y_p(x)$. Cylindrical functions of the third kind (Hankel's functions) which are defined by

$$\left. \begin{aligned} H_p^{(1)}(x) &= J_p(x) + iN_p(x), \\ H_p^{(2)}(x) &= J_p(x) - iN_p(x), \end{aligned} \right\} \quad (4)$$

are also frequently employed.

Any linear combination of any two linearly independent cylindrical functions will be a general solution of the Bessel equation. Such linearly independent functions will, in particular, be the Bessel functions $J_p(x)$ and $J_{-p}(x)$ when p is not an integer. When $p = n$, where n is an integer, the functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent since $J_{-n}(x) = (-1)^n J_n(x)$. When this is so, the general solution will be a linear combination of, say, J_n and N_n .

Cylindrical functions of a purely imaginary argument are known as modified Bessel functions. When n is an integer they are defined by

$$I_n(x) = i^{-n} J_n(ix), \quad K_n(x) = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix). \quad (5)$$

The function $K_n(x)$ is referred to as the Macdonald's function.

In physical problems it is frequently necessary to have approximate expressions for the cylindrical functions of small and large values of the argument. When $|x| \ll 1$, we have

$$\left. \begin{aligned} J_p(x) &\simeq \frac{x^p}{2^p \Gamma(p+1)}, & J_n(x) &\simeq \frac{x^n}{2^n n!}, & I_n(x) &\simeq \frac{x^n}{2^n n!}, \\ J_0(x) &\simeq 1 - \frac{x^2}{4}, & I_0(x) &\simeq 1 + \frac{x^2}{4}, \\ N_n(x) &\simeq -\frac{2^n (n-1)!}{\pi x^n}, & K_n(x) &\simeq \frac{2^{n-1} (n-1)!}{x^n}, \\ N_0(x) &\simeq \frac{2}{\pi} \ln \frac{\gamma x}{2}, & K_0(x) &\simeq \ln \frac{2}{\gamma x}, \end{aligned} \right\} \quad (6)$$

where $n \geq 1$, $\ln \gamma = 0.5772$.

The expressions for the Hankel functions for $|x| \ll 1$ may be obtained from (6) with the aid of (4). In particular,

$$H_0^{(1,2)}(x) \simeq 1 \pm \frac{2i}{\pi} \ln \frac{\gamma x}{2} = \pm \frac{2i}{\pi} \ln \left(\frac{\gamma x}{\pm 2i} \right). \quad (7)$$

Asymptotic expressions for the cylindrical functions ($|x| \ll 1$) are

$$\left. \begin{aligned} J_p(x) &\simeq \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{p\pi}{2} - \frac{\pi}{4} \right), \\ N_p(x) &\simeq \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{p\pi}{2} - \frac{\pi}{4} \right), \\ H_p^{(1,2)}(x) &\simeq \sqrt{\frac{2}{\pi x}} e^{\pm i \left(x - \frac{p\pi}{2} - \frac{\pi}{4} \right)}, \\ I_n(x) &\simeq \frac{e^x}{\sqrt{2\pi x}}, & K_n(x) &\simeq \sqrt{\frac{\pi}{2x}} e^{-x}. \end{aligned} \right\} \quad (8)$$

The following relationships may be shown to hold:

$$\left. \begin{aligned} Z_{p-1}(x) + Z_{p+1}(x) &= \frac{2p}{x} Z_p(x), \\ Z_{p-1}(x) - Z_{p+1}(x) &= 2Z'_p(x), \quad Z'_0(x) = -Z_1(x); \end{aligned} \right\} \quad (9)$$

$$\left. \begin{aligned} I_{p-1}(x) - I_{p+1}(x) &= \frac{2p}{x} I_p(x), & I_{p-1}(x) + I_{p+1}(x) &= 2I'_p(x), \\ K_{p-1}(x) - K_{p+1}(x) &= -\frac{2p}{x} K_p(x), \\ K_{p-1}(x) + K_{p+1}(x) &= -2K'_p(x). \end{aligned} \right\} \quad (10)$$

where Z_p represents J_p , N_p , or $H_p^{(1,2)}$.

The Bessel functions may also be written in integral form:

$$J_n(x) = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} e^{i(x \sin \varphi - n\varphi)} d\varphi = \frac{(-i)^n}{2\pi} \int_{\alpha}^{\alpha+2\pi} e^{i(x \cos \varphi - n\varphi)} d\varphi, \quad (11)$$

where α is a real number.

Integrals involving Bessel functions may be evaluated with the aid of (9) and (10). In particular,

$$\int x^p Z_p(x) dx = x^p Z_{p-1}(x), \quad \int x^{-p} Z_p(x) dx = -x^{-p} Z_{p+1}(x), \quad (12)$$

$$\int_0^1 x J_p(\alpha x) J_p(\beta x) dx = \frac{\alpha J'_p(\alpha) J_p(\beta) - \beta J'_p(\beta) J_p(\alpha)}{\beta^2 - \alpha^2}. \quad (13)$$

The following integral formulae are also employed in the present collection ($\text{Re } k > 0$):

$$(\rho^2 + z^2)^{-\frac{1}{2}} = \int_0^{\infty} e^{-k|z|} J_0(k\rho) dk, \quad (14)$$

$$\int_0^{\infty} \frac{\cos px \, dx}{(q^2 + x^2)^{s+1}} = \sqrt{\pi} \left(\frac{p}{2q}\right)^{s+\frac{1}{2}} \cdot \frac{K_{s+\frac{1}{2}}(pq)}{\Gamma(s+1)}, \quad (15)$$

$$\int_0^{\infty} \frac{J_0(xr) x \, dx}{x^2 + k^2} = k K_0(kr). \quad (16)$$

Spherical Bessel functions of the first kind and spherical Hankel functions of the first and second kind [denoted by the common symbol $z_l(\rho)$] are defined by

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+\frac{1}{2}}(\rho), \quad h_l^{(1,2)}(\rho) = \sqrt{\frac{\pi}{2\rho}} H_{l+\frac{1}{2}}^{(1,2)}(\rho). \quad (17)$$

For small ρ ,

$$j_l(\rho) = \frac{\rho^l}{(2l+1)!!}, \quad h_l^{(1,2)}(\rho) \text{ diverges as } \rho^{-l-1}. \quad (18)$$

For large ρ ,

$$\left. \begin{aligned} j_l(\rho) &= \frac{1}{\rho} \cos\left[\rho - \frac{(l+1)\pi}{2}\right], \\ h_l^{(1,2)}(\rho) &= \frac{1}{\rho} e^{\pm i\left[\rho - \frac{(l+1)\pi}{2}\right]}. \end{aligned} \right\} \quad (19)$$

When $R = |\mathbf{r} - \mathbf{r}'|$ (cf. Fig. 7) then

$$\frac{e^{ikR}}{R} = 4\pi ik \sum_{l,m} j_l(kr') h_l^{(1)}(kr) Y_{lm}(\vartheta, \alpha) Y_{lm}^*(\vartheta', \alpha') \quad (r > r'). \quad (20)$$

The functions $z_l(kr) Y_{lm}(\vartheta, \alpha)$ are special solutions of the wave equation for monochromatic waves: $\Delta\varphi + k^2\varphi = 0$. Any linear combination of these functions is also a solution.

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